Hadron Distribution Amplitudes from Lattice QCD

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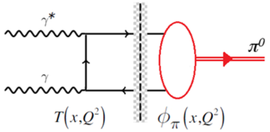
OUTLINE

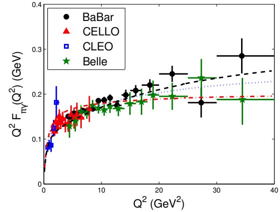
HARD EXCLUSIVE PROCESSES

Figure: The $\gamma\gamma^* \to \pi$ form factor



- Q^2 : Momentum transfer (large).
- x: Momentum fraction of the pion.

Figure: Experimental results (CLEO, BaBar, Belle) together with sample theoretical predictions.



- Example: pion transition form factor. At large momentum transfer Q, collinear factorisation is possible

$$F_{\pi\gamma}(Q^2) = \frac{2f_\pi}{3}\int_0^1 dx\, \underbrace{T^H_{\gamma\pi}(x,Q^2)}_{\text{hard}}\, \underbrace{\phi_\pi(x,Q^2)}_{\text{soft}}\,.$$

- **Distribution amplitudes (soft):**

$$\phi_\pi(x,Q^2) = 6x(1-x)\left[1 + \sum_{n=1}^{\infty} a^\pi_{2n}(Q^2)\, C^{3/2}_{2n}(2x-1)\right],$$

Interpreted as the probability amplitude for finding the $\bar{q}q$ valence state in the pion with momentum fraction x.

- **Hard process:**

$$T^H_{\gamma\pi}(x,Q^2) = \frac{1}{Q^2(1-x)}\left[1 + \mathrm{O}(\alpha_s)\right].$$

- At tree level in PT,

$$Q^2 F_{\pi\gamma}(Q^2) = 2f_\pi\left[1 + a^\pi_2(Q^2) + a^\pi_4(Q^2) + \ldots\right].$$

HADRON DISTRIBUTION AMPLITUDES.

- Provide information on the structure of bound hadronic states.
- **Difficult** to extract from experiment without contamination from other hadronic uncertainties.
- **Universal**. Involved in many processes:
  - weak exclusive B-meson and Λ_b decays (LHCb).
  - electric to magnetic nucleon form factor $Q^2 \sim 14$ GeV ^{2} (JLab, FAIR).
  - electric neutron form factor $Q^2 \sim 8$ GeV ^{2} (JLab, FAIR).
  - electroproduction of nucleon resonances at large $Q^2 \sim 14$ GeV^2 (JLab).
- DA moments $\Leftrightarrow$ **matrix elements of local operators** (computable on the lattice).

MOMENTS AND GEGENBAUER MOMENTS

- DA's defined through non-local matrix elements involving light-like quark separation, e.g.

$$\langle 0|\bar{d}(-z)\gamma_\mu\gamma_5[-z,z]u(z)|\pi^+\rangle = if_\pi p_\mu \int_{-1}^{1} d\xi e^{-i\xi p\cdot z}\phi_\pi(\xi,\mu),$$

  f_π is the pion decay constant, z is a lightlike vector, $\xi = 2x-1$ and $[-z,z]$ is a Wilson line connecting the u and $\bar{d}$ fields.

- Moments of DA's are related to matrix elements of leading twist local operators. One distinguishes,
  1. " Simply moments", $\langle \xi^n \rangle = \int d\xi \xi^n \phi(\xi,\mu), \quad \xi = 2x-1.$
  2. " Gegenbauer moments ",

$$a_n^\pi(\mu) = N\int d\xi C_n^{3/2}\phi(\xi,\mu) \implies \phi(\xi,\mu) = \tfrac{3}{4}\left(1+\sum_{n=1}^{\infty} a_n^\pi(\mu)C_n^{3/2}(\xi)\right).$$

  [Collinear conformal symmetry, SL$(2,\mathbb{R})$, $C_n^{3/2}(\xi)$ analogous to $Y^{lm}(\theta,\phi)$ in O(3).]

- They are connected by simple algebraic relations, e.g., $\frac{12}{7}a_2^\pi = 5\langle\xi^2\rangle - 1$.

- One expects $a_n^\pi \ll 1$, on the other hand $\langle\xi^n\rangle$ dominated by the 1 $\implies$ Loss of precision on the way $\langle\xi^n\rangle \to a_n^\pi$.

COMPUTED QUANTITIES ON THE LATTICE.

- Pseudoscalar mesons:
  - second moments for π, K **:** $\langle \xi^2 \rangle_{\pi/K} \Longleftrightarrow a_2^{\pi/K}(\mu)$,
  - first moments for K (No G-parity)**:** $\langle \xi \rangle_K \Longleftrightarrow a_1^K(\mu)$.
- Vector mesons:
  - longitudinal and transverse first moments for K^* **:** $\langle \xi \rangle^{\parallel}_{K^*}$, $\langle \xi \rangle^{\perp}_{K^*}$,
  - longitudinal second moments for K^*,ρ, ϕ **:** $\langle \xi^2 \rangle^{\parallel}_{K^*}$, $\langle \xi^2 \rangle^{\parallel}_{\rho}$, $\langle \xi^2 \rangle^{\parallel}_{\phi}$.
- Nucleon and N^*:
  - leading twist normalisation constant, **:** f_N,
    [probability amplitude to find the three valence quarks at one space point]
  - next-to-leading twist normalisation constants, **:** λ_1, λ_2,
    [related to the normalisation of the P-wave three-quark wave functions]
  - first and second moments **:** $\phi_i^{(1)} = (\phi^{100}, \phi^{010}, \phi^{001})$, $\phi_i^{(2)} = (\phi^{200}, \phi^{020}, \phi^{002}, \phi^{011}, \phi^{101}, \phi^{110})$.
  - momentum fractions carried by the 3 valence quarks in the proton **:** $\langle x_1 \rangle$, $\langle x_2 \rangle$, $\langle x_3 \rangle$.

All quantities related to ratios of matrix elements of local operators

STEPS REQUIRED FOR A LATTICE FORMULATION

- Find lattice discretised operators that transform according to irreducible representations of the (spinorial) **hypercubic group** H(4).
- Compute the non - perturbative renormalisation constants for these operators (RI-SMOM scheme).
- Compute bare ratios of matrix elements of these operators from suitable correlation functions.
- Perform the extrapolations:
  - Pion masses: $m_\pi \to m_\pi^{\text{phys}}$,
  - Lattice volume: $V \to \infty$,
  - Continuum limit: $a \to 0$.

CHALLENGES

- Bare ratios: Matrix elements with derivatives lead to poor signals.
  Large statistics needed.

- Renormalisation: Additional mixing among the discretised version
  of the operators is allowed. $\mathcal{Z}^{\text{mixing}}$ needed.

- Lattice extrapolations: need to be kept under control.

- Mesons: $a_2(\mu) \propto \langle \xi^2 \rangle - \frac{1}{5}$, and $\langle \xi^2 \rangle \sim 0.2$. Errors "blow up"
  [We propose a more "direct" calculation for $a_2(\mu)$].

- Baryons:
  - Problems get amplified.
  - Negative parity resonances ($N^*(1535)$, $N^*(1650)$) difficult.

$\langle \xi^2 \rangle$ AND $a_2(Q^2)$ (I)

[c.l. stands for continuum limit]

- Local operators involved:

$$\mathcal{O}^{5-}_{\{\mu\nu\rho\}}(x) = \bar{d}(x)\left(\gamma_{\{\mu}\vec{D}_\nu\vec{D}_{\rho\}} + \gamma_{\{\mu}\overleftarrow{D}_\nu\overleftarrow{D}_{\rho\}} - 2\gamma_{\{\mu}\overleftarrow{D}_\nu\vec{D}_{\rho\}}\right)\gamma_5 u(x),$$

$$\mathcal{O}^{5+}_{\{\mu\nu\rho\}}(x) = \bar{d}(x)\left(\gamma_{\{\mu}\vec{D}_\nu\vec{D}_{\rho\}} + \gamma_{\{\mu}\overleftarrow{D}_\nu\overleftarrow{D}_{\rho\}} + 2\gamma_{\{\mu}\overleftarrow{D}_\nu\vec{D}_{\rho\}}\right)\gamma_5 u(x),$$

$\{...\}$: symmetrisation of the Lorentz indices and subtraction of traces.

- Renormalised matrix elements:

$$\langle 0|\left[\mathcal{O}^{5-}_{\{\mu\nu\rho\}}\right]_R|\pi\rangle \underset{\text{c.l.}}{\propto} \int_{-1}^{1} d\xi \cdot \xi^2 \cdot \phi(\xi, Q^2) \equiv \langle \xi^2 \rangle_R,$$

$$\langle 0|\left[\mathcal{O}^{5+}_{\{\mu\nu\rho\}}\right]_R|\pi\rangle \underset{\text{c.l.}}{\propto} \int_{-1}^{1} d\xi \cdot 1 \cdot \phi(\xi, Q^2) \equiv 1.$$

- Second Gegenbauer Moment:

$$\frac{12}{7}[a_2^\pi]_R \propto \langle 0|\left[\mathcal{O}^{5-}_{\{\mu\nu\rho\}}\right]_R - \frac{1}{5}\left[\mathcal{O}^{5+}_{\{\mu\nu\rho\}}\right]_R|\pi\rangle \underset{\text{c.l.}}{\propto} \langle \xi^2 \rangle_R - \frac{1}{5}.$$

LATTICE ARTIFACTS ON $\mathcal{O}^{5+}_{\{4ij\}}$

Violations of the identity of, $[\mathcal{R}^{5+}_{\{4ij\};4}]_R = \langle 1^2 \rangle$

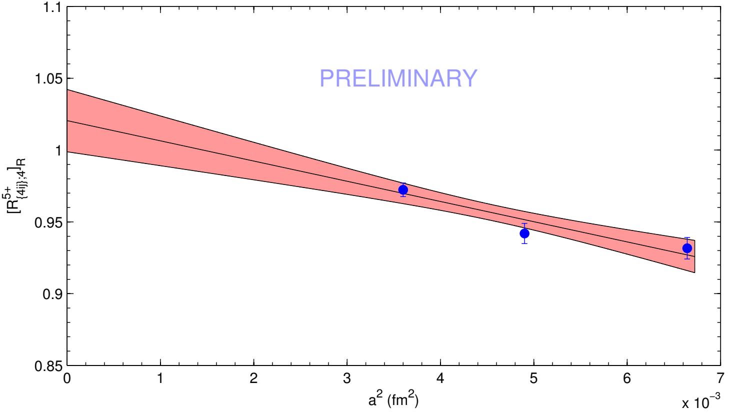
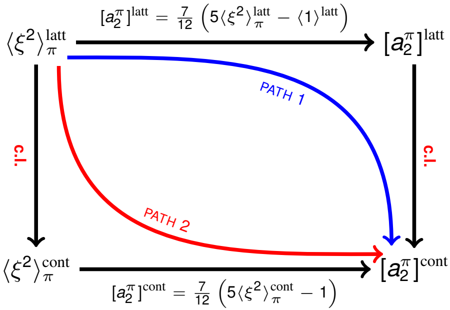


Figure: Study of $\mathcal{R}^{5+}_{\{4\mu\nu\};4} = \langle 1^2 \rangle$ for $m_\pi \sim 280$ MeV.

$\langle \xi^2 \rangle$ AND $a_2(Q^2)$ (II)



- Typical values of $\langle \xi^2 \rangle_\pi^{\rm cont} \sim 0.25 \pm 0.03 \Longrightarrow [a_2^\pi]^{\rm cont} \sim 0.146 \pm 0.087$.
- Errors “blow up”:

$$\frac{\Delta\left(\langle \xi^2 \rangle_\pi^{\rm cont}\right)}{\langle \xi^2 \rangle_\pi^{\rm cont}} \sim 0.12 \quad \Longrightarrow \quad \frac{\Delta [a_2^\pi]^{\rm cont}}{[a_2^\pi]^{\rm cont}} \sim 0.6.$$

- In the situation where continuum extrapolation is the main problem $\Longrightarrow$ PATH 1 leads to smaller errors than PATH 2.

SIMULATION DETAILS

β	N_f	κ_l	$L \times T$	$\#$meas $\times M$	a[fm]	m_π[MeV]	$m_\pi * L$
5.20	2	0.13596	32×64	1999×4	0.0815	280	3.7
5.29	2	0.13620	24×48	1170×2	0.07	430	3.7
5.29	2	0.13620	32×64	1998×2	0.07	422	4.8
5.29	2	0.13632	32×64	1999×1	0.07	294	3.4
5.29	2	0.13632	40×64	2028×2	0.07	289	4.2
5.29	2	0.13632	64×64	1237×2	0.07	285	6.7
5.29	2	0.13640	64×64	1599×3	0.07	150	3.5
5.40	2	0.13640	32×64	982×2	0.06	491	4.8
5.40	2	0.13647	32×64	1999×2	0.06	430	4.2
5.40	2	0.13660	48×64	2178×2	0.06	260	3.8

Table: M stands for the number of sources measured per configuration.

- Standard Wilson gauge action.
- $N_f = 2$ non-perturbative Wilson-clover fermions.
- Propagators smeared at the source (300 - 600 Wuppertal smearing steps).

RENORMALISATION

- $\mathcal{O}^{5\pm}_{\{\mu\nu\rho\}}$ need to be renormalised:
  - $\Rightarrow$ Lattice non - perturbative renormalisation: **RI-SMOM** scheme.
  - $\Rightarrow$ Conversion to $\overline{\mathrm{MS}}$ at high μ^2.

- Operator mixing ($i, j \in \{1, 2, 3\}$ with $i \neq j$):

$$\begin{pmatrix} [\mathcal{O}^{5-}_{\{4ij\}}]_R \\ [\mathcal{O}^{5+}_{\{4ij\}}]_R \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12}^{\mathrm{new}} \\ 0 & Z_{22} \end{pmatrix} \begin{pmatrix} \mathcal{O}^{5-}_{\{4ij\}} \\ \mathcal{O}^{5+}_{\{4ij\}} \end{pmatrix}$$

- Preliminary results $\mu^2 = 4\mathrm{GeV}^2$ in $\overline{\mathrm{MS}}$:

β	5.20	5.29	5.40
Z_{11}/Z_A	1.988(15)	2.011(11)	2.031(10)
Z_{12}/Z_A	−0.194(6)	−0.200(6)	−0.205(6)
Z_{22}/Z_A	1.519(4)	1.499(3)	1.476(2)

Z_A: renormalisation of the axial vector current.

VARIATIONAL METHOD

- Find an optimal interpolator for the π. Interpolators used,

$$P = \bar{q}\gamma_5 q, \quad A_0 = \bar{q}\gamma_4\gamma_5 q.$$

- Construct a cross correlation matrix,

$$C(t) = \begin{pmatrix} \langle P^\dagger | P \rangle & \langle P^\dagger | A_0 \rangle \\ \langle A_0^\dagger | P \rangle & \langle A_0^\dagger | A_0 \rangle \end{pmatrix}$$

- We solve the Generalised Eigenvalue Problem,

$$C^{-1/2}(t_0)C(t)C^{-1/2}(t_0)\psi^\alpha(t, t_0) = \lambda^\alpha(t, t_0)\psi^\alpha(t, t_0).$$

- Eigenvector corresponding to the lowest eigenvalue $\Longrightarrow$ optimal interpolation,

$$\mathcal{I}_{\text{opt}}(t) = P(t) + \gamma A_0(t).$$

$a_2(\mu)$ AND $\phi_\pi(x,\mu)$ AT $\mu = 2$ GeV

[Regensburg Lattice QCD collaboration]

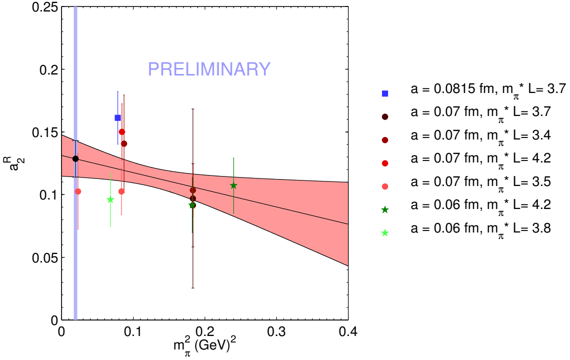


Figure: Gegenbauer Moment a_2 at the scale 2 GeV. No chiral logs [Chen and Stewart '04].

$$\phi(x) = 6x(1-x)\left[1 + a_2 C_2^{3/2}(2x-1)\right]$$

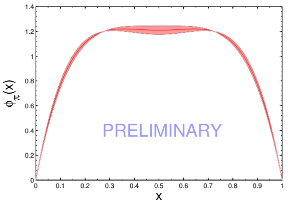


Figure: Momentum fraction distribution of valence quarks in the pion (distribution amplitude).

RESULTS

- **Pion**:

Reference	$\langle \xi^2 \rangle_\pi$	a_2^π	
[QCDSF/UKQCD coll, '06]	0.269(39)	0.201(114)	
[RBC/UKQCD coll, '10]	0.28(1)(2)	0.233(30)(60)	
[This work]	0.2344(47)(?)	0.129(15)(?)	PRELIMINARY

Table: Results for the pion second moments at $\mu = 2$ GeV.

- **Kaon** first moments:

Reference	$\langle \xi \rangle_K$	a_1^K
[QCDSF/UKQCD coll, '06]	0.0272(5)(16)	0.0453(9)(29)
[RBC/UKQCD coll, '10]	0.036(1)(2)	0.060(16)(33)

Table: Results for the kaon first moments at $\mu = 2$ GeV.

- **Kaon** second moments:

Reference	$\langle \xi^2 \rangle_K$	a_2^K
[QCDSF/UKQCD coll, '06]	0.260(6)(16)	0.175(18)(47)
[RBC/UKQCD coll, '10]	0.26(1)(2)	0.175(30)(58)

Table: Results for the kaon second moments at $\mu = 2$ GeV.

VECTOR MESONS

- Two types of distribution amplitudes at twist-2: $\phi_{||}(x,\mu)$, $\phi_{\perp}(x,\mu)$
- Longitudinally polarised vector mesons,

Quantity	[RBC/UKQCD coll, '10]	[QCDSF/UKQCD coll, '07]
$\langle\xi^2\rangle_\rho^{\|}$	0.27(1)(2)	
$\langle\xi^1\rangle_{K^*}^{\|}$	0.043(2)(3)	0.033(2)(5)
$\langle\xi^2\rangle_{K^*}^{\|}$	0.25(3)(2)	
$\langle\xi^2\rangle_\phi^{\|}$	0.25(2)(1)	

Table: Results for the moments of longitudinally polarised vector mesons at $\mu = 2$ GeV.

- Transversely polarised vector mesons,

Quantity	[QCDSF/UKQCD coll, '07]
$\langle\xi^1\rangle_{K^*}^{\perp}$	0.030(2)(8)

Table: Results for the moments of transversely polarised vector mesons at $\mu = 2$ GeV.

BARYONS

[Phys.Rev. D89 (2014) 094511]

- Momentum fractions in the proton,

$$\langle x_1 \rangle = 0.372(7),$$

$$\langle x_2 \rangle = 0.314(3),$$

$$\langle x_3 \rangle = 0.314(7).$$

Figure: Comparison of lattice results for the nucleon shape parameters (black circles) to QCD sum rule predictions (red symbols), light-cone sum rules (blue symbols) and the BK model (orange crosses).

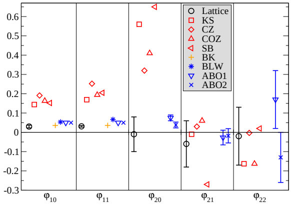
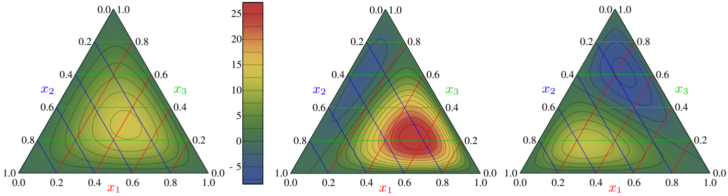


Figure: Barycentric plots of the nucleon [left], $N^*(1650?)$ [center] and $N^*(1535?)$ [right] wave functions. Only the first moments of the distribution amplitudes included.



PERSPECTIVES

- Finish the analysis for the PDA's: Extrapolate $V \to \infty, m_\pi \to m_\pi^{\mathrm{phys}}, a \to 0$.
- Explore the possibilities to compute DA's moments for the vector mesons.
- Set of ensembles with $N_f = 2 + 1$ open boundary conditions will be shortly available.
  - access to finer lattices $\Longrightarrow$ better control on the contnuum extrapolation
  - Repeat all program, including K mesons and full baryon octet.
- Explore η, η' DA's (disconnected diagrams involved).