The Next-to-leading Order JIMWLK Equation

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How scattering amplitudes and cross sections depend on the collision energy $\sqrt{s}$?

At low energy hadrons consists of relatively small number of partons. As the collision energy (rapidity) increases, new partons are emitted (*Weizsacker Williams radiation*).

As long as the density of partons remains small, new particles are created *linearly* (BFKL): the number of new partons due to the increase in collision energy is proportional to the number of emitting partons (so the density grows exponentially).

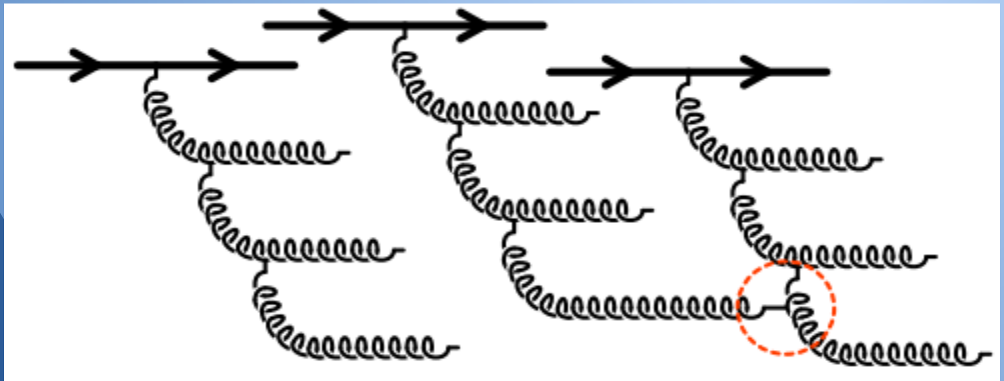
Unfortunately, BFKL suffers from two problems:

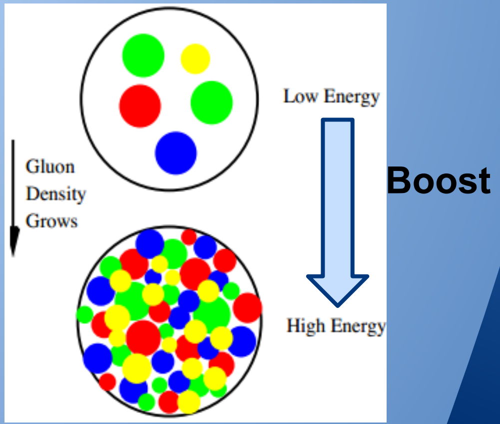
a. *Infrared Diffusion*

b. *Unitarity Violation* - the cross sections computed by using BFKL grow as $\sim s^{\alpha_P - 1}$ and thus violate the *Froissart bound*.

What happens if we push to higher collision energy?

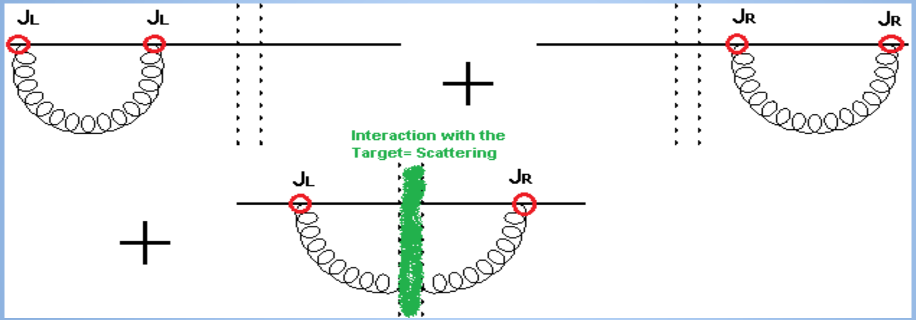
Eventually gluons start overlapping with each other and **new parton emission become a *collective process***.
Emission process becomes *non-linear* and leads to ***gluon saturation*** phenomena (also known as *Color Glass Condensate (CGC) / JIMWLK*). Gluon density grows logarithmically instead of exponentially (as in the case of BFKL).





Leading-Order JIMWLK

The JIMWLK equation, $\frac{d}{dY}\mathcal{O} = -H^{JIMWLK}\mathcal{O}$, describes the rapidity (denoted by Y) evolution of observables $\mathcal{O}$ in scattering process. It consists of three terms - interaction of the probe with the target fields and two virtual terms:



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The target is modeled by a fixed background field A.
The S-matrix of a fast particle interacting with a gluonic field A is the Wilson line (*Eikonal Approximation*):

$$S^{ab}(x) = \left[P\exp\left\{ig\int dx^+ T_c A_c^-(x^+, x)\right\}\right]^{ab}$$

x is a two dimensional transverse coordinate.

The LO JIMWLK Equation

The JIMWLK Hamiltonian can be obtained by computing the expectation value of the S-matrix operator (expanded to first order in longitudinal phase space):

$$H^{JIMWLK} = \langle\psi|\,\hat{S} - 1\,|\psi\rangle$$

The leading order wave function of fast hadron:

$$|\psi\rangle = (1 - g_s^2\,\kappa_0\,JJ)\,|\,no\,soft\,gluons\rangle + g_s\kappa_1\,J\,|\,one\,soft\,gluon\rangle$$

Then:

$$\mathcal{H}_{LO\,JIMWLK} = -\frac{\alpha_s}{2\pi}\int d^2x\,d^2y\,d^2z\,\frac{(x-z)(y-z)}{(x-z)^2(y-z)^2}\left[J_L^a(x)J_L^a(y) + J_R^a(x)J_R^a(y) - 2J_L^a(x)S^{ab}(z)J_R^b(y)\right]$$

J_L and J_R are operators acting on Wilson lines as rotations:

$$J_L^a(x)S^{ij}(w) = (t^a S(w))^{ij}\,\delta(w-x)$$

$$J_R^a(x)S^{ij}(w) = (S(w)t^a)^{ij}\,\delta(w-x)$$

This Hamiltonian describes the evolution of scattering amplitudes *for large values of rapidity*. It is a *non-linear functional* equation and it takes into account both linear growth as well as saturation effects.

Motivations for NLO JIMWLK Equation

The LO JIMWLK is only a first term in an infinite perturbative series:

$$H_{JIMWLK} = H_{LO}(\alpha_s) + H_{NLO}(\alpha_s^2) + H_{NNLO}(\alpha_s^3) + \ldots$$

The NLO term is necessary because:

a. NLO Corrections are *known to be large*.

b. Built-in information on the *running coupling* - better phenomenology. The running is known to slow down the evolution.

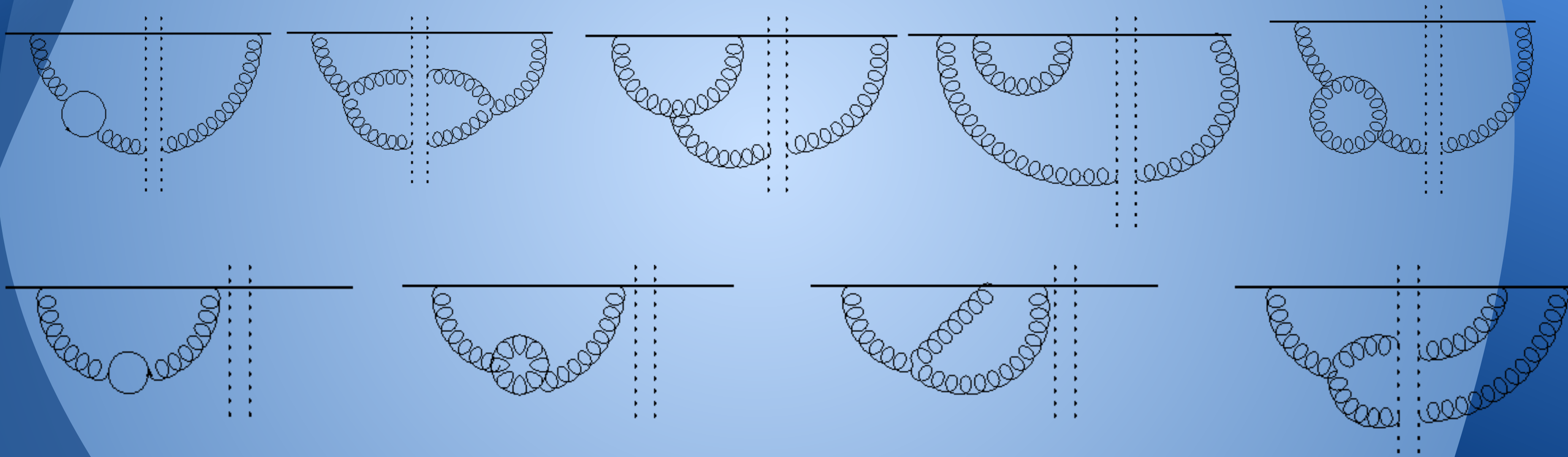
c. To get the *region of applicability* of the leading order equation.

d. Important step towards *all order resummation*.

NLO JIMWLK reduces to NLO BFKL in linear approximation.

NLO Correction from Perturbative Approach

The NLO JIMWLK sums up the following diagrams:



About 30 different diagrams. We managed to avoid fully calculating all these contributions.

Towards NLO JIMWLK

General structure of the wave-function up to g^3 (normalization up to g^4), which has the following structure:

$$|\psi\rangle = (1 - g_s^2\,\kappa_0\,JJ - g_s^4(\delta_1\,JJ + \delta_2\,JJJ + \delta_3\,JJJJ)\,|\,no\,soft\,gluons\rangle + \\ + (\,g_s\kappa_1\,J + g_s^3\epsilon_1\,J + g_s^3\,\epsilon_2\,J\,J)\,|\,one\,soft\,gluon\rangle + g_s^2(\epsilon_3\,J + \epsilon_4\,JJ)\,|\,two\,soft\,gluons\rangle + g_s^2\,\epsilon_5\,J\,|\,q\,\bar{q}\rangle$$

Combining this wave function with:

a. *Expected symmetries* of the NLO kernel, SU(N)xSU(N).

b. *Unitarity*, while S=1 we should get no evolution.

We can find the general form of the NLO JIMWLK Hamiltonian.

The General form of NLO JIMWLK Hamiltonian

$$H^{NLO\ JIMWLK} = \int_{x,y,z} K_{JSJ}(x,y;z)\,\left[J^a_L(x)\,J^a_L(y) + J^a_R(x)\,J^a_R(y) - 2\,J^a_L(x)\,S^{ab}_A(z)\,J^b_R(y)\right] +$$

$$+\int_{x\,y\,z\,z'} K_{JSSJ}(x,y;z,z')\left[f^{abc}\,f^{def}\,J^a_L(x)\,S^{be}_A(z)\,S^{cf}_A(z')\,J^d_R(y) - N_c\,J^a_L(x)\,S^{ab}_A(z)\,J^b_R(y)\right] +$$

$$+\int_{x,y,z,z'} K_{q\bar{q}}(x,y;z,z')\,[2\,J^a_L(x)\,tr[S^\dagger(z)\,T^a\,S(z')T^b]\,J^b_R(y) - J^a_L(x)\,S^{ab}_A(z)\,J^b_R(y)] +$$

$$+\int_{w,x,y,z,z'} K_{JJSSJ}(w;x,y;z,z')f^{acb}\left[J^d_L(x)\,J^e_L(y)\,S^{dc}_A(z)\,S^{eb}_A(z')\,J^a_R(w) - J^a_L(w)\,S^{cd}_A(z)\,S^{be}_A(z')\,J^d_R(x)\,J^e_R(y) +\right.$$

$$\left.+\frac{1}{3}[\,J^c_L(x)\,J^b_L(y)\,J^a_L(w) - J^c_R(x)\,J^b_R(y)\,J^a_R(w)]\right] +$$

$$+\int_{w,x,y,z} K_{JJSJ}(w;x,y;z)\,f^{bde}\left[J^d_L(x)\,J^e_L(y)\,S^{ba}_A(z)\,J^a_R(w) - J^a_L(w)\,S^{ab}_A(z)\,J^d_R(x)\,J^e_R(y) +\right.$$

$$\left.+\frac{1}{3}[\,J^d_L(x)\,J^e_L(y)\,J^b_L(w) - J^d_R(x)\,J^e_R(y)\,J^b_R(w)]\right]$$

Shortcut to the kernels

While we obtained the general form of the Hamiltonian the 5 kernels still have to be determined. How did we manage to find them?

Smart trick:

Evolution equation of ***quark dipole*** - $s(x,y) = \frac{1}{N}\mathrm{Tr}(S^{\dagger}(x)S(y))$

$$\frac{d}{dY}s(x,y) = -H_{NLO\ JIMWLK}\, s(x,y)$$

The NLO BK which was computed by I. Balitsky and G. A. Chirilli in hep-ph/**0710.4330 (PRD).**

Evolution equation of ***SU(3) Baryon*** - $B(x,y,z) = \epsilon^{ijk}\epsilon^{lmn}S^{il}(x)S^{jm}(y)S^{kn}(z)$

$$\frac{d}{dY}B(x,y,z) = -H_{NLO\ JIMWLK}\, B(x,y,z)$$

Connected part was computed by V. A. Grabovsky in **hep-ph/1307.5414**.

The Kernels for gauge invariant operators (color singlet amplitudes)

$$K_{JSJ}(x,y;z) = -\frac{\alpha_s^2}{16\pi^3}\frac{(x-y)^2}{X^2Y^2}\Big[b\ln(x-y)^2\mu^2 - b\frac{X^2-Y^2}{(x-y)^2}\ln\frac{X^2}{Y^2} + (\frac{67}{9}-\frac{\pi^2}{3})N_c - \frac{10}{9}n_f\Big] - \frac{N_c}{2}\int_{z'}\tilde{K}(x,y,z,z')$$

Here μ is the normalization point in the $\overline{MS}$ scheme and $b = \frac{11}{3}N_c - \frac{2}{3}n_f$ is the first coefficient of the β-function.

$$K_{JSSJ}(x,y;z,z') = \frac{\alpha_s^2}{16\,\pi^4}\Big[-\frac{4}{(z-z')^4} + \Big\{2\frac{X^2Y'^2+X'^2Y^2-4(x-y)^2(z-z')^2}{(z-z')^4[X^2Y'^2-X'^2Y^2]} + \frac{(x-y)^4}{X^2Y'^2-X'^2Y^2}\Big[\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\Big] + \frac{(x-y)^2}{(z-z')^2}\Big[\frac{1}{X^2Y'^2}-\frac{1}{X'^2Y^2}\Big]\Big\}\ln\frac{X^2Y'^2}{X'^2Y^2}\Big] + \tilde{K}(x,y,z,z')$$

$$\tilde{K}(x,y,z,z') = \frac{i}{2}\,[K_{JJSSJ}(x;x,y;z,z') - K_{JJSSJ}(y;x,y;z,z') - K_{JJSSJ}(x;y,x;z,z') + K_{JJSSJ}(y;y,x;z,z')]$$

$$K_{q\bar{q}}(x,y;z,z') = -\frac{\alpha_s^2\,n_f}{8\,\pi^4}\Big\{\frac{X'^2Y^2+Y'^2X^2-(x-y)^2(z-z')^2}{(z-z')^4(X^2Y'^2-X'^2Y^2)}\ln\frac{X^2Y'^2}{X'^2Y^2} - \frac{2}{(z-z')^4}\Big\}$$

$$K_{JJSJ}(w;x,y;z) = -i\frac{\alpha_s^2}{4\,\pi^3}\Big[\frac{X\cdot W}{X^2\,W^2} - \frac{Y\cdot W}{Y^2\,W^2}\Big]\ln\frac{Y^2}{(x-y)^2}\ln\frac{X^2}{(x-y)^2}$$

$$K_{JJSSJ}(w;x,y;z,z') = -i\frac{\alpha_s^2}{2\,\pi^4}\Big(\frac{X_iY'_j}{X^2Y'^2}\Big)\Big(\frac{\delta_{ij}}{2(z-z')^2} + \frac{(z'-z)_iW'_j}{(z'-z)^2W'^2} + \frac{(z-z')_jW_i}{(z-z')^2W^2} - \frac{W_iW'_j}{W^2W'^2}\Big)\ln\frac{W^2}{W'^2}$$

Kernels for color non-singlets operators

Most of our interest is in gauge invariant amplitudes. For the sake of completeness it is interesting to find the kernels applicable for action on non-gauge invariant structures. This has been done by comparison of the evolution equation of one, two and three Wilson lines $S^{ij}(x)$ $S^{ij}(x)S^{kl}(y)$ and with $S^{ij}(x)S^{kl}(y)S^{mn}(z)$ of ***I. I. Balitsky*** and ***G. A. Chirilli*** (hep-ph/1309.7644).

$$K_{JSJ}(x,y,z) \to \bar{K}_{JSJ}(x,y,z) \equiv K_{JSJ}(x,y,z) +$$
$$+ \frac{\alpha_s^2}{16\pi^3}\left\{\left[\frac{1}{X^2}+\frac{1}{Y^2}\right]\left[\left(\frac{67}{9}-\frac{\pi^2}{3}\right)N_c - \frac{10}{9}n_f\right] + \frac{b}{X^2}\ln X^2\mu^2 + \frac{b}{Y^2}\ln Y^2\mu^2\right\}$$

$$K_{JSSJ}(x,y;z,z') \to \bar{K}_{JSSJ}(x,y;z,z') \equiv K_{JSSJ}(x,y;z,z')$$
$$+ \frac{\alpha_s^2}{8\pi^4}\left[\frac{4}{(z-z')^4} - \frac{I(x,z,z')}{(z-z')^2} - \frac{I(y,z,z')}{(z-z')^2}\right]$$

$$K_{qq}(x,y;z,z') \to \bar{K}_{qq}(x,y;z,z') \equiv K_{qq}(x,y;z,z') - \frac{\alpha_s^2 n_f}{8\pi^4}\left[\frac{I_f(x,z,z')}{(z-z')^2} + \frac{I_f(y,z,z')}{(z-z')^2}\right]$$

$$I(x,z,z') = \frac{1}{X^2-(X')^2}\ln\frac{X^2}{(X')^2}\left[\frac{X^2+(X')^2}{(z-z')^2} - \frac{X\cdot X'}{X^2} - \frac{X\cdot X'}{(X')^2} - 2\right]$$
$$I_f(x,z,z') = \frac{2}{(z-z')^2} - \frac{2X\cdot X'}{(z-z')^2(X^2-(X')^2)}\ln\frac{X^2}{(X')^2}$$

These modifications are vanishing while acting on gauge invariant structures.

Conformal Properties of the NLO Hamiltonian hep-th/1401.0374

The leading order Hamiltonian is invariant under the following transformations (together known as conformal symmetry):

1. ***Scaling Symmetry:*** $x \to \alpha x$

2. ***Inversion:*** $x \to \frac{1}{x}$

We would like to address to address the question whether these symmetries are still preserved at NLO.

QCD is NOT a conformal theory. Therefore, we looked for NLO JIMWLK Hamiltonian for $\mathcal{N}=4$ theory, which is similar to QCD but with vanishing beta function.

Towards Conformal Symmetry

The NLO Hamiltonian is dilatationally invariant so we will focus on inversion, $\mathcal{I}_0$, which acts on Wilson line S in the following way:

$$\mathcal{I}_0: \ S(x_+, x_-) \to S(1/x_+, 1/x_-)$$

where: $x_\pm = x_1 \pm i\, x_2$

The NLO Hamiltonian is not invariant under the naive inversion $\mathcal{I}_0$ transformation:

$$\mathcal{I}_0 \; H^{JIMWLK} \; \mathcal{I}_0 \;=\; H^{JIMWLK} \;+\; \mathcal{A}$$

The anomalous piece $\mathcal{A}$ can be compensated if the Wilson lines S form a non-trivial representation of the conformal group such that

$$\mathcal{I}: S(x) \to S(1/x) \, + \, \delta S(x)$$

$$\mathcal{I}: H^{LO} \to H^{LO} \, - \, \mathcal{A}$$

where δS are perturbatively of the order α_s , such that the net anomaly is cancelled and the total Hamiltonian remains invariant at NLO:

$$\mathcal{I}\left(H^{LO} + H^{NLO}\right)\mathcal{I} = H^{LO} + H^{NLO}$$

Towards Conformal Symmetry

Our goal is to explicitly construct $\mathcal{I}$. We searched for it perturbatively in the form:

$$\mathcal{I} = (1 + \mathcal{C})\,\mathcal{I}_0$$

Inspired by a work of I. Balitsky and G. A. Chirilli (hep-th/ **0903.5326**) on the conformal dipole we used the ansatz:

$$\mathcal{C} = \int_{u,v,z'} F(u,v,z')\,\hat{h}(u,v,z') \qquad \hat{h} \equiv \left[J_L^a(x)\,J_L^a(y) + J_R^a(x)\,J_R^a(y) - 2J_L^a(x)\,S_A^{ab}(z)\,J_R^b(y)\right]$$

Where F is an unknown function. We managed to show that the anomaly A vanishes for the choice:

$$\mathcal{C} = -\frac{1}{2}\int_{x,y,z} M(x,y,z)\ln\left(\frac{z^2}{a^2}\right)\left[J_L^a(x)\,J_L^a(y) + J_R^a(x)\,J_R^a(y) - 2J_L^a(x)\,S_A^{ab}(z)\,J_R^b(y)\right]$$

Where: $M(x,y;z) = \frac{\alpha_s}{2\pi^2}\frac{(x-y)^2}{X^2Y^2}$. $\quad X \equiv x - z,\ Y \equiv y - z$.

Having $\mathcal{I}$ we can construct conformal extensions of any operator including the Hamiltonian itself.

Conclusions

1) We have constructed the NLO JIMWLK (suitable for acting on color singlet operators) based on symmetries and known results on dipole and baryon evolutions.

2) Generalization of the kernels applicable for non-color singlets has been done (to appear soon).

3) Full consistency with the Balitsky's hierarchy at NLO (I. Balitsky and A. Chirill, hep-ph/1309.7644) has been shown (a paper will appear during this week)

4) The NLO Hamiltonian (in $\mathcal{N} = 4$) was proven to preserve the conformal symmetry (which exists also in the leading-order Hamiltonian