

Redberry: a computer algebra system designed for tensor manipulation

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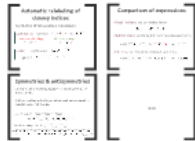
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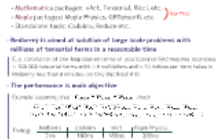
Outline

Specific features of tensorial CAS



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Redberry's place among tensor software



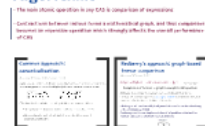
Redberry's place among tensor software

Key features of Redberry



Key features of Redberry & examples

Algorithms



Algorithms & performance

Specific features of tensorial CAS

Automatic relabeling of dummy indices

is critical for all CAS operations. Few examples:

Substitutions: substitute $R_{\alpha\beta} = R^{\mu}_{\alpha\mu\beta}$ into $R_{\rho\tau}R^{\tau\mu}$

without relabeling:

$$R^{\mu}_{\rho\mu\tau}R^{\mu\tau}_{\mu}$$

the correct way:

$$R^{\alpha}_{\rho\alpha\tau}R^{\beta\tau}_{\beta}$$

Expand: expand power $(p_{\mu}p^{\mu} + m^2)^2$

• gives: $p_{\mu}p^{\mu}p_{\alpha}p^{\alpha} + 2m^2p_{\mu}p^{\mu} + m^4$

Comparison of expressions

- **Simplifications:** reduce similar terms

$$A_{\mu}A^{\mu} + A_{\nu}A^{\nu} = 2A_{\mu}A^{\mu}$$

- **Substitutions:** matching both free and dummy indices

$$\text{apply } F^{\alpha\beta}F_{\alpha\gamma} \rightarrow T^{\beta}_{\gamma} \text{ to } F^{\beta\gamma}F^{\mu}_{\gamma}F_{\beta\mu} \Rightarrow T^{\gamma}_{\mu}F^{\mu}_{\gamma}$$

- **Functions:** matching both arguments and indices

$$\text{apply } f_{\mu\nu}(x_{\alpha\beta}) = x_{\mu}^{\alpha}x_{\nu}^{\beta} \text{ to } f_{\alpha\beta}(y_{\alpha}y_{\beta}) \Rightarrow y_{\alpha}y_{\mu}y^{\mu}y_{\beta}$$

Symmetries & antisymmetries

- Ability to define arbitrary symmetries is a key feature of tensorial CAS
- Both symmetries and antisymmetries must be considered in matching and simplification

For example, if $R_{abcd} = R_{cdba} = -R_{bacd}$

$$\Rightarrow R^{abcd}R_{c f d e}R^{e f}_{ab} + R_{r c}^{d f}R_{a b}^{r c}R_{f d}^{b a} = 0$$

Another example: if $W_{abcde} = W_{ebcda} = W_{cbade}$

$$\Rightarrow (W_{bdc}^{ij} + W_{bdc}^{ji} + W_{bcd}^{ij} + W_{bcd}^{ji} + W_{dbc}^{ij} + W_{dbc}^{ji})(W_{cfhji} + W_{chfji} + W_{cjhfi} + W_{fchij} + W_{fchji}) - (W_{bdc}^{ij} + W_{bdc}^{ji} + W_{bcd}^{ij} + W_{bcd}^{ji} + W_{dbc}^{ij} + W_{dbc}^{ji})(W_{cfhij} + W_{chfij} + W_{cjhfi} + W_{fchij} + W_{fchji}) = 0$$

...

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...

Redberry's place among tensor software

- **Mathematica** packages: xAct, Tensorial, Ricci, etc.
 - **Maple** packages: Maple Physics, GRTensorII, etc.
 - Standalone tools: Cadabra, Reduce etc.
-) Non free
- **Redberry is aimed at solution of large scale problems with millions of tensorial terms in a reasonable time**
- E. g. calculation of one-loop counterterms of gravitational field requires to process ~ 700 000 tensorial terms with ~ 8 multipliers and ~ 10 indices per term takes in Redberry less than 8 minutes (on this Mac Book Air)
- **The performance is main objective**

Example: assuming that $W_{abcde} = W_{ebcda} = W_{cbade}$ check

$$(W_{bde}^{ij} + W_{bde}^{ji} + W_{bed}^{ij} + W_{dbe}^{ji} + W_{de}^i{}_b{}^j)(W_{cfhji} + W_{chfji} + W_{cjhfi} + W_{fchij} + W_{fchji}) - \\ (W_{bde}^{ij} + W_{bde}^{ji} + W_{bed}^{ji} + W_{dbe}^{ij} + W_{de}^i{}_b{}^j)(W_{cfhij} + W_{chfij} + W_{cih fj} + W_{fchij} + W_{fchji}) = 0$$

Timing:	Redberry	Cadabra	xAct	Maple Physics
	2ms	180ms	180ms	200ms

Key features of Redberry

- ✓ tensor symmetries, multiple index types, dummy indices handling, LaTeX-style i/o
- ✓ a wide range of tensor-specific transformations and simplification routines
- ✓ HEP features: Dirac & SU(N) algebra, one-loop counterterms calculation etc.
- ✓ programming language with internal support of symbolic tensor algebra
- ✓ free & open-source with extensive API for developers

Toy example

1. Setup symmetries $R_{abcd} = R_{cdba} = -R_{bacd}$

2. Substitute

$$R_{abcd} = (2R_{abcd} + R_{acbd} - R_{adbc})/3 \quad \text{in} \quad 2R_{abcd}R^{abcd} - R_{abcd}R^{bacd}$$

3. Simplify and check that result is zero

Redberry code:

```
// Setup symmetries
addSymmetries 'R_abcd', [[0, 2], [1, 3]].p, -[[1, 0]].p
// Define substitution
subs = 'R_abcd = (2*R_abcd + R_acbd - R_adbc)/3'.t
// Input expression
expr = '2*R_abcd*R^abcd - R_abcd*R^bacd'.t
// Apply substitution and expand
expr = (subs & Expand) >> expr
println expr //prints zero
> 0
```

Scattering in gravity

Calculate matrix element of scalar particle scattering by exchanging one massive Fierz-Pauli graviton in D dimensions

Redberry code:

```
// Auxiliary tensor
P = 'P_ab[p_a] = g_ab + p_a*p_b/M**2'.t
// Fierz-Pauli graviton propagator
D = ''D_{mn ab}[p_a] = ((P_ma[p_a] * P_nb[p_a] + P_mb[p_a] * P_na[p_a])/2
- P_mn[p_a] * P_ab[p_a])/(D-1)/(p_a*p_a + M**2)'''.t
// Scalar-graviton vertex
V = 'V_{mn}[p_a, k_a] = p_m*k_n + p_n*k_m - (1/2)*g_mn*(p_a - k_a)*(p^a - k^a)'.t
// Matrix element
M = 'V_{ab}[p1_a, k1_a]*D^{ab cd}[p1_a - k1_a]+V_{cd}[p2_a, k2_a]'.t
// Apply substitutions
M = (D & V & P) >> M
// Set up Mandelstam variables
mShell = setMandelstam([p1_a: 'm', p2_a: 'm', k1_a: 'm', k2_a: 'm'])

// Simplifications of matrix element
M = (ExpandAll & EliminateMetrics & 'd^i_i = D'.t) >> M
M = (mShell & 'u = 4*m**2 - s - t'.t & Factor) >> M
println M

>
1
4(D-1)M^4(M^2+t) (16(D-2)m^4M^4-8m^2(M^4(2(D-1)s+(D-4)t)-2M^2t^2)
+M^4(4(D-1)s^2+4(D-1)st+(D-10)t^2)+4(D-4)M^2t^2+4(D-2)t^4)
```

Toy example

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```

$$\begin{aligned}
 &> \frac{1}{4(D-1)M^4(M^2+t)} \left(16(D-2)m^4M^4 - 8m^2 \left(M^4(2(D-1)s + (D-4)t) - 2M^2t^2 \right) \right. \\
 &\quad \left. + M^4(4(D-1)s^2 + 4(D-1)st + (D-10)t^2) + 4(D-4)M^2t^3 + 4(D-2)t^4 \right)
 \end{aligned}$$

Algorithms

- The main atomic operation in any CAS is comparison of expressions
- Contractions between indices forms a mathematical graph, and thus comparison becomes an expensive operation which strongly affects the overall performance of CAS

Common approach: canonicalisation

(Rodionov & Taranov 1989, Portugal 1999)

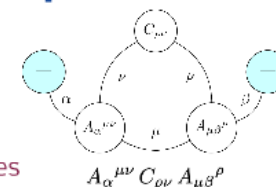
- Main idea: put indices of all expressions into "canonical" form (rename dummies, sort using symmetries, sort multipliers), then comparison becomes trivial:

$$\begin{aligned} R^{abcd} R_{efdc} R^{ef}_{ab} &\rightarrow -R^{ab}_{cd} R_{abef} R^{efcd} \\ R_{rc}^{df} R_{ab}^{rc} R_{fd}^{ba} &\rightarrow R^{ab}_{cd} R_{abef} R^{efcd} \end{aligned}$$

- The algorithm is equivalent to finding double coset representatives
 - ✗ NP-hard problem in general (Butler 1984, Luks 1993, Holt 2005)
 - ✗ Works only with products of simple tensors (need to expand if product contains sum)

Redberry's approach: graph-based tensor comparison

(Rodionov & Taranov 1989)



Graph: multipliers - vertexes, dummies - edges

Comparison of tensors = graph isomorphism (GI) problem

- Redberry does not perform expensive "canonicalisation" of terms
- It searches for isomorphisms ([mappings of indices](#)) between tensors and does not rely on canonical form of expressions
- **Although, GI is at least NP, in all practical cases it can be solved very efficiently (McKay, 1980)**
- **Redberry uses its own implementation of GI problem optimized for typical expressions arising in real calculations**



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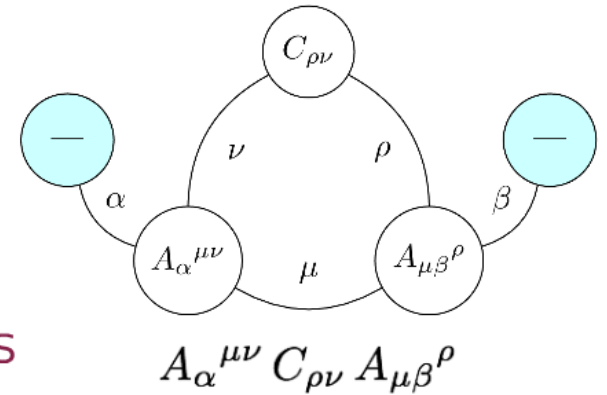
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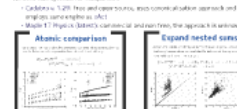
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Performance

Tensor problem: expand and collect product of sums and collect on the tensor.
- tensor reduction (internal) - the level of optimization
- strongly affects the overall performance of the calculations
- critical for ability of CAS to manage large scale problems through expressions
We used two options for comparison:



Performance

Typical problem: expand out product of sums and collect similar terms:

- involve nearly all internal low-level CAS routines
- strongly affects the overall performance of real calculations
- critical for ability of CAS to manage large-scale problems (huge expressions)

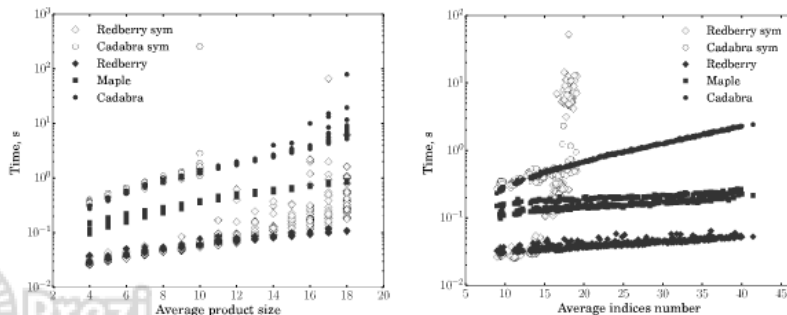
We used two systems for comparison:

- **Cadabra v. 1.29:** free and open-source, uses canonicalisation approach and employs same engine as xAct
- **Maple 17 Physics (latest):** commercial and non free, the approach is unknown

Atomic comparison

- Generate sum of 100 random different terms, rearrange indices (rename dummies and shuffle indices within symmetries) and subtract from itself, e. g. :

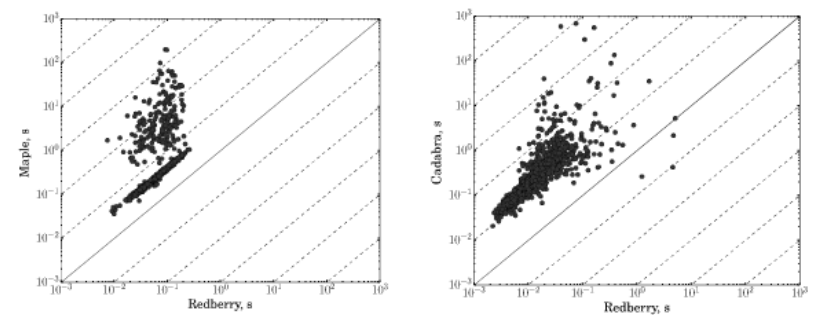
$$(F_{\mu\nu\alpha}T^{\nu}_{\beta}W^{\alpha\beta} + \dots (99 \text{ terms})) - (T^{\alpha}_{\rho}W^{\nu\rho}F_{\mu\alpha\nu} + \dots (99 \text{ terms})) = 0$$



Expand nested sums

- Generate random nested products of sums, expand, rearrange indices (rename dummies and shuffle indices within symmetries), and subtract from itself, e. g. :

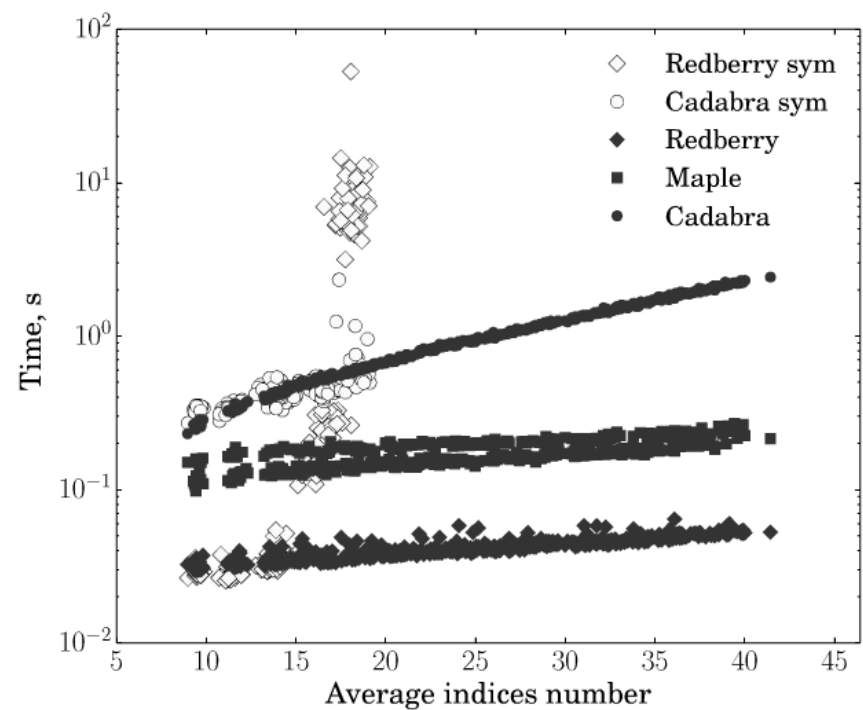
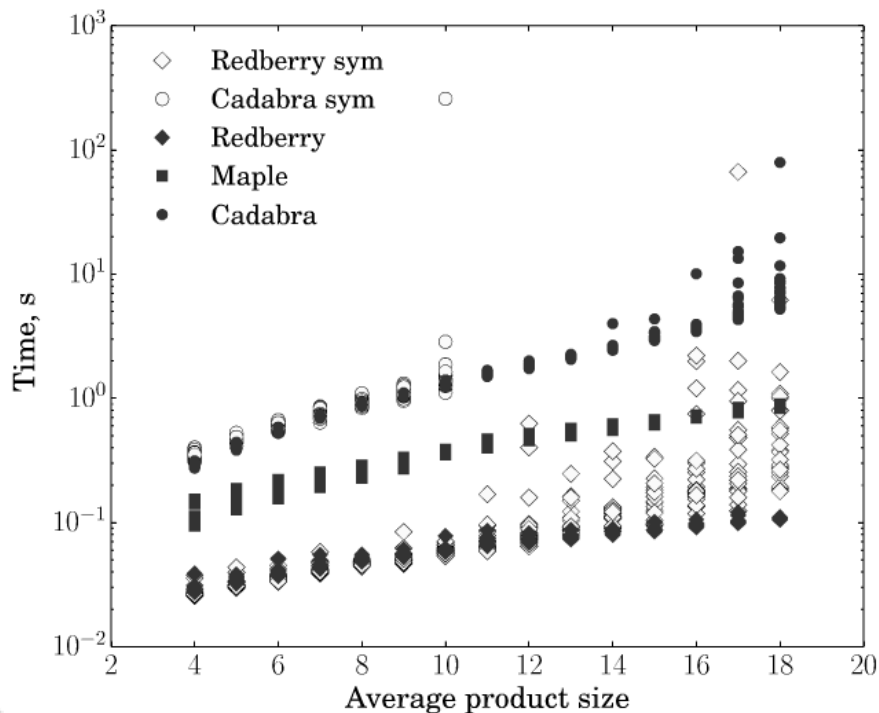
$$(F_{\mu\nu}(T^{\mu\alpha}R^{\nu\beta} + \dots) + \dots)(R_{\alpha\rho}(F^{\rho}_{\beta}R_{\tau\gamma} + \dots) + \dots) \times (\dots) - (F_{\nu\mu}T^{\nu\beta}R^{\mu\alpha}R_{\beta\rho}F^{\rho}_{\alpha}R_{\tau\gamma} + \dots) = 0$$



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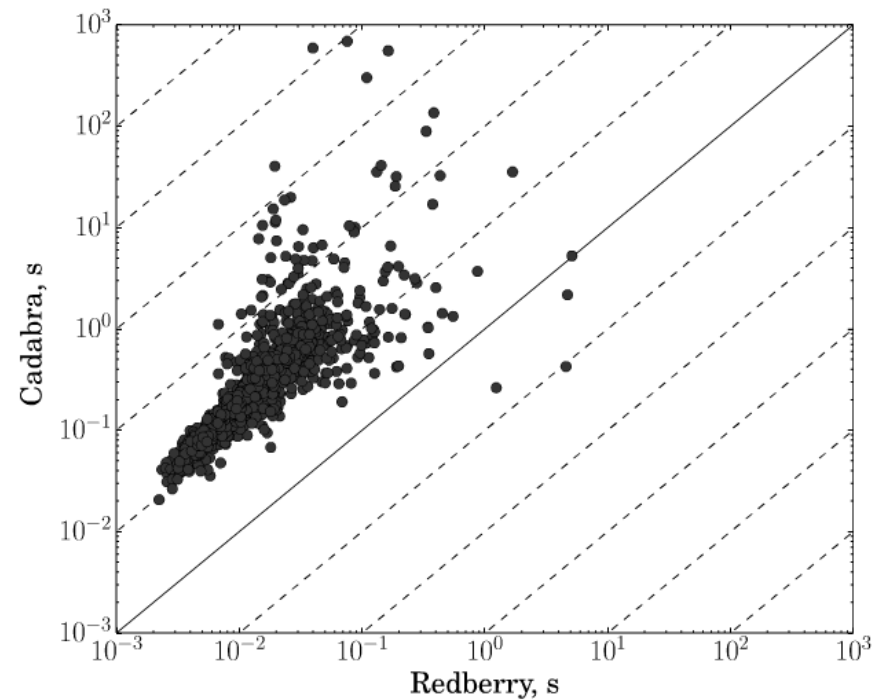
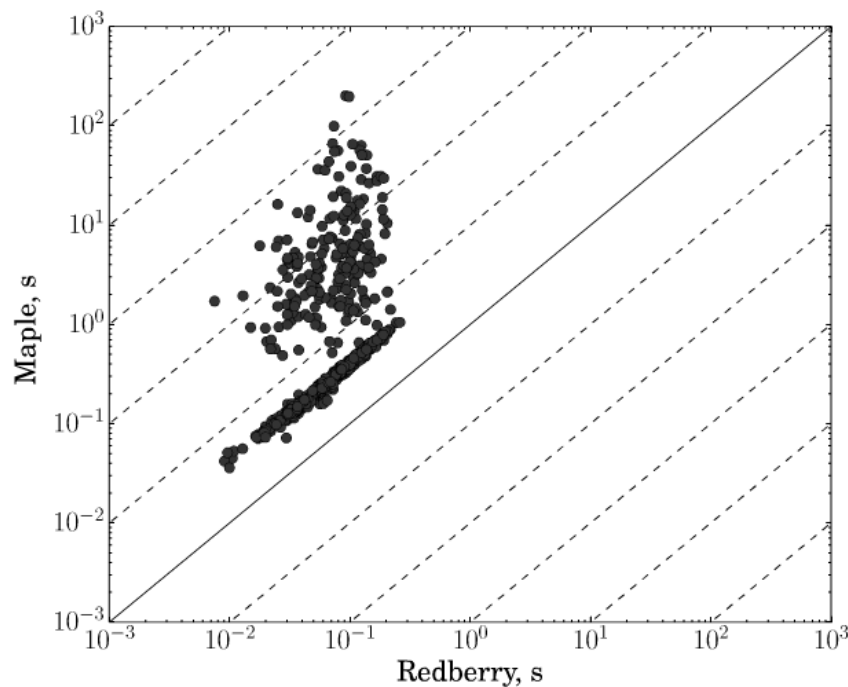
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Technical details

- Programming language: Java, Groovy (for user interface)
- Operating system: any Linux, Windows, Mac
- Lines of code: 132 148
- User interface: IntelliJ IDEA (syntax highlighting, code completion),
command line is also available
- License: GNU GPL v3 (free and open source)
- Requirements: Java 7+, Groovy 2+

Full overview and comprehensive documentation:

<http://redberry.cc>

Thank you!

References

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1. Setup symmetries $R_{abcd} = R_{cdba} = -R_{bacd}$
2. Check that $R^{ef}{}_{ab}R_{efdc}R^{abcd} + R_{rc}{}^{df}R_{ab}{}^{rc}R_{fd}{}^{ba} = 0$

Redberry code:

```
// Setup symmetries
addSymmetries 'R_abcd', [[0, 2], [1, 3]].p, -[[1, 0]].p
// Input expression
expr = 'R^abcd*R_efdc*R^ef_ab + R_rc^df*R_ab^rc*R_fd^ba'.t
println expr //prints zero

> 0
```

Mappings of indices

- The result of comparison is not just logical "true" or "false" but a complicated mapping:

$$F_{ab}G^{bc} \xrightarrow{\text{maps to}} F_{iq}G^{qj} = \left\{ \begin{array}{l} a \rightarrow i \\ c \rightarrow j \end{array} \right\}$$

- Several mappings can be found for a pair of tensors. E.g., if R_{ab} is antisymmetric, then

$$R_{ab}A_c + R_{bc}A_a \xrightarrow{\text{maps to}} R_{ab}A_c + R_{bc}A_a = + \left\{ \begin{array}{c} a \rightarrow a \\ b \rightarrow b \\ c \rightarrow c \end{array} \right\} \text{ and } - \left\{ \begin{array}{c} a \rightarrow c \\ b \rightarrow b \\ c \rightarrow a \end{array} \right\}$$

- When mapping tensor onto itself, we obtain permutational symmetries of its indices, so

⇒ Finding symmetries of tensors = graph automorphism (GA) problem

Examples

```

# find possible mappings between tensors
[1] (A1[i] - A2[i]*gamma[i])/gamma[i], A1[i]*gamma[i], and (A1[i] - A2[i]*gamma[i])/gamma[i] + A2[i]*gamma[i]

# create the symmetric rank-1 matrices
for (i in 1:n) {
  A1[i] = A1[i]*gamma[i]
  A2[i] = (A1[i] - A2[i]*gamma[i])/gamma[i] + A2[i]*gamma[i]
  A3[i] = (A1[i] - A2[i]*gamma[i])/gamma[i]
}

# for mapping in assigned
# print the mapping
[1] A1[i] = A1[i]*gamma[i], A2[i] = (A1[i] - A2[i]*gamma[i])/gamma[i] + A2[i]*gamma[i], A3[i] = (A1[i] - A2[i]*gamma[i])/gamma[i]

# find symmetries of [R1A1A2 + R2A2A3]A3A3A4 + R3A3
[1] symmetry: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 219, 220, 221, 222, 223, 224, 225, 226, 227, 228, 229, 230, 231, 232, 233, 234, 235, 236, 237, 238, 239, 240, 241, 242, 243, 244, 245, 246, 247, 248, 249, 250, 251, 252, 253, 254, 255, 256, 257, 258, 259, 260, 261, 262, 263, 264, 265, 266, 267, 268, 269, 270, 271, 272, 273, 274, 275, 276, 277, 278, 279, 280, 281, 282, 283, 284, 285, 286, 287, 288, 289, 290, 291, 292, 293, 294, 295, 296, 297, 298, 299, 300, 301, 302, 303, 304, 305, 306, 307, 308, 309, 310, 311, 312, 313, 314, 315, 316, 317, 318, 319, 320, 321, 322, 323, 324, 325, 326, 327, 328, 329, 330, 331, 332, 333, 334, 335, 336, 337, 338, 339, 340, 341, 342, 343, 344, 345, 346, 347, 348, 349, 350, 351, 352, 353, 354, 355, 356, 357, 358, 359, 360, 361, 362, 363, 364, 365, 366, 367, 368, 369, 370, 371, 372, 373, 374, 375, 376, 377, 378, 379, 380, 381, 382, 383, 384, 385, 386, 387, 388, 389, 390, 391, 392, 393, 394, 395, 396, 397, 398, 399, 400, 401, 402, 403, 404, 405, 406, 407, 408, 409, 410, 411, 412, 413, 414, 415, 416, 417, 418, 419, 420, 421, 422, 423, 424, 425, 426, 427, 428, 429, 430, 431, 432, 433, 434, 435, 436, 437, 438, 439, 440, 441, 442, 443, 444, 445, 446, 447, 448, 449, 450, 451, 452, 453, 454, 455, 456, 457, 458, 459, 460, 461, 462, 463, 464, 465, 466, 467, 468, 469, 470, 471, 472, 473, 474, 475, 476, 477, 478, 479, 480, 481, 482, 483, 484, 485, 486, 487, 488, 489, 490, 491, 492, 493, 494, 495, 496, 497, 498, 499, 500, 501, 502, 503, 504, 505, 506, 507, 508, 509, 510, 511, 512, 513, 514, 515, 516, 517, 518, 519, 520, 521, 522, 523, 524, 525, 526, 527, 528, 529, 530, 531, 532, 533, 534, 535, 536, 537, 538, 539, 540, 541, 542, 543, 544, 545, 546, 547, 548, 549, 550, 551, 552, 553, 554, 555, 556, 557, 558, 559, 560, 561, 562, 563, 564, 565, 566, 567, 568, 569, 570, 571, 572, 573, 574, 575, 576, 577, 578, 579, 580, 581, 582, 583, 584, 585, 586, 587, 588, 589, 590, 591, 592, 593, 594, 595, 596, 597, 598, 599, 600, 601, 602, 603, 604, 605, 606, 607, 608, 609, 610, 611, 612, 613, 614, 615, 616, 617, 618, 619, 620, 621, 622, 623, 624, 625, 626, 627, 628, 629, 630, 631, 632, 633, 634, 635, 636, 637, 638, 639, 640, 641, 642, 643, 644, 645, 646, 647, 648, 649, 650, 651, 652, 653, 654, 655, 656, 657, 658, 659, 660, 661, 662, 663, 664, 665, 666, 667, 668, 669, 670, 671, 672, 673, 674, 675, 676, 677, 678, 679, 680, 681, 682, 683, 684, 685, 686, 687, 688, 689, 690, 691, 692, 693, 694, 695, 696, 697, 698, 699, 700, 701, 702, 703, 704, 705, 706, 707, 708, 709, 710, 711, 712, 713, 714, 715, 716, 717, 718, 719, 720, 721, 722, 723, 724, 725, 726, 727, 728, 729, 730, 731, 732, 733, 734, 735, 736, 737, 738, 739, 740, 741, 742, 743, 744, 745, 746, 747, 748, 749, 750, 751, 752, 753, 754, 755, 756, 757, 758, 759, 760, 761, 762, 763, 764, 765, 766, 767, 768, 769, 770, 771, 772, 773, 774, 775, 776, 777, 778, 779, 780, 781, 782, 783, 784, 785, 786, 787, 78
```



Examples

- Find possible mappings between tensors

$$-(A_d^a + A_p^a A_d^p) F_{kq}^d - A_b^a A_q^b A_k^i \quad \text{and} \quad (A_m^n - A_m^p A_p^n) F_{nk}^i + A_{mn} A_j^n A_k^i$$

```
setAntiSymmetric 'A_mn', 'F_mnab'
from = '(A_m^n - A_m^p A_p^n) * F_{nk}^i + A_{mn} A_j^n A_k^i'.t
to   = '-(A_d^a + A_p^a A_d^p) * F_{kq}^d - A_b^a A_q^b A_k^i'.t
mappings = from % to
for (mapping in mappings)
    println mapping

-[_i->_i, _j->_q, _k->_k, _m->^a]
[_i->^k, _j->_q, _k->^i, _m->^a]
```

- Find symmetries of $(R_{abc} A_{de} + R_{bde} A_{ac}) A^{ce} + R_{adb}$

```
addSymmetry 'R_abc', -[[0, 1]].p
setSymmetric 'A_ab'
expr = '(R_abc * A_de + R_bde * A_ac) * A^{ce} + R_{adb}'.t
symmetries = findIndicesSymmetries('_abd'.si, expr)
for (sym in symmetries)
    println sym

+[]
-[[0, 2]]
```