Redberry: a computer algebra system designed for tensor manipulation

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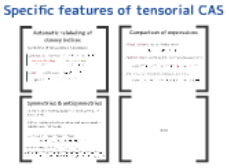
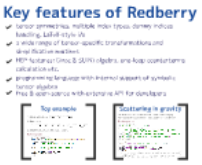
Outline

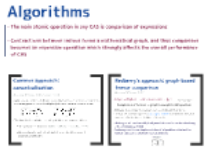


Specific features of tensorial CAS

Redberry's place among tensor software



Key features of Redberry & examples



Algorithms & performance

Specific features of tensorial CAS

Automatic relabeling of dummy indices

is critical for all CAS operations. Few examples:

Substitutions: substitute $R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta}$ into $R_{\rho\tau}R^{\tau\mu}$

without relabeling: $R^{\mu}{}_{\rho\mu\tau}R^{\mu\tau}{}_{\mu}{}^{\mu}$

the correct way: $R^{\alpha}{}_{\rho\alpha\tau}R^{\beta\tau}{}_{\beta}{}^{\mu}$

Expand: expand power $(p_\mu p^\mu + m^2)^2$

- gives: $p_\mu p^\mu p_\alpha p^\alpha + 2\,m^2\,p_\mu p^\mu + m^4$

Comparison of expressions

- Simplifications: reduce similar terms

$$A_\mu A^\mu + A_\nu A^\nu = 2\,A_\mu A^\mu$$

- Substitutions: matching both free and dummy indices

apply $F^{\alpha\beta}F_{\alpha\gamma} \to T^{\beta}{}_{\gamma}$ to $F^{\beta\gamma}F^{\mu}{}_{\gamma}F_{\beta\mu}$ $\Rightarrow$ $T^{\gamma}{}_{\mu}F^{\mu}{}_{\gamma}$

- Functions: matching both arguments and indices

apply $f_{\mu\nu}(x_{\alpha\beta}) = x_\mu{}^\alpha x_{\alpha\nu}$ to $f_{\alpha\beta}(y_\alpha y_\beta)$ $\Rightarrow$ $y_\alpha y_\mu y^\mu y_\beta$

Symmetries & antisymmetries

- Ability to define arbitrary symmetries is a key feature of tensorial CAS
- Both symmetries and antisymmetries must be considered in matching and simplification

For example, if $R_{abcd} = R_{cdba} = -R_{bacd}$

$$\Rightarrow\ R^{abcd}R_{cfdc}R^{ef}{}_{ab} + R_{rc}{}^{df}R_{ab}{}^{rc}R_{fd}{}^{ba} = 0$$

Another example: if $W_{abcde} = W_{ebcda} = W_{cbade}$

$$\Rightarrow (W_{bdc}{}^{ij}+W_{bdc}{}^{ji}+W_{bcd}{}^{ij}+W_{dbc}{}^{ji}+W_{dc}{}^{i}{}_{b}{}^{j})(W_{cfhji}+W_{chfji}+W_{cjhfi}+W_{fchij}+W_{fchji}) -$$
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Redberry's place among tensor software

- Mathematica packages: xAct, Tensorial, Ricci, etc.
- Maple packages: Maple Physics, GRTensorII, etc.
- Standalone tools: Cadabra, Reduce etc.

Non free

- **Redberry is aimed at solution of large scale problems with millions of tensorial terms in a reasonable time**

E. g. calculation of one-loop counterterms of gravitational field requires to process ~ 700 000 tensorial terms with ~ 8 multipliers and ~ 10 indices per term takes in Redberry less than 8 minutes (on this Mac Book Air)

- **The performance is main objective**

Example: assuming that $W_{abcde} = W_{ebcda} = W_{cbade}$ check

$$(W_{bde}{}^{ij}+W_{bde}{}^{ji}+W_{bed}{}^{ij}+W_{dbe}{}^{ji}+W_{de}{}^{i}{}_{b}{}^{j})(W_{cfhji}+W_{chfji}+W_{cjhfi}+W_{fchij}+W_{fchji}) -$$
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Timing:

Redberry	Cadabra	xAct	Maple Physics
2ms	180ms	180ms	200ms

Key features of Redberry

- ✔ tensor symmetries, multiple index types, dummy indices handling, LaTeX-style i/o
- ✔ a wide range of tensor-specific transformations and simplification routines
- ✔ HEP features: Dirac & SU(N) algebra, one-loop counterterms calculation etc.
- ✔ programming language with internal support of symbolic tensor algebra
- ✔ free & open-source with extensive API for developers

Toy example

1. Setup symmetries $R_{abcd} = R_{cdba} = -R_{bacd}$
2. Substitute

$$R_{abcd} = (2R_{abcd} + R_{acbd} - R_{adbc})/3 \quad \text{in} \quad 2R_{abcd}R^{acbd} - R_{abcd}R^{abcd}$$

3. Simplify and check that result is zero

Redberry code:

```
// Setup symmetries
addSymmetries 'R_abcd', [[0, 2], [1, 3]].p, -[[1, 0]].p
// Define substitution
subs = 'R_abcd = (2*R_abcd + R_acbd - R_adbc)/3'.t
// Input expression
expr = '2*R_abcd*R^acbd - R_abcd*R^abcd'.t
// Apply substitution and expand
expr = (subs & Expand) >> expr
println expr //prints zero
 > 0
```

Scattering in gravity

Calculate matrix element of scalar particle scattering by exchanging one massive Fierz-Pauli graviton in D dimensions

Redberry code:

```
// Auxiliary tensor
P = 'P_ab[p_a] = g_ab + p_a*p_b/M**2'.t
// Fierz-Pauli graviton propagator
D = '''D_{mn ab}[p_a] = ((P_ma[p_a] * P_nb[p_a] + P_mb[p_a] * P_na[p_a])/2
        - P_mn[p_a] * P_ab[p_a]/(D-1))/(p_a*p^a + M**2)'''.t
// Scalar-graviton vertex
V = 'V_{mn}[p_a, k_a] = p_m*k_n + p_n*k_m - (1/2)*g_mn*(p_a - k_a)*(p^a - k^a)'.t
// Matrix element
M = 'V_{ab}[p1_a, k1_a]*D^{ab cd}[p1_a - k1_a]*V_cd[p2_a, k2_a]'.t
// Apply substitutions
M = (D & V & P) >> M
// Set up Mandelstam variables
mShell = setMandelstam([p1_a: 'm', p2_a: 'm', k1_a: 'm', k2_a: 'm'])

// Simplifications of matrix element
M = (ExpandAll & EliminateMetrics & 'd^i_i = D'.t) >> M
M = (mShell & 'u = 4*m**2 - s - t'.t & Factor) >> M
println M
```

$$> \frac{1}{4(D-1)M^4(M^2+t)}\left(16(D-2)m^4M^4 - 8m^2\left(M^4(2(D-1)s+(D-4)t) - 2M^2t^2\right)\right.$$
$$\left. + M^4\left(4(D-1)s^2+4(D-1)st+(D-10)t^2\right) + 4(D-4)M^2t^3 + 4(D-2)t^4\right)$$

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$$\left.+M^4\left(4(D-1)s^2+4(D-1)st+(D-10)t^2\right)+4(D-4)M^2t^3+4(D-2)t^4\right)$$

Algorithms

- The main atomic operation in any CAS is comparison of expressions
- Contractions between indices forms a mathematical graph, and thus comparison becomes an expensive operation which strongly affects the overall performance of CAS

Common approach: canonicalisation

(Rodionov & Taranov 1989, Portugal 1999)

- Main idea: put indices of all expressions into "canonical" form (rename dummies, sort using symmetries, sort multipliers), then comparison becomes trivial:

$$R^{abcd}R_{efdc}R^{ef}{}_{ab} \quad \to \quad -R^{ab}{}_{cd}R_{abef}R^{efcd}$$
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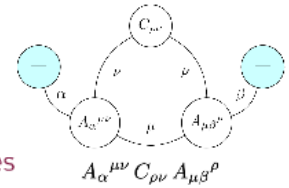
- The algorithm is equivalent to finding double coset representatives
  - × NP-hard problem in general (Butler 1984, Luks 1993, Holt 2005)
  - × Works only with products of simple tensors (need to expand if product contains sum)

Redberry's approach: graph-based tensor comparison

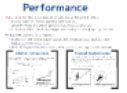
(Rodionov & Taranov 1989)



Graph: multipliers - vertexes, dummies - edges

Comparison of tensors = graph isomorphism (GI) problem

- Redberry does not perform expensive "canonicalisation" of terms
- It searches for isomorphisms (mappings of indices) between tensors and does not rely on canonical form of expressions
- **Althought, GI is at least NP, in all practical cases it can be solved very efficiently (McKay, 1980)**
- **Redberry uses its own implementation of GI problem optimized for typical expressions arising in real calculations**



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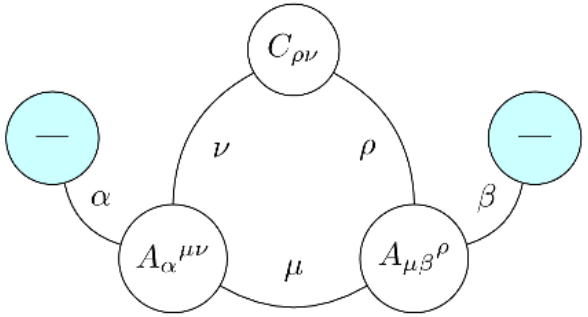

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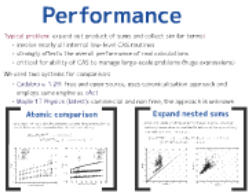


Performance

Typical problem: expand out product of sums and collect similar terms:

- involve nearly all internal low-level CAS routines
- strongly affects the overall performance of real calculations
- critical for ability of CAS to manage large-scale problems (huge expressions)

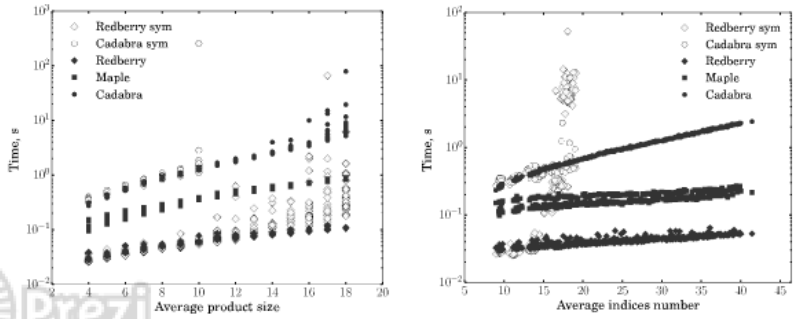
We used two systems for comparison:

- Cadabra v. 1.29: free and open-source, uses canonicalisation approach and employs same engine as xAct
- Maple 17 Physics (latest): commercial and non free, the approach is unknown

Atomic comparison

- Generate sum of 100 random different terms, rearrange indices (rename dummies and shuffle indices within symmetries) and subtract from itself, e. g. :

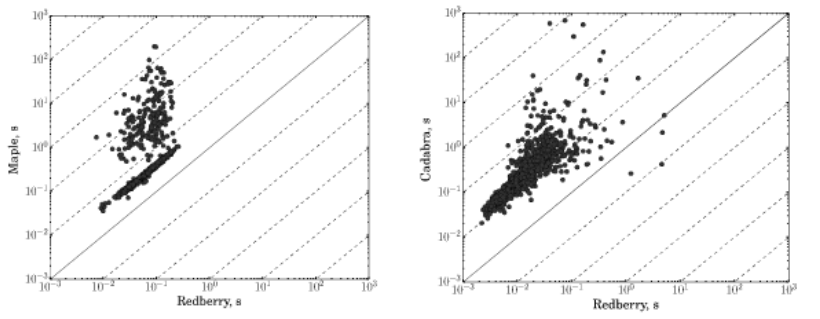
$$\left(F_{\mu\nu\alpha}T^{\nu}{}_{\beta}W^{\alpha\beta} + \ldots \textit{(99 terms)}\right) - \left(T^{\alpha}{}_{\rho}W^{\nu\rho}F_{\mu\alpha\nu} + \ldots \textit{(99 terms)}\right) = 0$$



Expand nested sums

- Generate random nested products of sums, expand, rearrange indices (rename dummies and shuffle indices within symmetries), and subtract from itself, e. g. :

$$\left(F_{\mu\nu}\left(T^{\mu\alpha}R^{\nu\beta} + \ldots\right) + \ldots\right)\left(R_{\alpha\rho}\left(F^{\rho}{}_{\beta}R_{\tau\gamma} + \ldots\right) + \ldots\right) \times (\ldots) - \left(F_{\nu\mu}T^{\nu\beta}R^{\mu\alpha}R_{\beta\rho}F^{\rho}{}_{\alpha}R_{\tau\gamma} + \ldots\right) = 0$$



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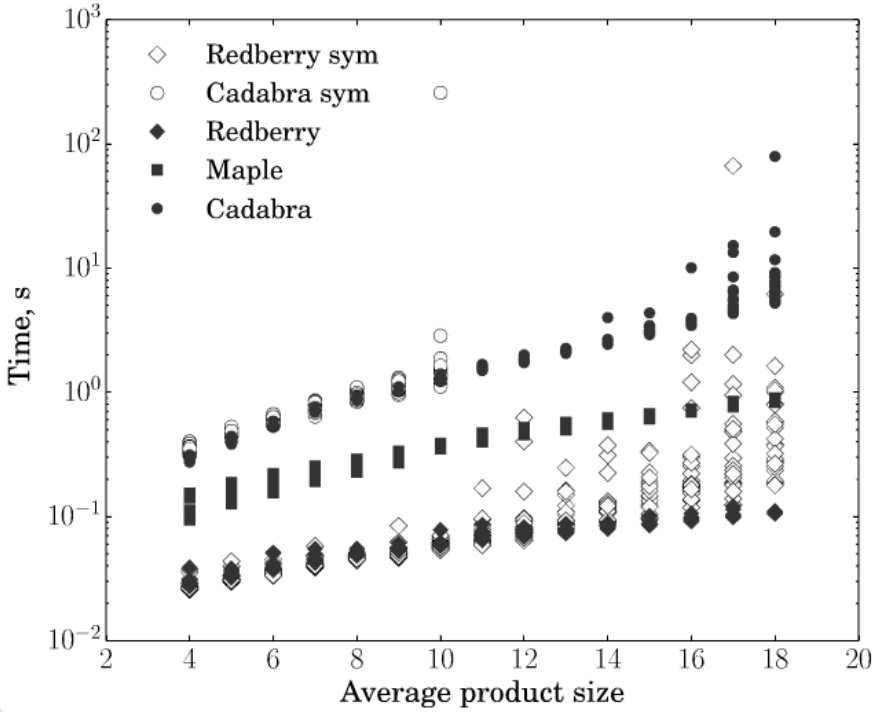

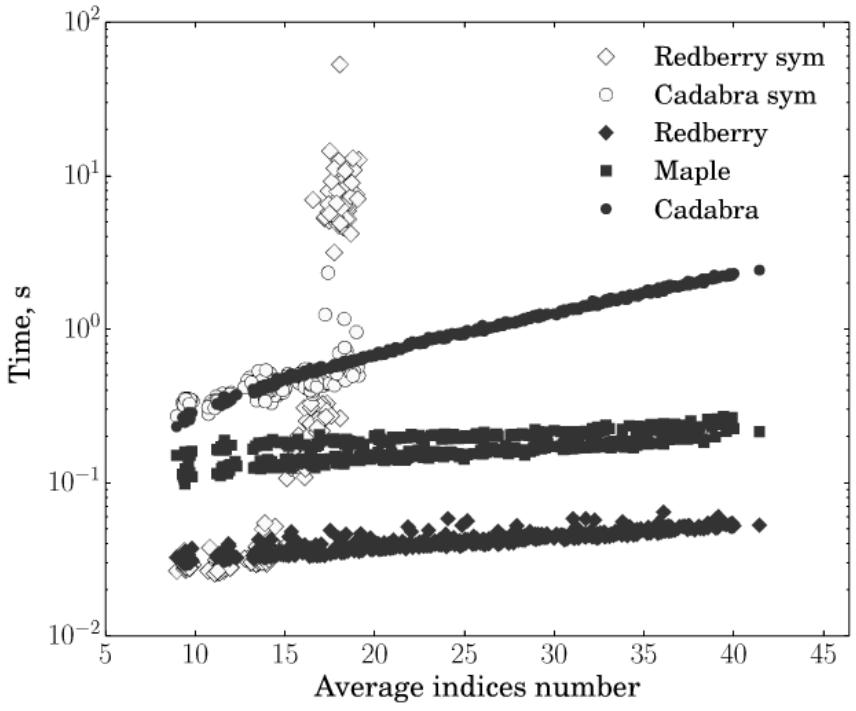

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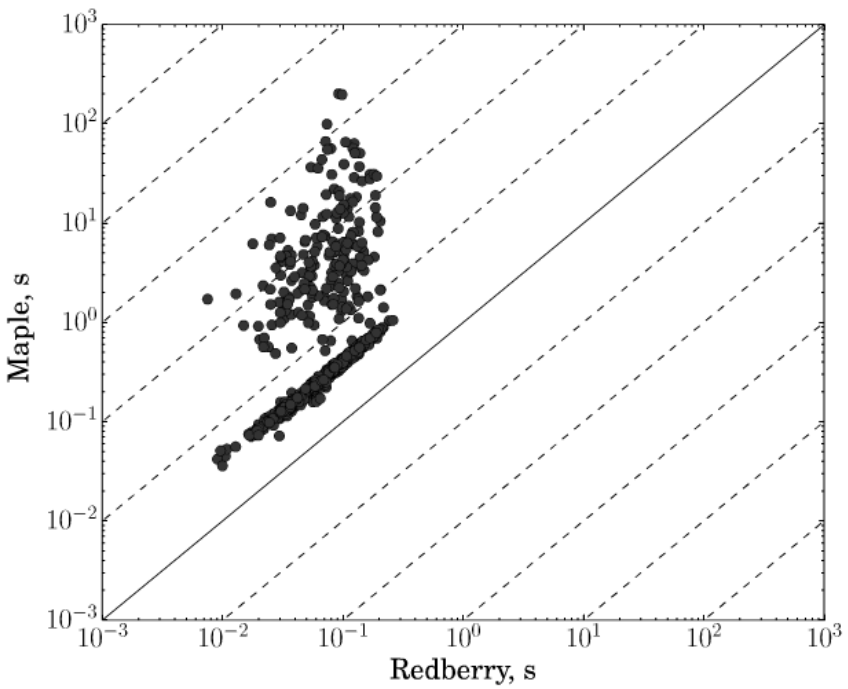

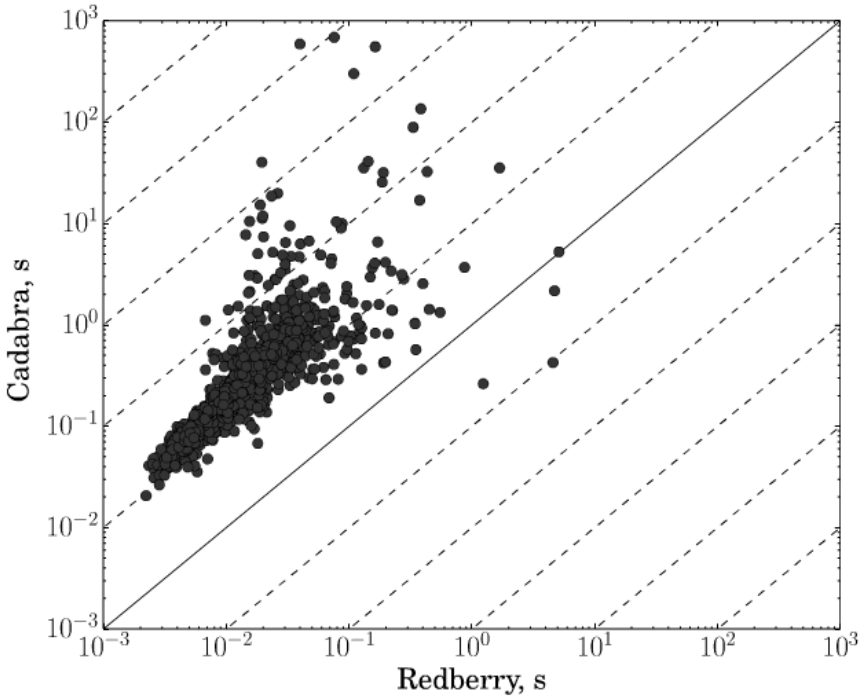

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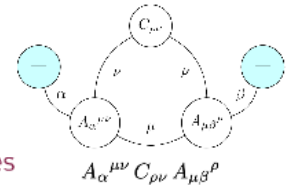

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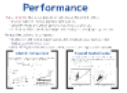


Technical details

- Programming language: Java, Groovy (for user interface)
- Operating system: any Linux, Windows, Mac
- Lines of code: 132 148
- User interface: IntelliJ IDEA (syntax highlighting, code completion), command line is also available
- License: GNU GPL v3 (free and open source)
- Requirements: Java 7+, Groovy 2+

Full overview and comprehensive documentation:

http://redberry.cc

Thank you!

References

- D. A. Bolotin, S. V. Poslavsky, "Introduction to Redberry: a computer algebra system designed for tensor manipulation", arXiv:1302.1219 [cs.SC]
- J. M. Martín-García et.al, "xAct: Efficient tensor computer algebra for Mathematica", http://xact.es/
- Kasper Peeters, "Introducing Cadabra: a symbolic computer algebra system for field theory problems", hep-th/0701238
- A. Ya. Rodionov, A. Yu. Taranov, "Combinatorial aspects of simplification of algebraic expressions", Eurocal '87, Lecture Notes in Computer Science Volume 378, 1989, pp 192-201
- R. Portugal, "Algorithmic simplification of tensor expressions", J. Phys. A 32 (1999) 7779-7789
- G. Butler, "On Computing Double Coset Representatives in Permutation Groups", in Computational Group Theory, ed. M. D. Atkinson, Academic Press (1984), 283--290
- Eugene M. Luks, "Permutation groups and polynomial-time computation", In Finkelstein and Kantor [FK93], pages 139– 175
- Derek F. Holt, Bettina Eick, Eamonn A. O'Brien, "Handbook Of Computational Group Theory", Chapman and Hall/CRC, 2005

1. Setup symmetries $R_{abcd} = R_{cdba} = -R_{bacd}$

2. Check that $R^{ef}{}_{ab} R_{efdc} R^{abcd} + R_{rc}{}^{df} R_{ab}{}^{rc} R_{fd}{}^{ba} = 0$

Redberry code:

```
// Setup symmetries
addSymmetries 'R_abcd', [[0, 2], [1, 3]].p, -[[1, 0]].p
// Input expression
expr = 'R^abcd*R_efdc*R^ef_ab + R_rc^df*R_ab^rc*R_fd^ba'.t
println expr //prints zero
```

```
> 0
```

Mappings of indices

- The result of comparison is not just logical "true" or "false" but a complicated mapping:

$$F_{ab}G^{bc} \xrightarrow[\text{maps to}]{} F_{iq}G^{qj} \;=\; \left\{ \begin{array}{l} a \to i \\ c \to j \end{array} \right\}$$

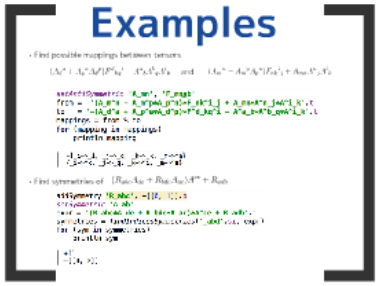
- Several mappings can be found for a pair of tensors. E.g., if R_{ab} is antisymmetric, then

$$R_{ab}A_c + R_{bc}A_a \xrightarrow[\text{maps to}]{} R_{ab}A_c + R_{bc}A_a \;=\; +\left\{ \begin{array}{l} a \to a \\ b \to b \\ c \to c \end{array} \right\} \text{ and } -\left\{ \begin{array}{l} a \to c \\ b \to b \\ c \to a \end{array} \right\}$$

- When mapping tensor onto itself, we obtain permutational symmetries of its indices, so

⇒ **Finding symmetries of tensors = graph automorphism (GA) problem**

Examples



Examples

- Find possible mappings between tensors

$$-(A_d{}^a + A_p{}^a A_d{}^p)F^d{}_{kq}{}^i - A^a{}_b A^b{}_q A^i{}_k \quad \text{and} \quad (A_m{}^n - A_m{}^p A_p{}^n)F_{nk}{}^i{}_j + A_{mn}A^n{}_j A^i{}_k$$

```
setAntiSymmetric 'A_mn', 'F_mnab'
from =  '(A_m^n - A_m^p*A_p^n)*F_nk^i_j + A_mn*A^n_j*A^i_k'.t
to   = '-(A_d^a + A_p^a*A_d^p)*F^d_kq^i - A^a_b*A^b_q*A^i_k'.t
mappings = from % to
for (mapping in mappings)
    println mapping
```

```
-{_i->_i, _j->_q, _k->_k, _m->^a}
{_i->^k, _j->_q, _k->^i, _m->^a}
```

- Find symmetries of $(R_{abc}A_{de} + R_{bde}A_{ac})A^{ce} + R_{adb}$

```
addSymmetry 'R_abc', -[[0, 1]].p
setSymmetric 'A_ab'
expr = '(R_abc*A_de + R_bde*A_ac)*A^ce + R_adb'.t
symmetries = findIndicesSymmetries('_abd'.si, expr)
for (sym in symmetries)
    println sym
```

```
+[]
-[[0, 2]]
```