# On calculations within the nonlinear sigma model[1]

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Outline:

- overview of present status
- Three examples of computation:
    1. standard 2-loop calculation
    2. leading logarithms
- *intermezzo: Gluon amplitudes*
    3. large-$n$ pion tree-level amplitudes
- summary

[1] in collaborations with Bachir Moussallam (Orsay); Hans Bijnens (Lund), Stefan Lanz (Bern); Jiří Novotný (Prague), Jaroslav Trnka (Caltech).

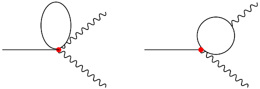
# Overview of low energy QCD

- history: current algebra
- modern form: ChPT [Weinberg'79], [Gasser,Leutwyler'84-'85]
- present status: ChPT for 2 and 3-flavours up to NNLO for the even sector, and up to NLO for the odd sector (chiral anomaly)
- extended for intermediate energy region: resonance chiral theory (RChT)
- overview on chpt beyond one loop e.g. [Bijnens '07]
- many applications: see e.g. recent [Bijnens, Ecker '14]

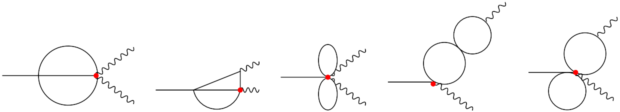
# Short example on 2-loop order calculation

# $\pi^0 \to \gamma\gamma$ at NNLO [K,Moussallam '09]

- NLO: a) One-loop diagrams with one vertex from $\mathcal{L}^{WZ}$, b) tree diagrams with one vertex from $\mathcal{L}^{WZ}$ and one vertex from $O(p^4)$ Lagrangian, c) tree diagrams with one vertex from $O(p^6)$ anomalous-parity sector
- $O(p^6)$ anomalous-parity sector from [Bijnens, Girlanda, Talavera '02]
- representation of chiral field: $U = \sigma + i\frac{\tau.\pi}{F}$, $\sigma = \sqrt{1 - \vec{\pi}^2/F^2}$ (no $\gamma 4\pi$ vertex at LO)
- one-loop



- two-loop



- verification of $Z$-factor, $F_\pi/F$ [Bürgi '96], [Bijnens, Colangelo, Ecker, Gasser, Sainio '02]
- double log checked by Weinberg consistency rel. [Colangelo '95]

# $\pi^0 \to \gamma\gamma$ at NNLO: some details, even sector

[Gasser, Leutwyler '84],[Bijnens, Colangelo, Ecker '99]

$$U = \sigma + i\frac{\phi}{F} \;, \sigma^2 + \frac{\phi^2}{F^2} = 1 \,, \phi = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix} = \phi^i \tau^i \,,$$

$$u_\mu = i u^\dagger \partial_\mu U u^\dagger \,, \chi_+ = u^\dagger \chi u^\dagger + u \chi^\dagger u \,, \chi = 2B(\hat{m}) \;, \hat{m} = \frac{1}{2}(m_u + m_d) \,,$$

- $O(p^2)$

$$\mathcal{L}_2 = \frac{F^2}{4}\langle u_\mu u^\mu + \chi_+ \rangle \,.$$

- $O(p^4)$

$$\mathcal{L}_4 = \frac{l_1}{4}\langle u_\mu u^\mu \rangle^2 + \frac{l_2}{4}\langle u_\mu u_\nu \rangle \langle u^\mu u^\nu \rangle + \frac{l_3}{16}\langle \chi_+ \rangle^2 + i\frac{l_4}{4}\langle u_\mu \chi_-^\mu \rangle - \frac{l_5}{2}\langle f_{-\mu\nu} f_-^{\mu\nu} \rangle + i\frac{l_6}{2}\langle f_{+\mu\nu} u^\mu u^\nu \rangle - \frac{l_7}{16}\langle \chi_- \rangle^2$$

$$l_i = l_i^r + \gamma_i (c\mu)^{d-4} \Lambda \,,$$

- $O(p^6)$

$$\mathcal{L}_6 = c_6 \langle \chi_+ h_{\mu\nu} h^{\mu\nu} \rangle + c_7 \langle u_\mu u^\mu \chi_+ \chi_+ \rangle + c_8 \langle u_\mu u^\mu \chi_+ \rangle \langle \chi_+ \rangle + c_9 \langle \chi_+ u_\mu \chi_+ u^\mu \rangle + \ldots$$

$$c_i = \frac{(c\mu)^{2(d-4)}}{F^2}\left(c_i^r - \gamma_i^{(2)} \Lambda^2 - (\gamma_i^{(1)} + \gamma_i^{(L)})\Lambda\right).$$

# Leading logarithms

# Renormalizable theories

- we calculate e.g. $F(M)$:

$$F = F_0 + F_1^1 L + F_0^1 + F_2^2 L^2 + F_1^2 L + F_0^2 + \dots$$
$$= \alpha + \alpha^2 f_1^1 L + \alpha^2 f_0^1 + \alpha^3 f_2^2 L^2 + \alpha^3 f_1^2 L + \alpha^3 f_0^2 + \dots$$

- where we have defined $L \equiv \log(\mu/M)$
- renormalization condition $\mu \frac{dF}{d\mu} = 0$
- non-trivial dependence on $\alpha$

$$\mu \frac{d\alpha}{d\mu} = \beta_0 \alpha^2 + \beta_1 \alpha^3 + \dots$$

- $\beta_0$ obtained from 1-loop diagrams
- renormalization condition $\Rightarrow$

$$f_1^1 = -\beta_0,\ f_2^2 = \beta_0^2,\ f_3^3 = -\beta_0^3 \quad \Rightarrow \quad F\Big|_{LL} = \frac{\alpha}{1 + \alpha\beta_0 L}$$

# Non-renormalizable theories

## What are the Leading Logarithms (LL)?

- We calculate e.g. $F(M)$:

$$F = F_0 + \underline{F_1^1 L} + F_0^1 + \underline{F_2^2 L^2} + F_1^2 L + F_0^2 + \dots$$

- where we have defined $L \equiv \log(\mu/M)$

## Why they are special?

- they are parameter-free
- to all orders from one-loop diagrams only (based on [Weinberg '79], [Büchler, Colangelo'03])

# $O(N)$ sigma model

- $O(N+1)/O(N)$ nonlinear sigma model

$$\mathcal{L}_{n\sigma} = \frac{F^2}{2}\partial_\mu \Phi^T \partial^\mu \Phi + F^2 \chi^T \Phi \,.$$

- explicit + spontaneous symmetry breaking

$$\langle \Phi^T \rangle = (1\; 0 \ldots 0) \qquad \chi^T = (M^2\; 0 \ldots 0)$$

- we have $N$ Goldstone bosons: $\phi$
- $N = 3$ equivalent to two-flavour ChPT

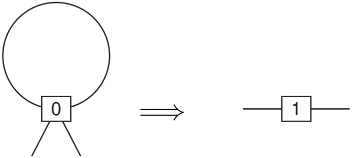
# $O(N)$ sigma model: physical mass $M_\pi$

LL for the physical mass [Bijnens, Carloni '10], [Bijnens,KK,Lanz '12]

- $\mathcal{L}_{n\sigma} \Rightarrow \boxed{0}$
- mass: two-point function
- schematically at LO:


$\Leftrightarrow \quad M_\pi = M$

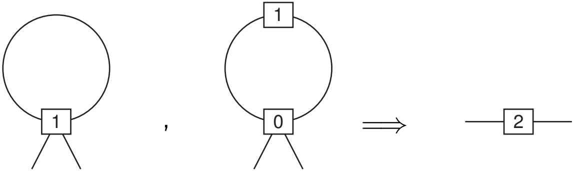
- NLO



# $O(N)$ sigma model: physical mass $M_\pi$

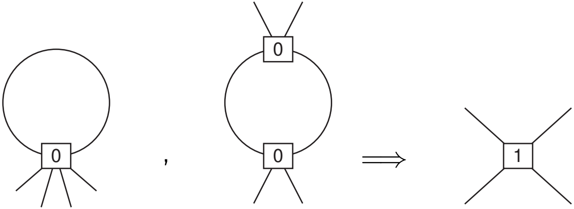
LL for physical mass [Bijnens, Carloni '10], [Bijnens,KK,Lanz '12]

- NNLO



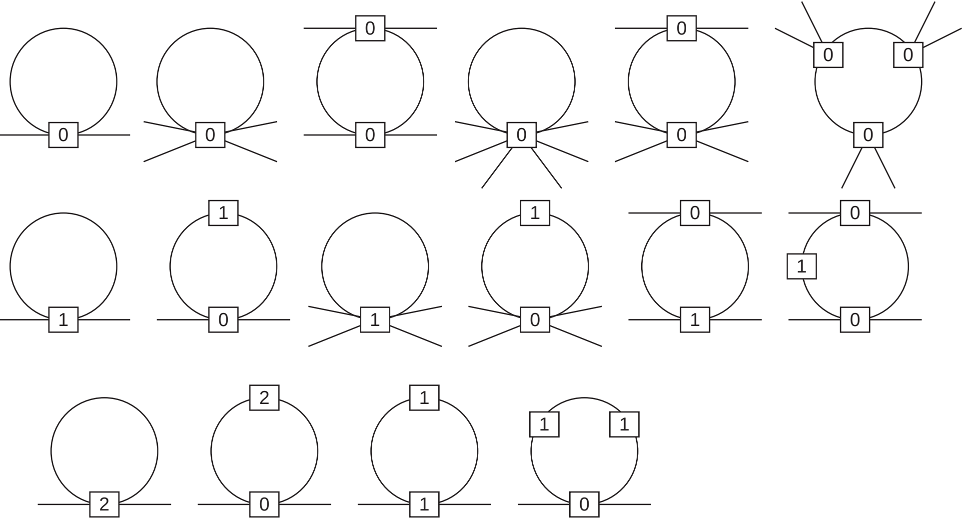

(1)

- we have to calculate first:

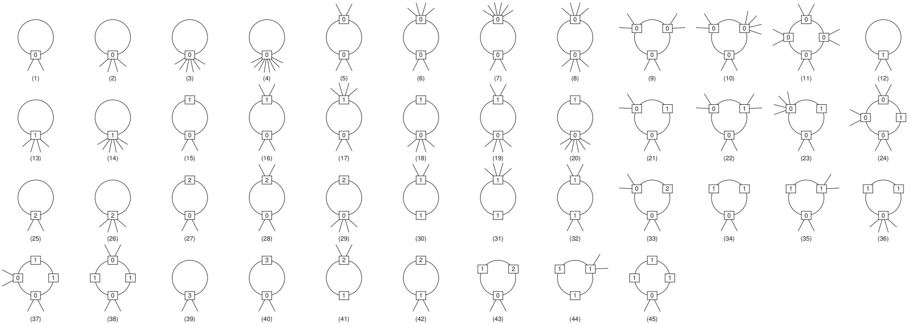



(2)

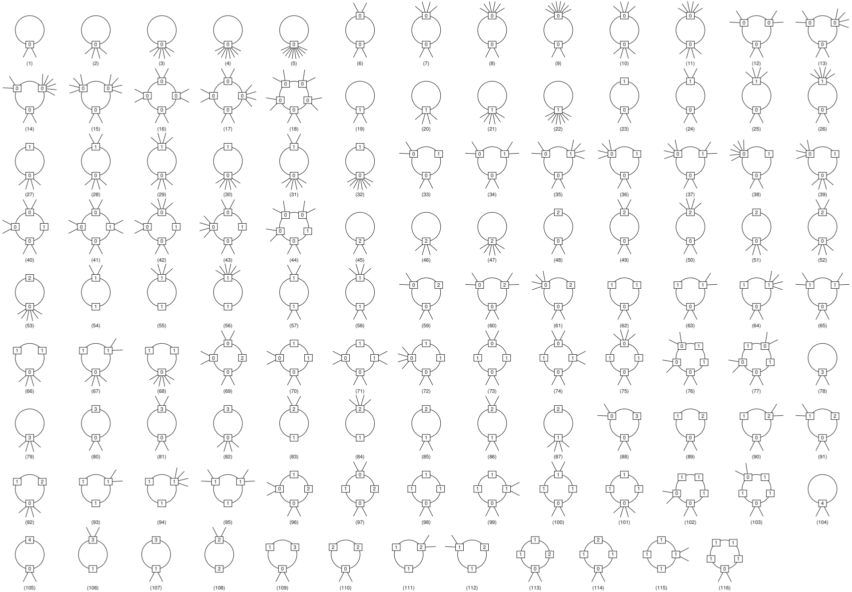
# mass up-to 3-loop order



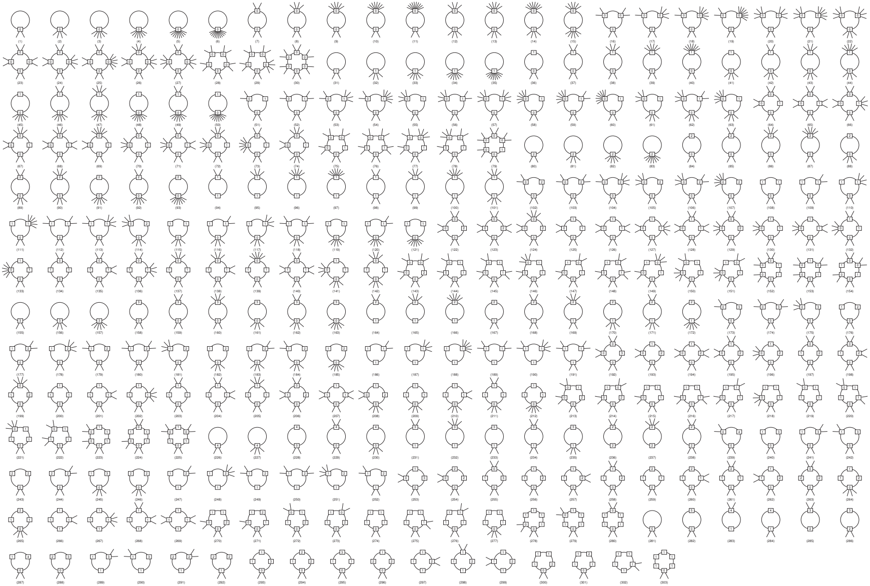
# mass up-to 4-loop order



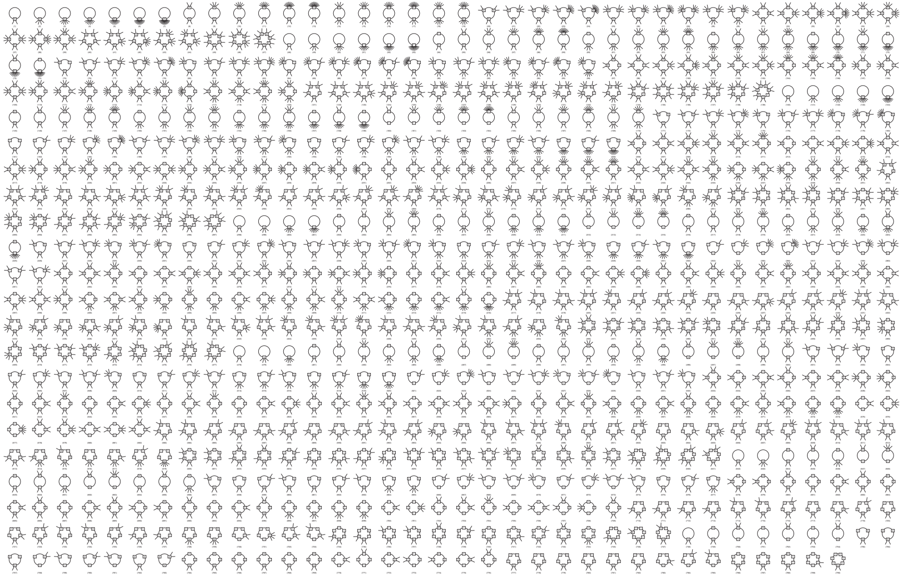
# mass up-to 5-loop order



# mass up-to 6-loop order



# mass up-to 7-loop order



# $O(N)$ sigma model: physical mass $M_\pi$, results

LL for physical mass [Bijnens, Carloni '10], [Bijnens,KK,Lanz '12]

- # of diagrams: $1, 5, 16, 45, 116, 303, 790, \ldots$
- $M_\pi^2 = M^2(1 + a_1 L_M + a_2 L_M^2 + \ldots)$
  $L_M = M^2/(16\pi^2 F^2)\log(\mu^2/M^2)$

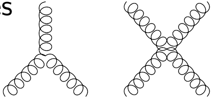
| i | $a_i$ for $N=3$ | $a_i$ for general $N$ |
|---|---|---|
| 1 | $-1/2$ | $1 - 1/2\ N$ |
| 2 | 17/8 | $7/4 - 7/4\ N + 5/8\ N^2$ |
| 3 | $-103/24$ | $37/12 - 113/24\ N + 15/4\ N^2 - N^3$ |
| 4 | 24367/1152 | $839/144 - 1601/144\ N + 695/48\ N^2 - 135/16\ N^3 + 231/128\ N^4$ |
| 5 | $-8821/144$ | $33661/2400 - 1151407/43200\ N + 197587/4320\ N^2 - 12709/300\ N^3 + 6271/320\ N^4 - 7/2\ N^5$ |
| 6 | $\frac{1922964667}{6220800}$ | $158393809/3888000 - 182792131/2592000\ N + 1046805817/7776000\ N^2 - 17241967/103680\ N^3 + 70046633/576000\ N^4 - 23775/512\ N^5 + 7293/1024\ N^6$ |
| 7 | $-\frac{1804453729667}{1714608000}$ | $1098817478897/8573040000 - 286907006651/1428840000\ N + 4533157401977/11430720000\ N^2 - 1986536871797/3429216000\ N^3 + 436238667943/762048000\ N^4 - 7266210703/21168000\ N^5 + 99977/896\ N^6 - 15\ N^7$ |

# intermezzo: Gluon amplitudes

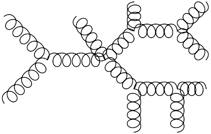
# Gluon amplitudes

standard method of calculating $n$-gluon scattering processes:

- dominated by pure-gluon interactions in QCD
- elementary 3pt and 4pt vertices



- construct all possible Feynman diagrams, e.g.:



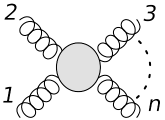
- complicated already for tree level diagrams even for small number of external legs

| $n$ | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|
| # diagrams (inc.crossing) | 1 | 4 | 25 | 220 | 2485 | 34300 |
| # diagrams (col.ordered) | 1 | 3 | 10 | 38 | 154 | 654 |

calculation tedious, however, some results known [Parke, Taylor '85-'86], [Berends, Giele '88] to be extremely simple

Is there some better way to calculate?

# Gluon amplitudes

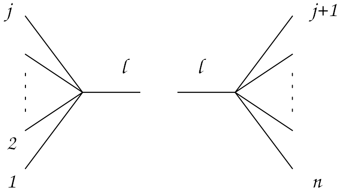



- colour ordering → stripped amplitude
- thanks to ordering the only possible poles are:

$$P_{ij}^2 = (p_i + p_{i+1} + \ldots + p_{j-1} + p_j)^2$$

- Pole structure — Weinberg's theorem (one-particle unitarity): on the factorization channel

$$\lim_{P_{1j}^2 \to 0} M(1, 2, \ldots n) = \sum_{h_l} M_L(1, 2 \ldots j, l) \cdot \frac{i}{P_{1j}^2} \cdot M_R(l, j+1, \ldots n)$$



# BCFW relations, preliminaries

[Britto, Cachazo, Feng, Witten '05]

Reconstruct the amplitude from its poles (in complex plane)

- shift in two external momenta

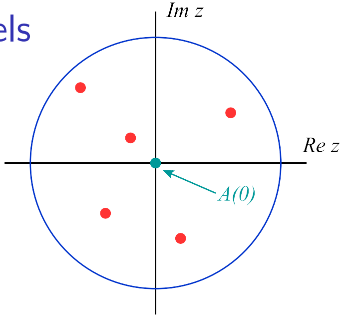
$$p_i \to p_i + zq, \qquad p_j \to p_j - zq$$

- keep $p_i$ and $p_j$ on-shell, i.e.

$$q^2 = q \cdot p_i = q \cdot p_j = 0$$

- amplitude becomes a meromorphic function $A(z)$
- only simple poles coming from propagators $P_{ab}(z)$
- original function is $A(0)$

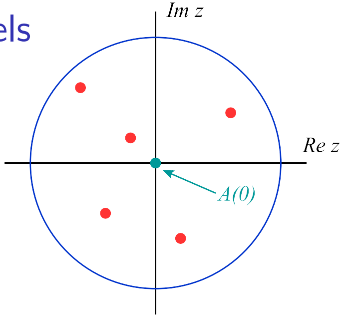
# BCFW relations: factorization channels




Cauchy's theorem

$$\frac{1}{2\pi\mathrm{i}}\int\frac{dz}{z}A(z) = A(0) + \sum_k \frac{\mathrm{Res}\,(A, z_k)}{z_k}$$

# BCFW relations: factorization channels




Cauchy's theorem

$$0 = \frac{1}{2\pi \mathrm{i}} \int \frac{dz}{z} A(z) = A(0) + \sum_k \frac{\mathrm{Res}\,(A, z_k)}{z_k}$$

If $A(z)$ vanishes for $z \to \infty$

$$A = A(0) = -\sum_k \frac{\mathrm{Res}\,(A, z_k)}{z_k}$$

# BCFW relations

$P_{ab}^2(z) = 0$ if one and only one $i$ (or $j$) in $(a, a+1, \dots, b)$.

Suppose $i \in (a, \dots, b) \not\ni j$

$$P_{ab}^2(z) = (p_a + \dots + p_{i-1} + p_i + zq + p_{i+1} + \dots + p_b)^2 = 0$$

solution

$$z_{ab} = -\frac{P_{ab}^2}{2(q \cdot P_{ab})}$$

and for allowed helicities it factorizes into two subamplitudes

$$\text{Res}(A, z_{ab}) = \sum_s A_L^{-s}(z_{ab}) \frac{i}{2(q \cdot P_{ab})} A_R^s(z_{ab})$$

Using Cauchy's formula, we have finally as a result

$$A = \sum_{k,s} A_L^{-s_k}(z_k) \frac{i}{P_k^2} A_R^{s_k}(z_k)$$

# Example: gluon amplitudes

# od diagrams for $n$-body gluon scatterings at tree level

| $n$ | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|
| # diagrams (inc.crossing) | 1 | 4 | 25 | 220 | 2485 | 34300 |
| # diagrams (col.ordered) | 1 | 3 | 10 | 38 | 154 | 654 |
| # BCFW terms | – | 1 | 2 | 3 | 6 | 20 |

Conclusion: it works and it is really fast.

Different directions were taken so far, e.g.

- implementation in multi-jet QCD computations
- extended to loop level [Bern, Dixon, Kosower'05]
- BCFW for planar loop integrands $\mathcal{N}=4$ SYM [Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka '10]
- Amplitudes as volumes of polytopes [Arkani-Hamed, Bourjaily, Cachazo, Hodges, Trnka '10]; Amplituhedron [Arkani-Hamed,Trnka'13]
- crucial are on-shell objects [Arkani-Hamed et al.'12]
- perturbative quantum gravity [Witten'03] [Cachazo, Skinner, Mason'12]

# Non-linear sigma model

# Leading order Lagrangian

- assume general simple compact Lie group $G$

$$G = SU(N), \quad SO(N), \quad Sp(N), \quad \ldots$$

- we will build a chiral non-linear sigma model, which will correspond to the spontaneous symmetry breaking ($G_L \simeq G_R \simeq G_V \simeq G$)

$$G_L \times G_R \to G_V$$

- consequence of the symmetry breaking: Goldstone bosons ($\equiv \phi$)

$$U = \exp\left(\sqrt{2}\frac{i}{F}\phi\right), \qquad \phi = \phi^i t^i$$

- transformation of $U$:

$$U \to V_R U V_L^{-1}$$

- their dynamics given by a Lagrangian (at leading order)

$$\mathcal{L} = \frac{F^2}{4}\langle \partial_\mu U \partial^\mu U^{-1} \rangle$$

where $\langle \ldots \rangle$ stands for a trace

# Stripping down

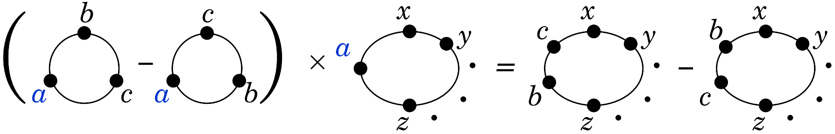
- "group structure" in structure constants, we will define

$$D_\phi^{ab} \equiv -i f^{abc} \phi^c$$

- and after some algebra the Lagrangian becomes

$$\mathcal{L} = -\partial\phi^T \cdot \left( \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{2}{F}\right)^{2n-2} D_\phi^{2n-2} \right) \cdot \partial\phi$$

- we can combine $f^{abc}$s into one trace, schematically




Tree level on-shell amplitude has a simple group structure

$$\mathcal{M}^{a_1\ldots a_n}(p_1,\ldots,p_n) = \sum_{\sigma\in S_n/Z_n} \langle t^{a_{\sigma(1)}} t^{a_{\sigma(2)}} \ldots t^{a_{\sigma(n)}} \rangle \mathcal{M}(p_{\sigma(1)},\ldots,p_{\sigma(n)})$$

# Properties of stripped amplitudes

- stripped amplitudes are unique
- $\mathcal{M}(p_1, \ldots, p_n)$ are thus "physical"
- we can study different parametrizations

  [Cronin'67], [Ellis,Renner'70], [Bijnens,KK,Lanz '13], [KK,Novotny,Trnka '13]
- most familiar: exponential parametrization
- there we can simply enlarge $G$ to $U(N)$ group
- $\phi^0$ however decouples
- results from less restricted $U(N)$ equal to $SU(N)$
- General form of the parametrization $U(\phi) \to f(x)$

$$f(x) = \sum_{k=0}^{\infty} a_k x^k, \qquad f(-x)f(x) = 1$$

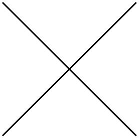
  - "exponential": $f_{\mathsf{exp}} = \mathrm{e}^x$
  - "minimal": $f_{\mathsf{min}} = x + \sqrt{1 + x^2}$
  - "Cayley": $f_{\mathsf{Caley}} = \frac{1+x/2}{1-x/2}$

# Explicit example: stripped 4pt amplitude

Natural parametrization for diagrammatic calculations: minimal (where off-shell and on-shell stripped vertices are equal) $\qquad s_{i,j} = P_{ij}^2$

4pt amplitude

$$2F^2\mathcal{M}(1,2,3,4) = -(s_{1,2} + s_{2,3})$$

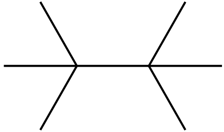


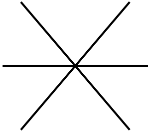
# Explicit example: stripped 6pt amplitude

$$4F^4\mathcal{M}(1,2,3,4,5,6) =$$
$$= \frac{(s_{1,2}+s_{2,3})(s_{1,4}+s_{4,5})}{s_{1,3}} + \frac{(s_{1,4}+s_{2,5})(s_{2,3}+s_{3,4})}{s_{2,4}}$$
$$+ \frac{(s_{1,2}+s_{2,5})(s_{3,4}+s_{4,5})}{s_{3,5}} - (s_{1,2}+s_{1,4}+s_{2,3}+s_{2,5}+s_{3,4}+s_{4,5})$$

This can be rewritten as

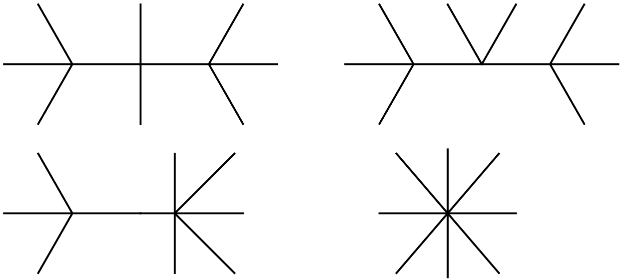
$$4F^4\mathcal{M}(1,2,3,4,5,6) = \frac{1}{2}\frac{(s_{1,2}+s_{2,3})(s_{1,4}+s_{4,5})}{s_{1,3}} - s_{1,2} + \text{cycl}\,,$$



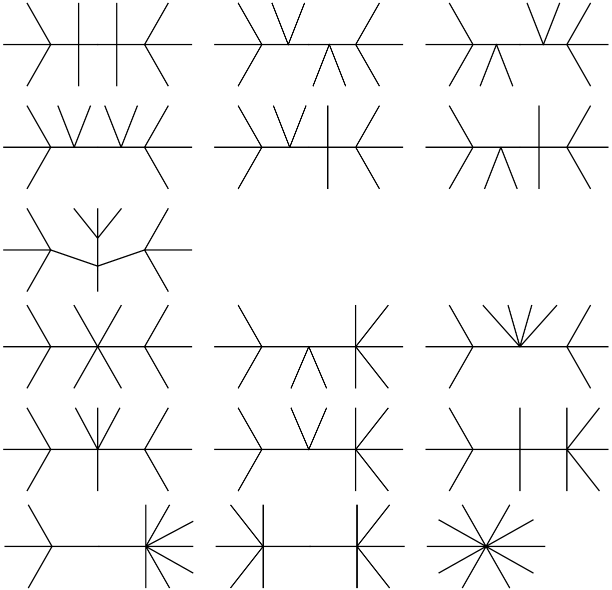


# Explicit example: stripped 8pt amplitude

$$8F^6\mathcal{M}(1,2,3,4,5,6,7) =$$
$$= -\frac{1}{2}\frac{(s_{1,2}+s_{2,3})(s_{1,4}+s_{4,7})(s_{5,6}+s_{6,7})}{s_{1,3}s_{5,7}} - \frac{(s_{1,2}+s_{2,3})(s_{1,4}+s_{4,5})(s_{6,7}+s_{7,8})}{s_{1,3}s_{6,8}}$$
$$+ \frac{(s_{1,2}+s_{2,3})(s_{4,5}+s_{4,7}+s_{5,6}+s_{5,8}+s_{6,7}+s_{7,8})}{s_{1,3}} - 2s_{1,2} - \frac{1}{2}s_{1,4} + \text{cycl}$$



# Explicit example: stripped 10pt amplitude



# Recursion relations

# Semi-on-shell amplitudes

Definition

$$J_n^{a,a_1,\ldots,a_n}(p_1,\ldots,p_n) = \langle 0|\phi^a(0)|\pi^{a_1}(p_1)\ldots\pi^{a_n}(p_n)\rangle$$

"off-shellness" in $p_{n+1}^2 \neq 0$

$$p_{n+1} = -\sum_{j=1}^{n} p_j$$

In analogy with vertex and on-shell amplitudes we can define flavour-stripped $J_n(p_1,\ldots,p_n)$. Then

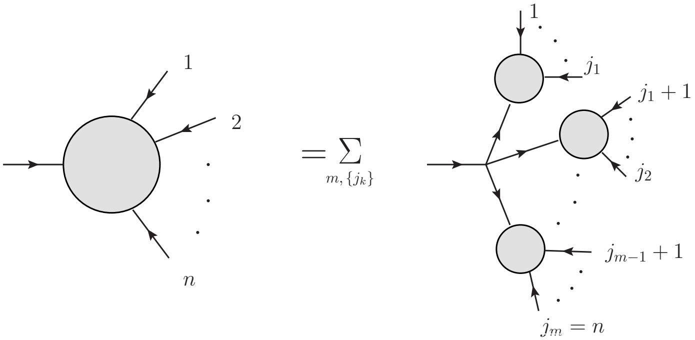
$$\mathcal{M}(p_1,p_2,\ldots,p_{n+1}) = -\lim_{p_{n+1}^2\to 0} p_{n+1}^2 J_n(p_1,p_2,\ldots,p_n)$$

Normalization

$$J_1(p) = 1$$

# Berends-Giele recursive relations

used for gluons [Berends,Giele'88]




$$J(1,2,\ldots,n) = \frac{\mathrm{i}}{p_{n+1}^2}\sum_{m=2}^{n}\sum_{j_1=1}^{n-m+1}\sum_{j_2=j_1+1}^{n-m+2}\cdots\sum_{j_{m-1}=j_{m-2}+1}^{n-m+(m-1)} \mathrm{i}V_{m+1}(p(1,j_1),p(j_1+1,j_2),\ldots p(j_{m-1}+1,n),-p(1,n)) \times J(1,\ldots,j_1)J(j_1+1,\ldots,j_2)\cdots J(j_{m-1}+1,\ldots,n).$$

# BCFW-like reconstruction

# Generalization of reconstruction formula: subtractions

[Benincasa, Conde '11] [Feng et al.'11] [KK,Novotny,Trnka '12]

In introduction: $A(z) \to 0$ for $z \to \infty$

Suppose some deformation of the external momenta $p_k \to p_k(z)$ so that

$$A(z) \sim z^k \qquad \text{for} \quad z \to \infty$$

we can assume the $(k+1)$-times subtracted Cauchy formula which leads to the desired generalization

$$A(z) = \sum_{i=1}^{n} \frac{\mathrm{Res}\,(A; z_i)}{z - z_i} \prod_{j=1}^{k+1} \frac{z - a_j}{z_i - a_j} + \sum_{j=1}^{k+1} A(a_j) \prod_{l=1, l\neq j}^{k+1} \frac{z - a_l}{a_j - a_l}$$

# Generalization of reconstruction formula: subtractions

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# Semi-on-shell amplitude $M$

- due to the derivative couplings the standard BCFW shift leads to $A(z) \sim z$ for $z \to \infty$
- we need <u>two</u> subtracted formula and two values $A(z_1)$ and $A(z_2)$

This is difficult to obtain. Way out: different continuation + give up on-shellness

We had

$$A(p_1, p_2, \ldots, p_{n+1}) = - \lim_{p_{n+1}^2 \to 0} p_{n+1}^2 J_n(1, \ldots, n)$$

# Semi-on-shell amplitude $M$

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This is difficult to obtain. Way out: different continuation + give up on-shellness
Now we define

$$M_n(p_1, p_2, \ldots, p_{n+1}) = \qquad p_{n+1}^2 J_n(1, \ldots, n)$$

and the complex deformation as (all even-line shift)

$$p_i \to p_i(z): \qquad p_{2i+1}(z) = p_{2i+1} \quad \text{and} \quad p_{2i}(z) = z p_{2i}$$

i.e.

$$M_n(z) \equiv M_{2n+1}(p_1, zp_2, p_3, \ldots zp_{2n}, p_{2n+1})$$

For its properties we can use directly the properties of $J$.

# Semi-on-shell amplitude $M$: properties

- $M(z)$ is meromorphic function of $z$ with single poles.
- physical amplitude for $z = 1$ and on-shell amplitude

$$A(p_1, \ldots, p_{2n+2}) = -\lim_{p_{2n+2}^2 \to 0} M_n(1)$$

- using scaling properties for $J$

1) $$\lim_{z\to 0} M_n(z) = \frac{(p_1 + p_3 + \ldots + p_{2n+1})^2}{(2F^2)^n} \equiv \frac{p_-^2}{(2F^2)^n}$$

2) $$M_n(z) = O(z^0) \qquad \text{as} \quad z \to \infty$$

- once subtracted reconstruction formula

$$M_n(z) = \frac{p_-^2}{(2F^2)^n} + \sum_P \frac{\text{Res}(M_n, z_P)}{z - z_P} \frac{z}{z_P}$$

# Summary

# Summary

Three different examples for non-linear sigma model

- standard two-loop calculation for $\pi^0 \to \gamma\gamma$
- leading logarithms up to 7th loop-order for $O(N)$ sigma model
- tree-level amplitudes for $n$-pion scatterings

Different computing techniques needed: Mathematica, Fortran, C++, Form...

Various well-established codes: FeynCalc, Fiesta, Fire...

New codes needed: Mathematica, Form

Thank you for your attention.