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The generalized BLM approach of fixing the scales in multiloop QCD expressions: the current status of investigations

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The BLM approach proposed at NLO Brodsky,Lepage,Mackenzie (1983), gauge-dependence in MOM schemes studied at NLO Celmaster, Stevenson (1983) and NNLO Chyla (1995) generalized in the MS-like schemes in NNLO Grunberg,Kataev (1992) and beyond : Beneke,Braun (1994), Mikhailov (2004) , Kataev,Mikhailov (2010-2012), Brosdky,Wu, Mojaza et al (2011-2014), Kataev, Mikhailov (arXiv:1408.0122). There are other applications : i.e. Jets characteristics: T. Gehrmann, N. Hafliger, P. F. Monni, (2014), BFKL at NLO in MOM scheme: S. J. Brodsky, V. S. Fadin, V. T. Kim, L. N. Lipatov and G. B. Pivovarov (1999)

The analytical α_s^4 results for the NS contributions to the Adler D-function of quark currents and to the Bjorken sum rule of the polarized lepton-nucleon scattering [Baikov, Chetyrkin, Kuhn 2010] are considered within β-expansion BLM extension approach with the multiple β-function generalization of the Crewther relation [Kataev, Mikhailov Quarks2010, TMP 2012]. New results for the Bjorken sum sule are obtained at $O(\alpha_s^3)$-level Kataev, Mikhailov (arXiv:1408.0122)] which confirm β-expansion parten

Studies of different ways of resummation and fixation of scale-scheme uncertainties of HO QCD PT predictions $\bigoplus$ understanding of basic features and symmetries beyond these representations are important theoretically and phenomenologically. Consider the **RG-invariant** quantities ($\overline{\mathrm{MS}}$)

$$D^{\mathrm{EM}}(Q^2/\mu^2, a_s(\mu^2)) = \left(\sum_i q_i^2\right) d_R D^{\mathrm{NS}}\left(Q^2/\mu^2, a_s(\mu^2)\right) + \left(\sum_i q_i\right)^2 d_R D^{\mathrm{S}}\left(Q^2/\mu^2, a_s(\mu^2)\right)$$

$$R_{e^+e^-}(s) \equiv R(s, \mu^2 = s) = \frac{1}{2\pi i}\int_{-s-i\epsilon}^{-s+i\epsilon} \frac{D^{\mathrm{EM}}(\sigma/\mu^2; a_s(\mu^2))}{\sigma} d\sigma \Bigg|_{\mu^2=s}$$

$$D^{\mathrm{NS}}(Q^2/\mu^2, a_s(\mu^2)) \stackrel{\mu^2=Q^2}{\longrightarrow} D^{\mathrm{NS}}(a_s(Q^2)) = 1 + \sum_{l\geq 1} d_l^{\mathrm{NS}} a_s^l(Q^2) \qquad (1)$$

$$S_{Bjp}(Q^2) = \int_0^1 [g_1^{lp}(x, Q^2) - g_1^{ln}(x, Q^2)]dx = \frac{g_A}{6}\; C_{Bjp}(Q^2/\mu^2, a_s(\mu^2))$$

$$C_{Bjp}(a_s) = C_{Bjp}^{\mathrm{NS}}(a_s) + \left(\sum_i q_i\right) C_{Bjp}^{\mathrm{S}}(a_s)$$ [Larin (2013), not confirmed yet directly]

$$C_{Bjp}^{\mathrm{NS}}(Q^2/\mu^2, a_s(\mu^2)) \stackrel{\mu^2=Q^2}{\longrightarrow} 1 + \sum_{l\geq 1} c_l^{\mathrm{NS}} a_s^l(Q^2) \qquad (2)$$

$$\beta(a_s) = \mu^2 \partial a_s(\mu^2)/\partial \mu^2 = -a_s^2(\mu^2)\sum_{l\geq 0} \beta_l a_s^l(\mu^2) \qquad (3)$$

β-expansions approach is the $\overline{MS}$-scheme generalization of the BLM method (1983) [Mikhailov, Quarks2004, JHEP(2007)] for all orders after NNLO generalization [Grunberg, Kataev (1992)] and works by [Beneke, Braun (1995)]. Similar problems also studied by [Brodsky, Wu et al (2012-2014)]
Basis - Instead of Scalar Representation

$$D^{NS} = 1 + \sum_{n \geq 1} a_s^n(Q^2) d_n(N_F) = 1 + (\overline{a_s d(N_F)}) \tag{4}$$

we use Matrix Representation

$$D^{NS} = 1 + \sum_{n \geq 1} \sum_{l} a_s^n(Q^2) D_{nl} B_l(N_F) \tag{5}$$

$B_l(N_F)$-products of β-function coefficients, $d_n(N_F) = D_{nl} B_l(N_F)$, terms D_{nl} do not depend on the numbers of flavours N_F and have the form

$$d_1 = d_1[0] = \frac{3}{4} C_F \quad , \quad d_2 = \beta_0(N_F) d_2[1] + d_2[0]$$

$$d_3 = \beta_0^2(N_F) d_3[2] + \beta_1(N_F) d_3[0,1] + \boxed{\beta_0(N_F) d_3[1]} + d_3[0]$$

$$d_4 = \beta_0^3 d_4[3] + \beta_1 \beta_0 d_4[1,1] + \beta_2(N_F) d_4[0,0,1] + \beta_0^2 d_4[2] + \boxed{\beta_1 d_4[0,1]} + \boxed{\beta_0 d_4[1]} + d_4[0]$$

The result for d_3 was obtained by [Mikhailov (2007)] using QCD + $n_{\tilde{g}}$ multiplet of massless gluino, contributing to $d_3(N_F, n_{\tilde{g}})$ from the result of [Chetyrkin (1997)]. Terms $d_n[0]$ are related to conformal symmetry limit. The phenomenological applications of these terms is the basis of the PMC method – [Brodsky, Wu et al (2012-...)]. $\boxed{\beta_0 d_3[1]}$, $\boxed{\beta_1 d_4[0,1]}$ $\boxed{\beta_0 d_4[1]}$ were neglected in the original expressions So ambiguity ? NO !

Why $\boxed{\beta_0(N_F) d_3[1]}$, $\boxed{\beta_1(N_F) d_4[0,1]}$ $\boxed{\beta_0(N_F) d_4[1]}$ were neglected by [Brodsky et al] ? Without neglecting these terms it was impossible to get used by **them (to our mind not perfect) variant of β-expansion**

$$d_1 = d_1[0] = \frac{3}{4} C_F \quad , \quad d_2 = \beta_0(N_F) d_2[1] + d_2[0]$$

$$d_3 = \beta_0^2(N_F) \tilde{d}_3[2] + \beta_1(N_F) \tilde{d}_3[0,1] + \tilde{d}_3[0]$$

$$d_4 = \beta_0^3 \tilde{d}_4[3] + \beta_1 \beta_0(N_F) \tilde{d}_4[1,1] + \beta_2(N_F) \tilde{d}_4[0,0,1] + \beta_0^2 \tilde{d}_4[2] + \tilde{d}_4[0]$$

Using complete N_F dependence of the $\beta_i(N_F)$ and $d_i(N_F)$

$$d_2(N_F) = N_F d_{21} + d_{20} \ , \ d_3(N_F) = N_F^2 d_{32} + N_F d_{31} + d_{30} \tag{6}$$

$$d_4(N_F) = N_F^3 d_{43} + N_F^2 d_{42} + N_F d_{41} + d_{40} \tag{7}$$

The β- representation is true for NS contribution to the Bjorken sum rule of the polarisez lepton-nucleon DIS [Kataev, Mikhailov(2010-2012)]. The β-expanded form for C^{NS}_{Bjp} was obtained from the $O(\alpha_s^4)$-generalization of the matrix representation for the $\overline{MS}$-scheme Crewther relation [Kataev, Mikhailov, Quarks2010, TMP(2012)]

$$D^{NS}(a_s)C^{NS}_{Bjp} = \mathbb{1} + \sum_{n\geq 1}\left(\frac{\beta(a_s)}{a_s}\right)^n P_n(a_s) \quad (8)$$

$$C^{NS}_{Bjp} = \sum_{n\geq 1}\sum_{l} a_s^n(Q^2)c_{nl}B_l(N_F) \quad (9)$$

$$c_1 = c_1[0] = -\frac{3}{4}C_F \quad , \quad c_2 = \beta_0(N_F)c_2[1] + c_2[0]$$

$$c_3 = \beta_0^2(N_F)c_3[2] + \beta_1(N_F)c_3[0,1] + \boxed{\beta_0(N_F)c_3[1]} + c_3[0]$$

$$c_4 = \beta_0^3 c_4[3] + \beta_1\beta_0 c_4[1,1] + \beta_2 c_4[0,0,1] + \beta_0^2 c_4[2] + \boxed{\beta_1 c_4[0,1]} + \boxed{\beta_0 c_4[1]} + c_4[0]$$

Note that the terms in **boxes** can not be eliminated

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Without them in [KM(2010-2012)] results the powers of β-function will be spoiled. Indeed the polynomial $P_n(a_s)$ at the powers of β-function contain these terms and they can not be neglected in the process of constructing Principle of Maximal Conformality by [Brodsky et al]

$$D^{NS}(a_s)C^{NS}_{Bjp} = 1 + \sum_{n\geq 1} \left(\frac{\beta(a_s)}{a_s}\right)^n P_n(a_s)$$

$$P_1(a_s) = -a_s\,(c_2[1] + d_2[1]) - a_s^2 \left(\boxed{c_3[1]} + \boxed{d_3[1]} + d_1(c_2[1] - d_2[1]) \right) - a_s^3\delta_1$$

$$\delta_1 = c_4[1] + d_4[1] + d_1 \left(\boxed{c_3[1]} - \boxed{d_3[1]} \right) + d_2[0]c_2[1] + d_2[1]c_2[0]$$

$$P_2(a_s) = a_s \left(c_3[2] + d_3[2] + a_s^2 \left(\boxed{c_4[2]} + \boxed{d_4[2]} - d_1(c_3[2] - d_3[2]) \right) \right)$$

If we neglect them the results of β expansion will not agree with the values of the factorized terms, which follow from the exact analytic calculations in the $\overline{\mathrm{MS}}$-scheme.

Explicit results for $O(a_s^3)$ – QCD+$n_{\tilde{g}}$ (multiplet of massless gluino $\tilde{g}$).
Additional degree of freedom, [Mikhailov(2007)], $\beta_i \to \tilde{\beta}_i\,(N_F, n_{\tilde{g}})$,
help to separates $\tilde{\beta}_0(N_F, n_{\tilde{g}})$ and $\tilde{\beta}_1(N_F, n_{\tilde{g}})$-terms:

$$D^{NS}(a_s, N_F, n_{\tilde{g}}) = 1+\frac{a_s}{4}(3C_F)+\left(\frac{a_s}{4}\right)^2(3C_F)\left[\frac{\mathrm{C_A}}{3}-\frac{\mathrm{C_F}}{2}+\left(\frac{11}{2}-4\zeta_3\right)\tilde{\beta}_0\right]+\left(\frac{a_s}{4}\right)^3(3C_F)d_3$$

$$d_3 = \tilde{\beta}_0^2 \cdot d_3[2] + \tilde{\beta}_1 \cdot d_3[0,1] + \tilde{\beta}_0 \cdot d_3[1] + d_3[0]$$

$$\begin{aligned}
d_3[2] &= \frac{302}{9} - \frac{76}{3}\zeta_3 \\
d_3[0,1] &= \frac{101}{12} - 8\zeta_3 \\
d_3[1] &= C_A\left(\frac{3}{4} + \frac{80}{3}\zeta_3 - \frac{40}{3}\zeta_5\right) - C_F\,(18 + 52\zeta_3 - 80\zeta_5) \\
d_3[0] &= \left(\frac{523}{36} - 72\zeta_3\right) C_A^2 + \frac{71}{3} C_A C_F - \frac{23}{2} C_F^2
\end{aligned}$$

The result agree withg the calculatrions by [Chetyrkin (1997)]. QCD result is obtained in case of taking $n_{\tilde{g}} = 0$ with substituting $\tilde{\beta}_j(N_F, n_{\tilde{g}}) \to \beta_j(N_F)$ so we get all terms in the β-expansion of D-function, while in the [Brodsky et al] the term proportional to β_0 is absent

New prediction: [Kataev, Mikhailov(2010-2012)] from the β-expansion for C_{Bjp} with the help of $\overline{\mathrm{MS}}$-generalised Crewther relation of order of $O(a_s^3)$ we obtain its expression in QCD $+n_{\tilde{g}}$, $\beta_i \to \tilde{\beta}_i\,(N_F, n_{\tilde{g}})$

$$C^{NS}(a_s, N_F, n_{\tilde{g}}) = 1 - \frac{a_s}{4}(3C_F) - \left(\frac{a_s}{4}\right)^2 (3C_F)\left[\frac{\mathrm{C_A}}{3} - \frac{7}{2}\mathrm{C_F} - 2\tilde{\beta}_0\right] - \left(\frac{a_s}{4}\right)^3 (3C_F)c_3$$

$$c_3 = \tilde{\beta}_0^2 \cdot c_3[2] + \tilde{\beta}_1 \cdot c_3[0,1] + \tilde{\beta}_0 \cdot c_3[1] + c_3[0]$$

$$\begin{aligned} c_3[2] &= \frac{115}{18}; \quad c_3[0,1] = \frac{59}{12} - 4\zeta_3 \\ c_3[1] &= -\left[C_A\left(\frac{215}{36} - 32\zeta_3 + \frac{40}{3}\zeta_5\right) + C_F\left(\frac{166}{9} - \frac{16}{9}\zeta_3\right)\right] \\ c_3[0] &= \left[\left(\frac{523}{36} - 72\zeta_3\right)C_A^2 + \frac{65}{3}C_A C_F + \frac{1}{2}C_F^2\right] \end{aligned}$$

These results can be checked by direct analytical calculations in the QCD$+n_{\tilde{g}}$ multiplets of light gluinos in the $\overline{\mathrm{MS}}$-scheme.

Next - PMC by [Brodsky et al(2011-2014)] in our realization: we consider first β expansion for $D^{NS}(a_s(t = Q^2/\mu^2))$ and find $a_s(t_1, t)$ to cancel a part of β-expansion. In each new order of PT we define the new scale Q_i^2, absorbing the β-function coefficients into the scale(s). So we have the consequent approximations with $Q^2 \to Q_1^2 \to Q_2^2 \to Q_3^2 \dots$:

$$\begin{aligned} N^2L: \quad D^{NS}(Q_1^2) &= 1 + d_1[0]a_s(Q_1^2) + d_2[0]a_s^2(Q_1^2) \\ N^3L: \quad D^{NS}(Q_2^2) &= 1 + d_1[0]a_s(Q_2^2) + d_2[0]a_s^2(Q_2^2) + d_3[0]a_s^3(Q_2^2) \end{aligned}$$

At presnt we are able to fix all scales and coefficients in these approximation of PT for $D^{NS}(a_s)$ and $C^{NS}(a_s)$ at the $O(a_s^3)$-approximation. The coefficients respect conformal symmetry- so this is the analog of PMC by [Brodsky et al.]. In higher-order level we should have

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$$N^4L: \; D^{NS}(Q_3^2) = 1 + d_1[0]a_s(Q_3^2) + d_2[0]a_s^2(Q_3^2) + d_3[0]a_s^3(Q_3^2) + d_4[0]a_s^4(Q_3^2) \quad (10)$$

In the [Brodsky et al] works the results are

$$D^{NS}(Q_i) = 1 + d_1[0]a_s(\tilde{Q}_1{}^2) + \tilde{d}_2[0]a_s^2(\tilde{Q}_2{}^2) + \tilde{d}_3[0]a_s^3(\tilde{Q}_3^2) + \tilde{d}_4[0]a_s^4(\tilde{Q}_4^2) \quad (11)$$

The PMC ideas are correct, but PMC original values of coefficients should be coordinated with our considerations, adding in the initial expansion additional neglected by [Brodsky, Wu, Mojaza] terms. The scales of PMC can be also related using thed RG-group and the neglected initial non-conformal terms, which can be also resummed. The common understanding contain the basis of important theoretical approach which contain the knowledge on the Conformal Symmetry of $SU(N_c)$ (in the values of the undressed by coefficients of β-function coefficiednts) which due to our common efforts [Kataev, Mikhailov, Brodsky, Mojaza, Wu] There are extra arguments that the form used by KM $\{\beta\}$-expansion is correct and follows from multiple power β-function represntation [Broadhurst,Kataev,Maxwell 2013-2014, work in progress]. Preliminary results reported by Broadhurst at Loops and Legs in 2014