Non-planar Feynman diagrams and Mellin-Barnes representations with AMBRE

Tord Riemann, DESY

in collaboration with

I. Dubovyk, DESY

and

J. Gluza, University of Silesia

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Track 3: Computations in Theoretical Physics: Techniques and Methods

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Outline

AMBRE

AMBRE is a Mathematica package for the evaluation of Feynman integrals with the aid of their Mellin-Barnes representations.

(i) New package version **AMBRE v.3** (I. Dubovyk, J. Gluza, K. Kajda, T. Riemann): Proper and efficient Mellin-Barnes presentation of non-planar topologies.
arXiv:1407.7832 [1], arXiv:1312.5603 [2], http://www.us.edu.pl/ gluza/ambre

(ii) New package **MBsums** (M. Ochman, DESY, unpublished):
For an automatic derivation of a subsequent representation by multiple sums, allowing a linking of packages for automatic summation like XSUMMER, SIGMA
No details here, see PoS(LL2014)052, arXiv:1407.7832.

Introduction

L–loop n–point functions

Consider an arbitrary L–loop integral $G(X)$ with loop momenta k_l , with E external legs with momenta p_e and with N internal lines with masses m_i and propagators $1/D_i$

$$G(X) = \frac{1}{(i\pi^{d/2})^L} \int \frac{d^d k_1 \dots d^d k_L \, X(k_1, \dots, k_L)}{D_1^{n_1} \dots D_i^{n_i} \dots D_N^{n_N}}$$

$$d = 4 - 2\epsilon$$

$$D_i = q_i^2 - m_i^2 = \left[\sum_{l=1}^{L} c_i^l k_l + \sum_{e=1}^{M} d_i^e p_e \right] - m_i^2$$

$X(k_1, \dots, k_L)$ stands for tensors in the loop momenta.

Two representations for integrals

Feynman parameter representation ($N_\nu = n_1 + \ldots + n_N$):

$$\frac{1}{D_1^{n_1} D_2^{n_2} \ldots D_N^{n_N}} = \frac{\Gamma(n_1 + \ldots + n_N)}{\Gamma(n_1) \ldots \Gamma(n_N)} \int_0^1 dx_1 \ldots \int_0^1 dx_N \frac{x_1^{n_1-1} \ldots x_N^{n_N-1} \delta(1 - x_1 - \ldots - x_m)}{(x_1 D_1 + \ldots + x_N D_N)^{N_\nu}}$$

Alpha parameter representation:

$$\frac{1}{D_1^{n_1} D_2^{n_2} \ldots D_N^{n_N}} = \frac{i^{-N_\nu}}{\Gamma(n_1) \ldots \Gamma(n_N)} \int_0^\infty d\alpha_1 \ldots \int_0^\infty d\alpha_N \alpha_1^{n_1-1} \ldots \alpha_N^{n_N-1} e^{i[\alpha_1 D_1 + \ldots + \alpha_N D_N]}$$

Using the identity

$$1 = \int_0^\infty \frac{d\lambda}{\lambda} \delta\left(1 - \frac{1}{\lambda} \sum_{i=1}^{N} \alpha_i\right)$$

and by change of variables from α_i to $\alpha_i = \lambda x_i$, one can prove their identity.

- Both representations have been used extensively since the beginning of one-loop calculations.
- There are also other methods of Feynman integral calculations, e.g.
  – Differential equations (which got lately a new strong kick initiated by J. Henn) [3, 4, 5]
  – Expansions by regions [6]
  – Sector decomposition [7, 8, 9, 10, 11, 12, 13, 14]
- However, in our opinion one method is still not exploited sufficiently, namely solving Feynman integrals by Mellin-Barnes representations [15, 16, 17, 18, 19, 20].
  There is steady progress in its automatization.
  See e.g. [21, 22, 23, 24, 25, 2] and software at http://projects.hepforge.org/mbtools.
  But the method is less developed so far than, e.g., that of differential equations.

J. Gluza and T. Riemann, *Simple Feynman diagrams and simple sums*, RISC-DESY Meeting, May 2012, Linz [26]
$\longrightarrow$
It would be wonderful to have an algorithm for automatic analytical evaluation of all the scalar (and tensor) integrals by infinite multiple sums!

For not too involved classes of functions, see Summer, at http://www.nikhef.nl/ t68/ and XSUMMER [27].
For automation we have to

1. know about planarity or non-planarity see hep-ph/1312.5603, [2]
2. construct MB representations this talk: non-planar case (I. Dubovyk et al.)
3. change them into nested sums MBsums package (M. Ochman et al., unpubl., see talk at LL2014)
4. try to perform the multiple sums analytically
   see talks by J. Gluza (LL2014_052.pdf, [1]) and C. Raab (LL2014_020.pdf, [28])

$$Certainly,\ there\ are\ limitations:$$

- Number of loops: One-loop, two-loop,...?
- Number of scales: Massive, off-shell?
- Number of legs: 2-,3-,.... point functions?
- ... and the complexity, e.g. due to non-planarity

Starting point

$$G(X) = \frac{(-1)^{N_\nu}\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\prod_{i=1}^{N}\Gamma(n_i)} \int \prod_{j=1}^{N} dx_j\, x_j^{n_j-1}\delta(1-\sum_{i=1}^{N} x_i)\frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

The functions U and F are called graph or Symanzik polynomials.

General Mellin-Barnes relation (not shown) can be applied to polynomials U and F

$$\frac{1}{(A_1+\ldots+A_n)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{(2\pi i)^{n-1}}\int_{-i\infty}^{i\infty} dz_1 \ldots dz_{n-1}$$
$$\times \prod_{i=1}^{n-1} A_i^{z_i}\, A_n^{-\lambda - z_1 - \ldots - z_{n-1}} \prod_{i=1}^{n-1}\Gamma(-z_i)\,\Gamma(\lambda + z_1 + \ldots + z_{n-1})$$

Then we get multidimensional MB integrals.

MB integrals and the iterative loop–by–loop (LA) approach

Examples, description, links to basic tools and literature:
http://us.edu.pl/∼ gluza/ambre/

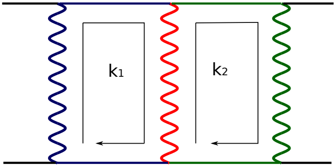


Figure 1 : Loop–by–loop (LA) example

Here, because we have a sequence of on-loop steps: $U(x) \equiv 1$
Input:

$$\mathrm{PR}[\mathrm{k1},\mathrm{m},\mathrm{n1}]\mathrm{PR}[\mathrm{k1}+\mathrm{p1},0,\mathrm{n2}]\mathrm{PR}[\mathrm{k1}+\mathrm{p1}+\mathrm{p2},\mathrm{m},\mathrm{n3}]\mathrm{PR}[\mathrm{k1}-\mathrm{k2},0,\mathrm{n4}]$$
$$\mathrm{PR}[\mathrm{k2},\mathrm{m},\mathrm{n5}]\mathrm{PR}[\mathrm{k2}+\mathrm{p1}+\mathrm{p2},\mathrm{m},\mathrm{n6}]\mathrm{PR}[\mathrm{k2}-\mathrm{p3},0,\mathrm{n7}]$$

Integration over k_2:

$$\mathrm{PR[k1-k2,0,n4]PR[k2,m,n5]PR[k2+p1+p2,m,n6]PR[k2-p3,0,n7]}$$

$$\mathrm{F[X]} = \mathrm{m}^2\,(\mathrm{X[2]+X[3]})^2 - \mathrm{PR[k1,m]X[1]X[2]} - \mathrm{PR[k1+p1+p2,m]X[1]X[3]} - \mathrm{sX[2]X[3]} - \mathrm{PR[k1-p3,0]X[1]X[4]}$$

Integration over k_1:

$$\mathrm{PR[k1,m,}\alpha\mathrm{]PR[k1+p1,0,n2]PR[k1+p1+p2,m,}\beta\mathrm{]PR[k1-p3,0,}\gamma\mathrm{]}$$

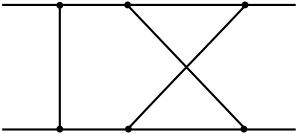
$$\mathrm{F[X]} = \mathrm{m}^2\,(\mathrm{X[1]+X[3]})^2 - \mathrm{sX[1]X[3]} - \mathrm{tX[2]X[4]}$$

	Massless cases				Massive cases			
number of loops (L)	1	2	3	4	1	2	3	4
dimensions, no BFL	1	4	7	10	3	8	13	18
dimensions, with BFL	1	4	7	10	2 (1+1)	6 (4+2)	10 (7+3)	14 (10+4)

Optimal results: Dimensions of ladder planar MB integrals before and after applying Barnes First Lemma (BFL).

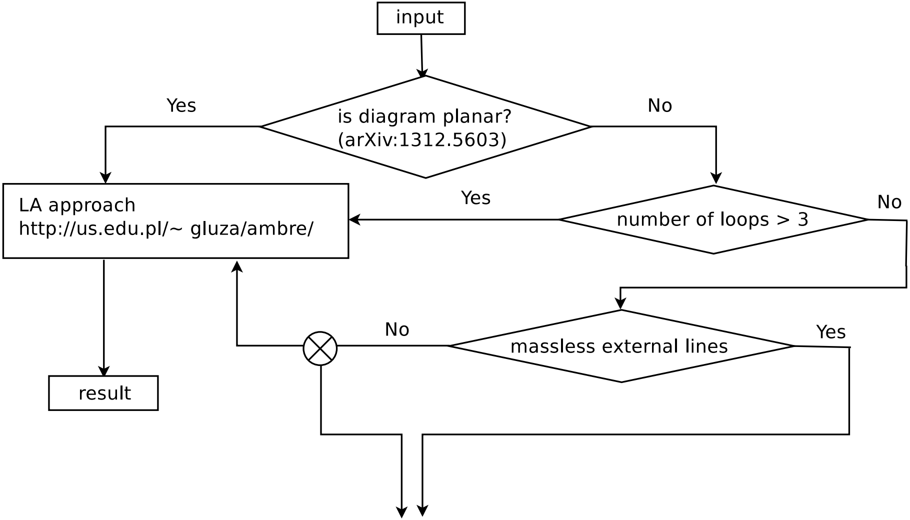
Global approach - GA

It happens for non-planar integrals that the loop-by-loop approach failes to give correct results.
Then one may alternatively derive a MB representation using the complete U and F polynomials in one step, and not loop-by-loop,
e.g.

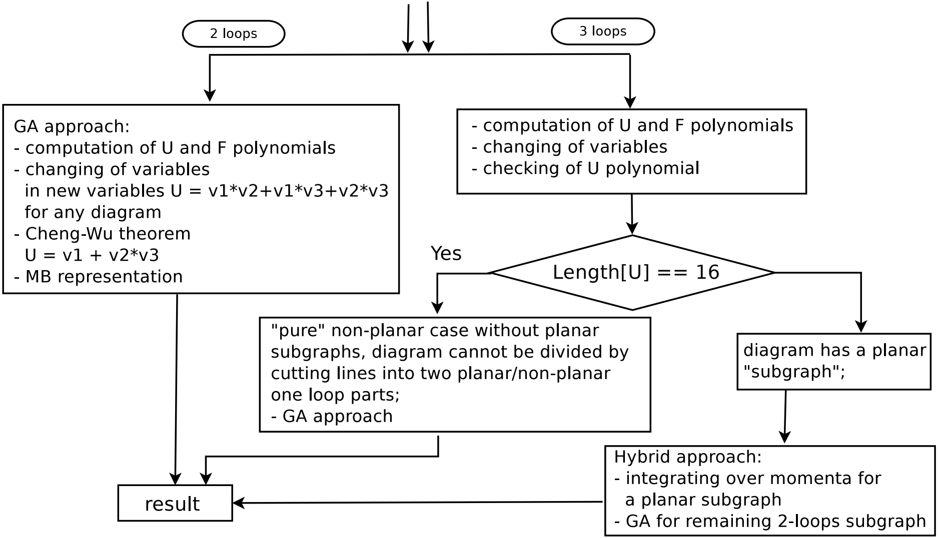


1. massless case: 4-dim MB - GA
2. massive case: 8-dim MB - LA
   with GA not less then 10-dim MB (Heinrich, Smirnov, PLB 2004 [11])

1. AMBRE reloaded - non-planar version, basic chart (I)



Basic chart (II), GA



Cheng–Wu Theorem

$$G(X) = \frac{(-1)^{N_\nu}\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\prod_{i=1}^{N}\Gamma(n_i)} \int \prod_{j=1}^{N} dx_j \, x_j^{n_j-1} \delta(1 - \sum_{i=1}^{N} x_i) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

The Cheng–Wu theorem [29, 30] states that the same formula holds with the delta function

$$\delta\left(1 - \sum_{i\in\Omega} x_i\right)$$

where Ω is an arbitrary **subset** of the lines $1, \ldots, L$, when the integration over the rest of the variables, i.e. for $i \notin \Omega$, is extended to the integration from 0 to ∞. One can prove this theorem in a simple way starting from the alpha representation using

$$1 = \int_0^\infty \frac{d\lambda}{\lambda}\delta\left(1 - \frac{1}{\lambda}\sum_{i=1}^{N}\alpha_i\right) \Leftrightarrow 1 = \int_0^\infty \frac{d\lambda}{\lambda}\delta\left(1 - \frac{1}{\lambda}\sum_{i\in\Omega}\alpha_i\right)$$

and change variables from α_i to $\alpha_i = \lambda x_i$ as shown above.

Non–Planar DoubleBox

J. B. Tausk,
"Nonplanar massless two loop Feynman diagrams with four on-shell legs",
Phys. Lett. B **469** (1999) 225; [hep-ph/9909506], [19]

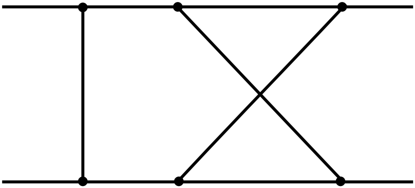


Figure 2 : The non-planar double box.

$$B_7^{NP} = \iint d^dk_1 d^dk_2 \frac{1}{[(k_1+k_2+p_1+p_2)^2]^{n_1}[(k_1+k_2+p_2)^2]^{n_2}[(k_1+K_2)^2]^{n_3}} \frac{1}{[(k_1-p_3)^2]^{n_4}[(k_1)^2]^{n_5}[(k_2-p_4)^2]^{n_6}[(k_2)^2]^{n_7}}$$

$$U(x) = x[1]x[2] + x[1]x[4] + x[2]x[4] + x[1]x[5] + x[2]x[5] + x[2]x[6] + x[4]x[6] + x[5]x[6] + x[1]x[7] + x[4]x[7] + x[5]x[7] + x[6]x[7]$$

$$F(x) = -s\, x[1]x[2]x[5] - s\, x[1]x[3]x[5] - s\, x[2]x[3]x[5] - u\, x[2]x[4]x[6] - s\, x[3]x[5]x[6] - t\, x[1]x[4]x[7] - s\, x[3]x[5]x[7] - s\, x[3]x[6]x[7]$$

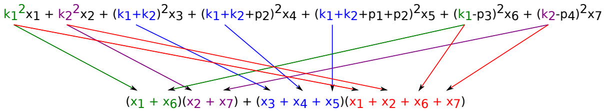


Figure 3 : Factorization scheme

$$U(x) = (x[1]+x[6])(x[2]+x[7]) + (x[3]+x[4]+x[5])(x[1]+x[2]+x[6]+x[7])$$

$$F(x) = -t\, x[1]x[4]x[7] - u\, x[2]x[4]x[6] - s\, x[1]x[2]x[5] - s\, x[3]x[6]x[7] - s\, x[3]x[5](x[1]+x[2]+x[6]+x[7])$$

Now we can apply the Cheng-Wu theorem and integrations will look as follows

$$B_7^{NP} = \frac{(-1)^{N_\nu}\Gamma\left(N_\nu - d\right)}{\Gamma(n_1)\ldots\Gamma(n_N)} \int\limits_0^\infty dx_3 dx_4 dx_5 \int\limits_0^1 dx_1 dx_2 dx_6 dx_7 \delta(1-(x_1+x_2+x_6+x_7)) \frac{((x_1+x_6)(x_2+x_7)+x_3+x_4+x_5)^{N_\nu-\frac{3d}{2}}}{(-t\, x_1x_4x_7 - u\, x_2x_4x_6 - s\, x_1x_2x_5 - s\, x_3x_6x_7 - s\, x_3x_5)^{N_\nu-d}}$$

$$B_7^{NP} = \frac{(-1)^{N_\nu}}{\Gamma(n_1)\ldots\Gamma(n_N)} \int\limits_{-i\infty}^{i\infty} dz_1\ldots dz_4 \int dx_1\ldots dx_7\, (-s)^{-N_\nu+d-z2-z3}(-t)^{z2}(-u)^{z3} \times \Gamma(-z1)\Gamma(-z2)\Gamma(-z3)\Gamma(-z4)\Gamma(N_\nu-d+z1+z2+z3+z4) \times x_1^{-N_\nu+d-z1-z2-z3} x_2^{z2+z3} {x_3}^{-N_\nu+d-z2-z3-z4} {x_4}^{z1+z3} {x_5}^{z2+z4} x_6^{z1+z2} x_7^{z3+z4} \times (x_3+x_4+x_5+(x_1+x_6)(x_2+x_7))^{N_\nu-\frac{3d}{2}}$$

Integration over Cheng–Wu variables

$$\int_0^\infty dx\, x^{N_1} (x+A)^{N_2} = \frac{A^{1+N_1+N_2}\Gamma(1+N_1)\Gamma(-1-N_1-N_2)}{\Gamma(-N_2)} \qquad (1)$$

4-dim result:

$$B_7^{NP} = \frac{(-1)^{N_\nu}}{\Gamma(n_1)\ldots\Gamma(n_7)} \int_{-i\infty}^{i\infty} dz_1 \ldots dz_4 (-s)^{4-2\epsilon-N_\nu-z_{23}} (-t)^{z_3} (-u)^{z_2}$$
$$\frac{\Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_4)\Gamma(2-\epsilon-n_{45})\Gamma(2-\epsilon-n_{67})}{\Gamma(4-2\epsilon-n_{4567})\Gamma(n_{45}+z_{1234})\Gamma(n_{67}+z_{1234})\Gamma(6-3\epsilon-N_\nu)}$$
$$\Gamma(n_2+z_{23})\Gamma(n_4+z_{24})\Gamma(n_5+z_{13})\Gamma(n_6+z_{34})\Gamma(n_7+z_{12})\Gamma^3(-2+\epsilon+n_{4567}+z_{1234})$$
$$\Gamma(4-2\epsilon-n_{124567}-z_{123})\Gamma(4-2\epsilon-n_{234567}-z_{234})\Gamma(-4+2\epsilon+N_\nu+z_{1234})$$

with notations $z_{i\ldots j\ldots k} = z_i + \ldots + z_j + \ldots + z_k$
and $n_{i\ldots j\ldots k} = n_i + \ldots + n_j + \ldots + n_k$

AMBRE - another way, by suitable change of variables

$$B_7^{NP} = \iint d^dk_1 d^dk_2 \frac{1}{[(k_1+k_2+p_1+p_2+p_3)^2]^{n_1}[(k_1+k_2)^2]^{n_2}}$$

$$\frac{1}{[(k_1)^2]^{n_3}[(k_1+p_1)^2]^{n_4}[(k_1+p_1+p_2)^2]^{n_5}[(k_2)^2]^{n_6}[(k_2+p_3)^2]^{n_7}}$$

$$\begin{aligned} m^2 = \sum x_i D_i \quad &= \quad x_1(k_1+k_2+p_1+p_2+p_3)^2 && x_1 \to v_1 C_1 \\ &+ \quad x_2(k_1+k_2)^2 && x_2 \to v_1 C_2 \\ &+ \quad x_3(k_1)^2 && x_3 \to v_2 A_1 \\ &+ \quad x_4(k_1+p_1)^2 && x_4 \to v_2 A_2 \\ &+ \quad x_5(k_1+p_1+p_2)^2 && x_5 \to v_2 A_3 \\ &+ \quad x_6(k_2)^2 && x_6 \to v_3 B_1 \\ &+ \quad x_7(k_2+p_3)^2 && x_7 \to v_3 B_2 \end{aligned}$$

$$\delta\left(1-\sum_{i=1}^{7} x_i\right) \Rightarrow \delta(1-v_1-v_2-v_3)\delta(1-A_1-A_2-A_3)\delta(1-B_1-B_2)\delta(1-C_1-C_2)$$

Jacobian of the transformation:

$$J = v_1^{N_C-1} v_2^{N_A-1} v_3^{N_B-1} = v_1 v_2^2 v_3$$

1. Using $\delta(1-A_1-A_2-A_3)\delta(1-B_1-B_2)\delta(1-C_1-C_2)$ we can write for U and F

$$U = v_1v_2 + v_1v_3 + v_2v_3 \qquad F = -s\,A_1A_3v_1v_2^2 - s\,A_1A_3v_2^2v_3 - u\,A_2B_1C_1v_1v_2v_3$$
$$-s\,A_1B_2C_1v_1v_2v_3 - s\,A_3B_1C_2v_1v_2v_3 - t\,A_2B_2C_2v_1v_2v_3$$

2. Choose now v_2 as Cheng-Wu variable $\int_0^\infty dv_2 \int_0^1 dv_1dv_3\delta(1-v_1-v_3)$

$$U = v_2 + v_1v_3 \quad F = -s\,A_1A_3v_2^2 - u\,A_2B_1C_1v_1v_2v_3 - s\,A_1B_2C_1v_1v_2v_3$$
$$-s\,A_3B_1C_2v_1v_2v_3 - t\,A_2B_2C_2v_1v_2v_3$$

3. Apply MB relation for F
4. Integrate over v_2 using again eqn. (1)

$$\int_0^\infty dx\, x^{N_1}(x+A)^{N_2} = \frac{A^{1+N_1+N_2}\Gamma(1+N_1)\Gamma(-1-N_1-N_2)}{\Gamma(-N_2)}$$

5. Integrate over each subset of variables $\{v, A, B, C\}$ separately using

$$\int_0^1 \prod_{i=1}^{N} dx_i\, x_i^{n_i-1}\,\delta(1-x_1-\ldots-x_N) = \frac{\Gamma(n_1)\ldots\Gamma(n_N)}{\Gamma\left(n_1+\ldots+n_N\right)}$$

and get 4-dim representation, as before.

Some remarks

Change of variables in Symanzik polynomials U and F is effective as:

- They are homogeneous in the Feynman parameters, U is of degree L, F is of degree $L+1$
- U is linear in each Feynman parameter. If all internal masses are zero, then also F is linear in each Feynman parameter
- In expanded form each monomial of U has coefficient $+1$

Further:

1. Note that on a basic chart II, 3-loops cases, we have already $\mathrm{Length[U]==16}$ for GA (independently of both the multileg topology and mass configurations!), that is why much more effort must be done for simplifying U and F polynomials at this level
2. In general, it is not true that $dim(\mathtt{MB}[planars]) \simeq dim(\mathtt{MB}[non-planars])$,
3. Cases of massive external legs are completely different, e.g. such a factorization is not always the best:

$$F = F_0 + U \sum_{n=1}^{N} x_n \frac{m_n^2}{\{\mathrm{scales}\}^2}$$

Hybrid method for 3–loops

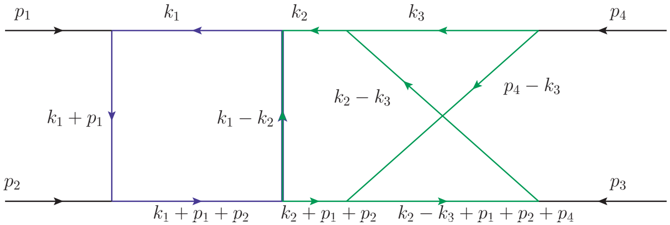


Figure 4 : Hybrid method: LA– $\{k_1\}$; GA– $\{k_2, k_3\}$.

Input:

```
PR[k1, 0, n1] PR[k1 + p1, 0, n2] PR[k1 + p1 + p2, 0, n3]
PR[k1 - k2, 0, n4] PR[k2, 0, n5] PR[k2 + p1 + p2, 0, n6]
PR[p1 + p2 + p4 + k2 - k3, 0,n7] PR[k2 - k3, 0, n8]
                         PR[k3, 0, n9] PR[p4 - k3, 0, n10]
```

step 1 input (LA– $\{k_1\}$):

```
PR[k1, 0, n1] PR[k1 + p1, 0, n2] PR[k1 + p1 + p2, 0, n3]
                                        PR[k1 - k2, 0, n4]
```

step 1 output:

```
((-1)^(2-eps-z3) (-s)^z3 Gamma[-z1] Gamma[2-eps-n1-n2-n4-z1-z2]
  Gamma[-z2] Gamma[n2+z2] Gamma[2-eps-n1-n2-n3-z3] Gamma[-z3]
     Gamma[n1+z1+z3] Gamma[-2+eps+n1+n2+n3+n4+z1+z2+z3])
/(Gamma[n1] Gamma[n2] Gamma[n3]
                           Gamma[4-2 eps-n1-n2-n3-n4] Gamma[n4])
PR[k2, 0, -z1] PR[k2+p1, 0, -z2]
                 PR[k2+p1+p2,0,-2+eps+n1+n2+n3+n4+z1+z2+z3]
```

step 2 input (GA– $\{k_2, k_3\}$):

```
   PR[k2, 0, n5-z1] PR[k2-k3, 0, n8] PR[k2+p1, 0, -z2]
     PR[k2+p1+p2, 0, -2+eps+n1+n2+n3+n4+n6+z1+z2+z3]
PR[k2-k3+p1+p2+p4, 0, n7] PR[k3, 0, n9] PR[p4-k3, 0, n10]
```

step 2 output ($n_i \to 1$):

```
((-s)^(-4 - 3 eps - z6 - z7) (-t)^z6 (-u)^
  z7 Gamma[-eps]^2 Gamma[-z1] Gamma[-1 - eps - z1 - z2]
  Gamma[1 + z2] Gamma[-1 - eps - z3] Gamma[-z3]
  Gamma[1 + z1 + z3]  Gamma[2 + eps + z1 + z2 + z3]
  Gamma[-2 - 2 eps - z3 - z4] Gamma[-z4] Gamma[-z5]
  Gamma[3 + eps + z1 + z2 + z3 + z4 + z5]
  Gamma[-3 - 3 eps - z3 - z4 - z5 - z6] Gamma[-z6]
  Gamma[1 + z5 + z6] Gamma[-3 - 3 eps - z3 - z4 - z5 - z7]
  Gamma[-3 - 3 eps - z1 - z3 - z5 - z6 - z7] Gamma[-z7]
  Gamma[1 + z5 + z7] Gamma[-z2 + z6 + z7]
  Gamma[4 + 3 eps + z3 + z4 + z5 + z6 + z7])/
(Gamma[-2 eps]^2 Gamma[1 - z1] Gamma[-2 - 4 eps - z3]
  Gamma[3 + eps + z1 + z2 + z3] Gamma[-2 - 3 eps - z3 - z4]^2)
```

Some non-planar MB box diagrams

Massless				Massive			
1-loop	2-loop	3-loop	4-loop	1-loop	2-loop	3-loop	4-loop
1	6	19	?	3	10	?	?
1	6	19	?	2	8	?	?

AMBRE, LA: Dimensions of some $2 \to 2$ non–planar topologies before and after applying Barnes' first Lemma.

Massless				Massive			
1-loop	2-loop	3-loop	4-loop	1-loop	2-loop	3-loop	4-loop
1	4	11	?	3	14	?	?
1	4	11	?	2	14	?	?

AMBRE, GA: Dimensions of some $2 \to 2$ non–planar topologies before and after applying Barnes' first Lemma.

General structure of the MB integrals

$$\frac{1}{(2\pi i)^n} \int_{-i\infty}^{i\infty} \dots \int_{-i\infty}^{i\infty} \prod_i^n dz_i \; \mathbf{F}(Z, S, \vec{n}, \epsilon) \frac{\prod_j \mathbf{G_1}(Z_j, N_j, \epsilon)}{\prod_k \mathbf{G_2}(Z_k, N_k, \epsilon)}$$

F depends on: Z – linear combinations of n complex variables z_i,
S – kinematic parameters and masses;
$\vec{n} \in \{n_1, \dots, n_N\}$ – powers of the N propagators;

$\mathbf{G_1}, \mathbf{G_2}$: Gamma and PolyGamma functions
N_i : linear combinations of z_i, n_i (and ϵ)

In practice F is a product of powers of S, with exponents being linear combinations of z_i, n_i:

$$\mathbf{F} \sim \prod_k X_k^{\prod_{i,j} (\alpha z_i + \beta n_j)}$$

$$\alpha, \beta \in \mathbf{R}, \quad e.g. \quad \mathbf{X} = \{\tfrac{s}{t}, \tfrac{m^2}{s}, \dots\}.$$

Changing MB integrals into sums: general remarks

`Under development: package MBsums`

- Takes from AMBRE the `MBInt[]` (regulated, expanded in ϵ) and transforms it to some multiple series,
  by closing the contours and calculating residues
- Depending on closing contours to the left or right, and on the order of integrations, convergent infinite series suitable for further processing can be obtained

Simple one-dimensional example:

$$I = \int_{-i\infty-1/2}^{i\infty-1/2} dz_1 (-1/s)^{z_1} \frac{\Gamma[-z_1]^3 \Gamma[1+z_1]}{\Gamma[-2z_1]}$$

might look convergent for $s = -3$ when closing z_1 contour to the left and taking residues, but in fact the Gamma functions change the sum into a chain of alternating and increasing numbers (it converges above the threshold, $|s| > 4$)

Some details for the massive QED one-loop vertex function

The integral comes from the massive QED vertex. It was discussed in detail e.g. at ACAT2007 (Gluza, Haas, Kajda, Riemann, [31]).

$$\frac{1}{2(-s)2\pi i} I = \texttt{V312m}[-1, \texttt{y}] = \frac{1}{2}\frac{1}{2\pi i}\int_{-i\infty+u}^{+i\infty+u} dz(-t)^{-1-z}\,\frac{\Gamma^3[-z]\Gamma[1+z]}{\Gamma[-2z]}$$

The resulting representation is a inverse binomial sum:

$$\texttt{V312m}[-1, \texttt{y}] = \frac{1}{2}\sum_{n=0}^{\infty}\frac{s^n}{\binom{2n}{n}(2n+1)},$$

It can be done using Mathematica's built-in summation:

$$\texttt{V312m}[-1, \texttt{y}] = \frac{1}{2}\frac{4\arcsin(\sqrt{s/2})}{\sqrt{4-s}\sqrt{s}} = \frac{1}{2}\frac{-2y}{1-y^2}\ln y.$$

with

$$y = \frac{\sqrt{-s+4}-\sqrt{-s}}{\sqrt{-s+4}+\sqrt{-s}}.$$

The more complicated sums appearing for that vertex may be obtained from sums listed in Table 1 of Appendix D in [32].

Package MBsums - Changing MB integrals into sums

Two dimensional massive example:

```
int = MBint[((-s/m^2)^(z1 - z2) Gamma[1 - z1] Gamma[-z1]
      Gamma[z1] Gamma[1 + z1] Gamma[1 + z1 - z2]^2
      Gamma[-z2] Gamma[-z1 + z2])/Gamma[2 + z1 - 2 z2],
      {{eps -> 0}, {z1 -> -(13/32), z2 -> -(5/32)}}]
```

Input:

```
      MBIntToSum[int,kinematics,options]
e.g.  MBIntToSum[int,{s->-1, m->10},{z2 -> L, z1 -> L}]
```

Output:

```
{(B^n2 n2! (n1 + n2)! (HarmonicNumber[n1 + n2] -
     HarmonicNumber[2 + n1 + 2 n2]))/(2 + n1 + 2 n2)!,
     n1 >= 0 && n2 >= 0}
```

to be summed over `n1, n2`.
Another condition, e.g.: `n1 >= 0 && n2 >= 0 && n1 > n2`.

Example

$$\sum_{n_2=0}^{\infty}\sum_{n_1=0}^{\infty}\frac{n_2!(n_1+n_2)!}{(n_1+2n_2+2)!}\left(S_1(n_1+n_2)-S_1(n_1+2n_2+2)\right)S^{n_2}$$

1. Compute inner sum using `Sigma`

$$\sum_{n_2=0}^{\infty}\frac{S_1(n_2)-S_1(2n_2+1)}{(n_2+1)(2n_2+1)\binom{2n_2}{n_2}}S^{n_2}$$

2. Rewrite in terms of iterated integrals using rewrite rules [1]
3. Bring to a canonical form [1]
4. Remove square-roots by $S=-\frac{(1-y)^2}{y}$

$$\frac{y}{(1-y)^2}\left(4\zeta_3+2\zeta_2\ln(y)+\frac{\ln(y)^3}{6}+2\ln(y)\mathrm{Li}_2(y)-4\mathrm{Li}_3(y)\right)$$

Licence

We made some bad experiences in recent years concerning fair quotations of source-open software.

See e.g.:
http://zfitter.com (ZFITTER collab. 2011)
http://zfitter.education (ZFITTER collab. 2014)
A review is [33].

See also:
http://fh.desy.de/projekte/gfitter01/Gfitter01.htm (DESY General Director 2013)
http://zfitter-gfitter.desy.de (DESY Research Director 2014)

The generally accepted rules for using software in high energy physics were discussed at the round table discussion at ACAT 2013, see the summary:
F. Carminati, D. Perret-Gallix, T. Riemann:
"Summary of the ACAT Round Table Discussion: Open-source, knowledge sharing and scientific collaboration"
hep-ph/1407.0540, DESY-14-079, J. Phys. Conf. Ser. 523 (2014) 012066 [34]

For these reasons, we distribute only Mathematica **executables on request** of AMBRE v.3, as it is already common for software from the RISC/LINZ group, like the SIGMA package.

Conclusions

1. Basic non-planar version of AMBRE is ready
2. MB integrals are exploited towards general analytic solutions
3. Changing MB integrals into infinite sums package is yet under development
4. Work is undertaken on rewriting multiple infinite sums for massive MB integrals into iterated integrals over suitably alphabet and by applying rewrite rules (cooperation with RISC, Linz group)

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