General formulation of the sector-improved residue subtraction

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Abstract. The main theoretical tool to provide precise predictions for scattering cross sections of strongly interacting particles is perturbative QCD. Starting at next-to-leading order (NLO) the calculation suffers from unphysical IR-divergences that cancel in the final result. At NLO there exist general subtraction algorithms to treat these divergences during a calculation. Since the LHC demands for more precise theoretical predictions, general subtraction methods at next-to-next-to-leading order (NNLO) are needed.
These proceedings outline the four-dimensional formulation of the sector improved residue subtraction. The subtraction scheme STRIPPER and in particular its extension to arbitrary multiplicities is explained. Therefore, it furnishes a general framework for the calculation of NNLO cross sections in perturbative QCD.

1. Introduction

We are interested in predicting the hadronic cross section, which is known to factorize into parton distribution functions and the partonic cross section

$$\sigma_{h_1h_2}(P_1,P_2) = \sum_{ab} \iint_0^1 \mathrm{d}x_1\mathrm{d}x_2\, f_{a/h_1}(x_1,\mu_\mathrm{F})\, f_{b/h_2}(x_2,\mu_\mathrm{F})\, \hat{\sigma}_{ab}(x_1P_1, x_2P_2;\, \alpha_s(\mu_\mathrm{R}),\, \mu_\mathrm{R},\, \mu_\mathrm{F})\,. \tag{1}$$

The summation runs over initial state partons $\{a,b\}$, i.e. massless quarks and gluons. The parton distribution function $f_{a/h_1}(x_1,\mu_\mathrm{F})$ can be understood as the probability density for finding parton a inside hadron h_1 carrying the momentum $p_1 = x_1P_1$. Parton distribution functions are non-perturbative objects and have to be determined experimentally.
In contrast, the partonic cross section $\hat{\sigma}_{ab}$ can be calculated using perturbative QCD. Including terms up to next-to-next-to-leading order, its expansion in the strong coupling α_s reads

$$\hat{\sigma}_{ab} = \hat{\sigma}_{ab}^{(0)} + \hat{\sigma}_{ab}^{(1)} + \hat{\sigma}_{ab}^{(2)}\,. \tag{2}$$

The leading order contribution is known as the Born approximation and reads

$$\hat{\sigma}_{ab}^{(0)} = \hat{\sigma}_{ab}^{\mathrm{B}} = \frac{1}{2\hat{s}}\frac{1}{N_{ab}} \int \mathrm{d}\boldsymbol{\Phi}_n\, \langle \mathcal{M}_n^{(0)} | \mathcal{M}_n^{(0)} \rangle\, \mathrm{F}_n\,, \tag{3}$$

where n is the number of final state particles and $\mathrm{d}\boldsymbol{\Phi}_n$ the phase space measure. The measurement functions F_n defines the infrared safe observable and prevents n massless partons

from becoming soft or collinear. The i-loop matrix element is denoted by $|\mathcal{M}_n^{(i)}\rangle$. For details of the notation we refer to [1].
Beyond leading order, we decompose the cross section according to the number of particles in the final state. At next-to-leading order (NLO) we have

$$\hat{\sigma}_{ab}^{(1)} = \hat{\sigma}_{ab}^{\mathrm{R}} + \hat{\sigma}_{ab}^{\mathrm{V}} + \hat{\sigma}_{ab}^{\mathrm{C}} \,, \tag{4}$$

with

$$\hat{\sigma}_{ab}^{\mathrm{R}} = \frac{1}{2\hat{s}} \frac{1}{N_{ab}} \int \mathrm{d}\mathbf{\Phi}_{n+1} \, \langle \mathcal{M}_{n+1}^{(0)} | \mathcal{M}_{n+1}^{(0)} \rangle \, \mathrm{F}_{n+1} \,, \quad \hat{\sigma}_{ab}^{\mathrm{V}} = \frac{1}{2\hat{s}} \frac{1}{N_{ab}} \int \mathrm{d}\mathbf{\Phi}_{n} \, 2\mathrm{Re} \, \langle \mathcal{M}_{n}^{(0)} | \mathcal{M}_{n}^{(1)} \rangle \, \mathrm{F}_{n} \,. \tag{5}$$

Starting at this order, separate parts suffer from soft and collinear (infrared) divergences [1]. They appear as poles in the regulator ϵ after setting the space-time dimension to $d = 4 - 2\epsilon$. We distinguish between explicit virtual poles, that emerge in the one-loop matrix element $|\mathcal{M}_n^{(1)}\rangle$ of the virtual contribution $\hat{\sigma}_{ab}^{\mathrm{V}}$, and real poles, that appear after integrating the phase space of the additional parton of the real contribution $\hat{\sigma}_{ab}^{\mathrm{R}}$. All poles cancel in the sum (4).
At next-to-next-to leading order (NNLO) we get

$$\hat{\sigma}_{ab}^{(2)} = \hat{\sigma}_{ab}^{\mathrm{RR}} + \hat{\sigma}_{ab}^{\mathrm{RV}} + \hat{\sigma}_{ab}^{\mathrm{VV}} + \hat{\sigma}_{ab}^{\mathrm{C1}} + \hat{\sigma}_{ab}^{\mathrm{C2}} \,, \tag{6}$$

where

$$\begin{aligned}
\hat{\sigma}_{ab}^{\mathrm{RR}} &= \frac{1}{2\hat{s}} \frac{1}{N_{ab}} \int \mathrm{d}\mathbf{\Phi}_{n+2} \, \langle \mathcal{M}_{n+2}^{(0)} | \mathcal{M}_{n+2}^{(0)} \rangle \, \mathrm{F}_{n+2} \,, \\
\hat{\sigma}_{ab}^{\mathrm{RV}} &= \frac{1}{2\hat{s}} \frac{1}{N_{ab}} \int \mathrm{d}\mathbf{\Phi}_{n+1} \, 2\mathrm{Re} \, \langle \mathcal{M}_{n+1}^{(0)} | \mathcal{M}_{n+1}^{(1)} \rangle \, \mathrm{F}_{n+1} \,, \\
\hat{\sigma}_{ab}^{\mathrm{VV}} &= \frac{1}{2\hat{s}} \frac{1}{N_{ab}} \int \mathrm{d}\mathbf{\Phi}_{n} \left(2\mathrm{Re} \, \langle \mathcal{M}_{n}^{(0)} | \mathcal{M}_{n}^{(2)} \rangle + \langle \mathcal{M}_{n}^{(1)} | \mathcal{M}_{n}^{(1)} \rangle \right) \mathrm{F}_{n} \,.
\end{aligned} \tag{7}$$

The double-real contribution $\hat{\sigma}_{ab}^{\mathrm{RR}}$ contains two additional massless partons in the final state that can become unresolved and lead to poles in ϵ after the phase space integration is performed. The real-virtual contribution $\hat{\sigma}_{ab}^{\mathrm{RV}}$ consists of the one-loop amplitude integrated over the $n+1$ particle phase space. In addition to virtual poles of the one-loop matrix element, it develops real poles by integrating the phase space of one unresolved particle. The double-virtual contribution $\hat{\sigma}_{ab}^{\mathrm{VV}}$ contains only explicit virtual poles in the two-loop amplitude and the squared one-loop amplitude. The sum of all contributions in (6) is finite.
In general it is not possible to perform phase space integrations analytically. Furthermore, to be able to compare a predicition with experimental data the phase space integration should be implemented in a flexible Monte-Carlo software to adapt the observable to the experimental setup easily.
Subtraction methods at NLO have been established to handle infrared singularities before numerical integrations are performed. Catani-Seymour subtraction [2] and FKS subtraction [3] are commonly used schemes.
At NNLO, subtraction schemes become more involved. Antenna subtraction [4] and q_T - subtraction [5] are the most advanced proposals and have already been applied to $e^+e^- \to 3\,\text{jets}$ [6], $pp \to 2\,\text{gluonic jets}$ [7, 8], Higgs production [5] and vector boson pair production [9, 10] and other non-trivial examples [11, 12].
In these schemes poles cancel analytically. We illustrate this by taking the real and virtual

[1] For simplicity initial state collinear counterterms $\hat{\sigma}_{ab}^{\mathrm{C}}$ at NLO and $\hat{\sigma}_{ab}^{\mathrm{C1}}$, $\hat{\sigma}_{ab}^{\mathrm{C2}}$ at NNLO are not mentioned in the further discussion. For details see [1].

contribution of a NLO cross section in (4). The real-radiation cross section is made integrable in four dimensions by a suitable subtraction term that mimics the behaviour of the squared matrix element in the singular limits. Adding the subtracted term back and integrating it analytically over the unresolved one-particle phase space provides poles that cancel the poles of the virtual contribution. Finally, both phase space integrals are numerically integrable in four dimensions. At NNLO the procedure is similar: Subtraction terms for real unresolved particles are introduced to render the phase space integrable. Analytically integrated subtraction terms cancel the explicit poles of virtual contributions.

Here we present STRIPPER (SecToR ImProved PhasE space for Real radiation), a NNLO subtraction scheme that is completely numerical and avoids cumbersome analytic integrations. The scheme was introduced in [13] and generalized to arbitrary final states in [1]. It has been first applied to top-quark pair production [14], and subsequently to other processes of low multiplicity: Z decay [15], including some conceptual improvements, Higgs + jet [16], charmless bottom quark decay [17], top quark decay [18], single top quark production [19] and muon decay [20].

The scheme was initially formulated using conventional dimensional regularization (CDR), where momenta and spin degrees of freedom of resolved and unresolved particles are treated in $d = 4-2\epsilon$ dimensions. Unresolved particles are either virtual particles of loop contributions or real-radiated particles that can become soft or collinear. In contrast to analytic subtraction schemes, momenta of resolved particles are explicitly parameterized in $d > 4$ dimensions. The explicit dimension increases as the multiplicity of the final state rises, e.g. for top-quark pair production already five dimensions have been parameterized explicitly. It turns out that STRIPPER in CDR is not applicable for high multiplicities. In addition, tree-level Matrix elements, that appear in subtraction terms, have to be provided to several powers in ϵ. Available software only provides them up to ϵ^0.

Thus, it has been necessary to reformulate the scheme in 't Hooft-Veltman regularization (HV), where momenta and spin degrees of freedom of resolved particles are four-dimensional.

In these proceedings, we explain the general idea of STRIPPER in order to obtain a Laurent series in ϵ for $\hat{\sigma}_{ab}^{\mathrm{RR}}$, where each coefficient can be calculated numerically. Afterwards, we shortly point out how it can be reformulated in HV to provide a self-contained subtraction scheme for NNLO calculations. The detailed description is to be found in [1].

2. STRIPPER

The subtraction scheme STRIPPER is an algorithmic method to extract real singularities of different contributions in (6). Each part will be given as a Laurent series in ϵ, where each coefficient has been calculated numerically. The final result, after summing the different parts, is finite.

We outline the method for $\hat{\sigma}_{ab}^{\mathrm{RR}}$ as given in (7). Since it contains two additional, potentially unresolved, partons, the phase space integral has the most complicated infrared structure.

First, we split the phase space into double-collinear and triple-collinear sectors. In a triple-collinear sector singularities are generated as three specific partons become collinear to each other and/or two of them soft. In a double-collinear sector singularities emerge as two specific pairs of partons become collinear and/or two of them soft.

In a next step, we parameterize the collinear particles in each sector separately using energies and angles. We illustrate the parametrization on the basis of a triple-collinear sector, where the three particles are in the final state: The reference momentum indicating the triple-collinear direction is denoted by r^μ. The momenta of the unresolved partons are named u_1^μ and u_2^μ. Each momentum is parameterized by its energy and a $(d-1)$-dimensional unit vector in spherical coordinates

$$r^\mu \equiv r^0 \begin{pmatrix} 1 \\ \hat{\boldsymbol{q}}_{\mathbf{1}} \end{pmatrix} , \quad u_1^\mu \equiv u_1^0 \begin{pmatrix} 1 \\ \hat{\boldsymbol{u}}_{\mathbf{1}} \end{pmatrix} , \quad u_2^\mu \equiv u_2^0 \begin{pmatrix} 1 \\ \hat{\boldsymbol{u}}_{\mathbf{2}} \end{pmatrix} . \tag{8}$$

The unresolved particles' energies are rescaled by their maximal value E, $u_i^0 = \hat{\xi}_i E$, for $i = 1, 2$. The soft limit is approached as $\hat{\xi}_1 \to 0$ and/or $\hat{\xi}_2 \to 0$. Using rotations in $(d-1)$-dimensions, the scalar products between the three given momenta take the following form

$$\hat{\boldsymbol{u}}_{\boldsymbol{1}} \cdot \hat{\boldsymbol{r}} = \cos\theta_1 = 1 - 2\hat{\eta}_1 \ , \quad \hat{\boldsymbol{u}}_{\boldsymbol{2}} \cdot \hat{\boldsymbol{r}} = \cos\theta_2 = 1 - 2\hat{\eta}_2 \ , \quad \hat{\boldsymbol{u}}_{\boldsymbol{1}} \cdot \hat{\boldsymbol{u}}_{\boldsymbol{2}} = \cos\theta_1 \cos\theta_2 + \cos\phi_2 \sin\theta_1 \sin\theta_2 \ . \tag{9}$$

This parametrization is shown in figure 1. The limit of $\hat{\eta}_i$ at zero indicates the collinear limit

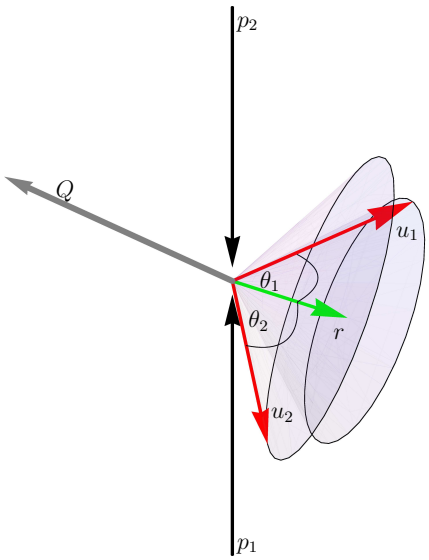

Figure 1. Parametrization of a triple-collinear sector, where the three partons are in the final state. The reference momentum is r, the unresolved momenta are denoted by u_1 and u_2. The sum of momenta of remaining resolved particles is Q.

of one of the unresolved partons and the reference parton. $\hat{\boldsymbol{u}}_{\boldsymbol{1}}$ is collinear to $\hat{\boldsymbol{u}}_{\boldsymbol{2}}$ when $\hat{\eta}_1 = \hat{\eta}_2$ and $\phi_2 = 0$. A non-linear transformation of ϕ_2 to ζ,

$$\phi_2 \to \phi_2(\hat{\eta}_1, \hat{\eta}_2, \zeta) \ , \tag{10}$$

then ensures that ϕ_2 always vanishes as $\hat{\eta}_1 = \hat{\eta}_2$. Accordingly, all possible collinear limits are indicated only by two variables: $\hat{\eta}_1$ and $\hat{\eta}_2$. Furthermore, if we find a formulation in HV, meaning that all other momenta and the reference momentum are four-dimensional, at most six dimensions are needed to parameterize all possible scalar products consistently.
At this point, four physical variables $\{\hat{\eta}_1, \hat{\eta}_2, \hat{\xi}_1, \hat{\xi}_2\}$ parameterize all possible soft and collinear limits in a given sector. Additional sector decompositions [21] in those variables factorize all possible overlapping singularities that appear at NNLO. In practice this amounts to split each sector again. For example double-soft overlapping singularities are disentangled using

$$1 = \theta\left(\hat{\xi}_1 - \hat{\xi}_2\right) + \theta\left(\hat{\xi}_2 - \hat{\xi}_1\right) . \tag{11}$$

The decomposition of the phase space due to soft overlapping singularities is sufficient to factorize all possible limits in a double-collinear sector. In a triple-collinear sector the phase space is split into five additional sectors to factorize collinear and soft-collinear overlapping singularities. This splitting is accompanied by a transition from physical variables $\{\hat{\eta}_1, \hat{\eta}_2, \hat{\xi}_1, \hat{\xi}_2\}$ to corresponding sector variables $\{\eta_1, \eta_2, \xi_1, \xi_2\}$.
Finally, the full double-real radiation cross section can be written as a sum over different decomposed triple- and double-collinear sectors

$$\hat{\sigma}_{ab}^{\mathrm{RR}} = \sum_S \hat{\sigma}_{ab}^{\mathrm{RR},S} \ . \tag{12}$$

Each contribution has the following form

$$\hat{\sigma}_{ab}^{\mathrm{RR},S} = \int_0^1 \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\eta_1 \mathrm{d}\eta_2 \frac{F_S\left(\xi_1, \xi_2, \eta_1, \eta_2\right)}{\xi_1^{1-b_1\epsilon} \xi_2^{1-b_2\epsilon} \eta_1^{1-b_3\epsilon} \eta_2^{1-b_4\epsilon}}. \tag{13}$$

The function $F_S\left(\xi_1, \xi_2, \eta_1, \eta_2\right)$ is finite in the limit of vanishing arguments. All appearing singularities are factorized in the sector variables. Poles are extracted by an iterative usage of the plus distribution in each variable

$$\int_0^1 \mathrm{d}x \, \frac{f\left(x\right)}{x^{1-b\epsilon}} = \frac{f\left(0\right)}{b\epsilon} + \int_0^1 \mathrm{d}x \, \frac{f\left(x\right) - f\left(0\right)}{x^{1-b\epsilon}} \, . \tag{14}$$

We obtain a Laurent series in ϵ and all coefficients are calculated numerically. The described procedure is process independent, since it is possible to use the known universal infrared-limits of QCD amplitudes for the subtraction terms in (14).

3. Formulation in the 't Hooft-Veltman regularization scheme

The above procedure is carried out in CDR, i.e. resolved and unresolved momenta are d-dimensional. To get a transition to the 't Hooft-Veltman regularization scheme, we have to identify unresolved particles in different contributions of (6): Either one particle is unresolved, what we call the single-unresolved contribution, or two particles are unresolved, what we call the double-unresolved contribution. By adding process independent correction terms to the single-unresolved contribution and subtracting them from the double-unresolved contribution we make sure that the two contributions are finite separately. These corrections are process independent, since they depend just on the form of the splitting and soft functions and the integration region of unresolved momenta, that do not enter the matrix elements. Now we are able to identify unresolved and resolved particles unambiguously. Since each contribution is finite we set the momenta of resolved particles to $d = 4$ dimensions, the mistake is of order ϵ. This makes sure that explicit parameterized dimensions are limited to six for arbitrary multiplicities. Setting also their spin degrees of freedom to the physical dimensions allows us to use matrix elements at order ϵ^0.

To avoid contractions between d-dimensional vectors of unresolved momenta and four-dimensional matrix elements, we replace spin-correlated splitting functions in collinear subtraction terms by their azimuthal averaged counterparts.

This procedure leads to a fully general subtraction algorithm for NNLO calculations.

4. Summary and conclusion

In these proceedings, we presented the general formulation of STRIPPER, a subtraction scheme to calculate cross sections at NNLO in perturbative QCD for arbitrary multiplicities. The full subtraction scheme, completely described in [1], can readily be implemented in a computer code.

Acknowledgments

This research was supported by the German Research Foundation (DFG) via the Sonderforschungsbereich/Transregio SFB/TR-9 "Computational Particle Physics". The author thanks the organizers of ACAT 2014 for remission of the conference fee.

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