On calculations within the nonlinear sigma model

Karol Kampf

Institute of Particle and Nuclear Physics, Charles University,
V Holešovičkách 2, Prague, Czech Republic.

E-mail: `karol.kampf@mff.cuni.cz`

Abstract.
   The chiral $SU(N)$ nonlinear sigma model represents one of the simplest case of effective field theory. For the last several decades it has played an extremely important role not only for the low energy phenomenology but also in many other areas of theoretical physics. In this talk we will focus on the tree-level scattering amplitudes of the n-Goldstone bosons. It will be shown that it can be reconstructed using the BCFW-like recursion relations. This method, which does not rely on the Lagrangian description, is much more efficient than the standard Feynman diagram techniques.

1. Introduction

It is well known that if we want to study the properties of quantum chromodynamics (QCD) at low energies we cannot directly use this theory as for example in the case of quantum electrodynamics. Main reason is that the degrees of freedom of QCD, quarks and leptons are not the ones we measure in experiments. One of the possible way how to tackle this problem is an effective field theory approach. This is historically connected with the name current algebra and non-linear sigma model developed later into the systematic theory called nowadays Chiral perturbation theory (ChPT) [1]. This theory describes the behaviour of the lightest pseudoscalar mesons (i.e. pions, kaons and eta) at low energies. The natural expansion parameters are thus momenta and masses of these particles and the standard choice is the same importance of them. A rough estimate of the theoretical error leads to the value of 30% for each step in the chiral series. In order to be competitive with an experimental error of the order of ten percent we are forced to provide the given calculation up to next-to-next-to-leading order (NNLO). This is true not only for the even-intrinsic-parity sector but in some cases also for the odd anomalous sector which arises due to the chiral anomaly (see e.g.[2]). In both sectors we are thus facing not only the complicated inevitable two-loop calculations but also an increase of the number of unknown low-energy constants (LECs). Their values can be estimated using the influence of resonances from the so-called intermediate energy region (for the systematic studies at NLO for both sectors see [3]). The typical calculation within ChPT thus includes complicated (usually two loop order) calculations and the part devoted to the estimation of used parameters. In this work we will further discuss two different strategies for study of the non-linear sigma model properties. Both of them will rely only on the leading order Lagrangian of ChPT and we will thus avoid the phenomenological problem connected with the estimation of LECs.

2. Leading logarithms

The first way is connected with the concept of (non)renormalizability in the effective field theories. Details can be found in the [4], here we will merely summarized main points.

The leading logs (LL) are the logarithms with highest power at the given order, i.e. LL^1 at the one-loop order, LL^2 at the two-loop order and so on. As already mentioned they are parameter-free and can be calculated, in principle, to all orders from the one-loop diagrams only. In [4] the procedure of the one-loop calculations was automatized and high orders of LL were presented. The actual automatization of the calculation was performed within cshell using C++ and the most important part, the algebraic treatment was done in FORM [5].

It is clear that main motivation for LLs is theoretical as it can hint to some deeper understanding of calculation within effective field theories. The eventual resummation of LL then leads to the same effect as is the concept of running coupling constant for the renormalizable theories. However, their application can be also phenomenological. As already stated, they are not proportional to some unknown parameters. We can define LL using the lowest-order parameters of the Lagrangian as

$$L = \frac{M^2}{16\pi^2 F^2} \log \frac{\mu^2}{M^2} \,,$$

(1)

or using the physical mass and decay constant

$$L_\pi = \frac{M_\pi^2}{16\pi^2 F_\pi^2} \log \frac{\mu^2}{M_\pi^2} \,.$$

(2)

The general notation of the expansion is as follows

$$O_{\text{phys}} = O_0(1 + \sum_i a_i L^i), \qquad O_{\text{phys}} = O_0(1 + \sum_i c_i L_\pi^i) \,.$$

(3)

The high orders of LL of several quantities relevant for ChPT were already calculated: expansion of mass and decay constant, vacuum expectation value, vector and scalar formfactor, $\pi\pi$ scattering and also $\gamma\gamma \to \pi\pi$ process. Concerning the anomalous processes, LL for the quantities connected with $\pi\gamma \to \pi\pi$ and $\pi^0 \to \gamma\gamma$ were calculated up to the 6th and 7th loop respectively. The similar program can be extended in different areas of particle physics, as done recently for the nucleon mass [6]. Let us also stress that though the calculation is automatic now, due to the time and computer limitations, it is still impossible to extend it to very high orders (see discussion in [7]).

3. Tree-level amplitudes

We have started the introduction with the phenomenologically relevant calculations within the non-linear sigma model. This includes one and two-loop diagram calculations. Further extension, in principle up to all orders, was performed using the renormalization group properties. Our next object of interest will go in the other direction. Instead of increasing number of loops we will increase number of legs. To simplify the problem we will study only tree-level diagrams and we stay in the massless limit. Let us first briefly summarize the method used in the case of the renomalizable theory which we will generalize to our model. The toy model will be the pure gluonic sector of QCD.

3.1. Motivation: gluon amplitudes and BCFW reconstruction

The standard method to calculate the n-gluon amplitudes relies on the Feynman diagrams. We have to combine two possible elementary vertices (in the case of the studied model) and draw

all possible diagrams. The number of diagrams grows exponentially with the number of external legs. What is interesting, however, after long and tedious calculation some results known to be extremely simple. In the so-called helicity formalism we have for all tree-level maximally helicity violating gluonic amplitudes close formula [8]

$$A_n(- - + \ldots +) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle}, \qquad \text{with} \qquad \langle ij \rangle \sim \sqrt{|2p_i \cdot p_j|}. \qquad (4)$$

The standard method via Feynman diagrams thus leads to the big cancellations among different diagrams and is thus not very well suited for this kind of calculations. We will briefly recapitulate a new alternative method which relies on the BCFW (Britto, Cachazo, Feng and Witten) recursion relations. It reconstructs the on-shell tree-level amplitudes from its poles using the simple analytic properties [9]. First step is the possibility to define a colour-ordered stripped amplitude

$$M^{a_1 \ldots a_n}(p_1, \ldots p_n) = \sum_{\sigma/Z_n} \text{Tr}(t^{a_{\sigma(1)}} \ldots t^{a_{\sigma(n)}}) M_\sigma(p_1, \ldots, p_n). \qquad (5)$$

The existence of ordering is important for the following discussion of the pole structure. The only possible poles are:

$$P_{ij}^2 = (p_i + p_{i+1} + \ldots + p_{j-1} + p_j)^2. \qquad (6)$$

The one particle unitarity is telling us that if we go with one of the above poles to zero the amplitude will factorize into two parts (left/right), schematically:

$$\lim_{P_{1j}^2 \to 0} M(1, 2, \ldots n) = \sum_{h_l} M_L(1, 2 \ldots j, l) \cdot \frac{i}{P_{1j}^2} \cdot M_R(l, j+1, \ldots n). \qquad (7)$$

This can be trivially verified using e.g. the Feynman diagram representation.

In order to reconstruct the amplitude from its poles in complex plane we will make use of the following continuation. We will shift *two* (usually adjacent) external momenta

$$p_i \to p_i + zq, \qquad p_j \to p_j - zq, \qquad (8)$$

so that the new p_i and p_j remain on-shell. The amplitude thus becomes a meromorphic function $A(z)$ of the complex parameter z. Taking $z = 0$ we should get the original function. It is important to notice that we will have only simple poles coming from propagators $P_{ab}(z)$. Employing the Cauchy's theorem one gets

$$\frac{1}{2\pi i} \int \frac{dz}{z} A(z) = A(0) + \sum_k \frac{\text{Res}\,(A, z_k)}{z_k}. \qquad (9)$$

Assuming that $A(z)$ vanishes for $z \to \infty$ we get for the original amplitude

$$A = A(0) = -\sum_k \frac{\text{Res}\,(A, z_k)}{z_k}. \qquad (10)$$

The propagator $P_{ab}^2(z) = 0$ if one and only one of the two shifted momenta is in $(a, a+1, \ldots, b)$. Assuming the solution is z_{ab} we get the factorization into two subamplitudes

$$\text{Res}(A, z_{ab}) = \sum_s A_L^{-s}(z_{ab}) \frac{i}{2(q \cdot P_{ab})} A_R^s(z_{ab}), \qquad (11)$$

(for the allowed helicities). Using Cauchy's formula, we have finally

$$A = \sum_{k,s} A_L^{-s_k}(z_k) \frac{i}{P_k^2} A_R^{s_k}(z_k). \qquad (12)$$

3.2. BCFW-like reconstruction of non-linear sigma model

Let us turn back to the non-linear sigma model. Such a model corresponds in general to the spontaneous symmetry breaking of the chiral group $G_L \times G_R$ where $G_{L,R} = G$ to its diagonal subgroup $G_V = G$. In the following we will focus only on the group $G = SU(N)$ (which is the case of ChPT). The dynamics of the Goldstone bosons is governed by the Lagrangian (note we are in the massless limit)

$$\mathcal{L} = \frac{F^2}{4} \langle \partial_\mu U \partial^\mu U^{-1} \rangle \,, \tag{13}$$

where $\langle \ldots \rangle$ stands for the trace in the flavour space. In order we can use the analyticity properties for the amplitude reconstruction also in the case of the non-linear sigma model one important generalization is needed. The Cauchy theorem expressed by (10) was based on an assumption that the amplitude vanishes for $z \to \infty$. However, in a general case $A(z) \approx z^k$. Then $k+1$ subtractions in a_i points are needed:

$$A(0) = -\sum_i \frac{\mathrm{Res}(A, z_i)}{z_i} \prod_{j=1}^{k+1} \frac{a_j}{a_j - z_i} + \sum_{j=1}^{k+1} A(a_j) \prod_{l=1, l \neq j}^{k+1} \frac{a_l}{a_l - a_j} \,. \tag{14}$$

We will present here a reconstruction of the semi-on-shell amplitude M_n,

$$M_n(p_1, p_2, \ldots, p_{n+1}) = p_{n+1}^2 J_n(1, \ldots, n) \,, \tag{15}$$

where the semi-off-shell J_n is the amplitude with the $(n+1)$th particle off-shell. The main reason why we have focus on the off-shell amplitude lays in the fact that the unphysical objects have different behaviour in different parametrization. For different way how to proceed see also [10]. The complex deformation which we propose is the all even-line shift, i.e.

$$p_i \to p_i(z): \qquad p_{2i+1}(z) = p_{2i+1} \quad \text{and} \quad p_{2i}(z) = z p_{2i} \,. \tag{16}$$

The shifted amplitude (15) will be denoted as

$$M_n(z) \equiv M_{2n+1}(p_1, z p_2, p_3, \ldots z p_{2n}, p_{2n+1}) \,. \tag{17}$$

The advantage of working with the off-shell amplitude is that in a specific parametrization we can show that $M_n(z) = O(z^0)$ (as $z \to \infty$). Once subtracted reconstruction formula (14) must be thus employed

$$M_n(z) = M_n(0) + \sum_P \frac{\mathrm{Res}(M_n, z_P)}{z - z_P} \frac{z}{z_P} \,. \tag{18}$$

The amplitude for the non-linear sigma model can be reconstructed similarly to the BCFW reconstruction of gluon amplitudes. The most important practical advantage consists in the reduction of the number of diagrams. It can be shown that the number of poles, i.e. terms in (18) is at most $\sim n^2$

The recursive relations represent an equivalent description of the nonlinear $SU(N)$ sigma model to the standard Lagrangian formulation. Apart from its computational efficiency we can use it also in the investigation of the further properties. One such an application is the proof of the double Adler zero of the soft limit as conjectured in [11]. For details and formal proof see [12].

4. Summary and outlook

We have presented three different directions used within the non-linear sigma model. First, relevant for the phenomenology relies on a systematic calculation of the studied quantity perturbatively up to a given order. This order is dictated by the theoretical and experimental details of the problem. In practice it usually includes two-loop diagrams and they represent the most difficult part of the calculation. The question if even the NNLO order is enough is partly studied in the second direction: calculation of the leading logarithms. We were capable to automatized their calculations for different objects and showed some results up the seven order. This might be useful in resummation of the series and can thus lead to the similar concept as is the running coupling constant known for the renormalizable theories. In the third direction we stress the importance of the tree-level diagrams. We have introduced the BCFW-like reconstruction for the $SU(N)$ non-linear sigma model i.e. for model with infinite number of interaction vertices. New techniques developed primary for renormalizable models can be thus used for a broader class of theories than was expected before. One gains much faster way how to calculate amplitudes and new view on these quantities. The latter is useful in obtaining better understanding of new properties and was demonstrated on the proofs of the so-called Adler zeros. The BCFW-like reconstruction was limited so far to the tree-level. It should be possible to extend it up to one-loop level [13] or used in completely different theories (as are for example galileons, cf. [14]).

References

[1] S. Weinberg, Physica A **96** (1979) 327; J. Gasser and H. Leutwyler, Annals Phys. **158** (1984) 142; J. Gasser and H. Leutwyler, Nucl. Phys. B **250** (1985) 465.

[2] K. Kampf and B. Moussallam, Phys. Rev. D **79** (2009) 076005 [arXiv:0901.4688 [hep-ph]]; J. Bijnens and K. Kampf, Nucl. Phys. Proc. Suppl. **207-208** (2010) 220 [arXiv:1009.5493 [hep-ph]].

[3] G. Ecker, J. Gasser, A. Pich and E. de Rafael, Nucl. Phys. B **321** (1989) 311; K. Kampf and J. Novotny, Phys. Rev. D **84** (2011) 014036 [arXiv:1104.3137 [hep-ph]].

[4] J. Bijnens and L. Carloni, Nucl. Phys. B **843** (2011) 55 [arXiv:1008.3499 [hep-ph]]. J. Bijnens, K. Kampf and S. Lanz, Nucl. Phys. B **860** (2012) 245 [arXiv:1201.2608 [hep-ph]]. J. Bijnens, K. Kampf and S. Lanz, Nucl. Phys. B **873** (2013) 137 [arXiv:1303.3125 [hep-ph]].

[5] J. A. M. Vermaseren, math-ph/0010025.

[6] J. Bijnens and A. A. Vladimirov, arXiv:1409.6127 [hep-ph].

[7] K. Kampf, PoS CD **12** (2013) 039 [arXiv:1302.7311 [hep-ph]].

[8] S. J. Parke and T. R. Taylor, Phys. Rev. Lett. **56** (1986) 2459.

[9] R. Britto, F. Cachazo and B. Feng, Nucl. Phys. B **715** (2005) 499 [hep-th/0412308]; R. Britto, F. Cachazo, B. Feng and E. Witten, Phys. Rev. Lett. **94** (2005) 181602 [hep-th/0501052].

[10] K. Kampf, J. Novotny and J. Trnka, Phys. Rev. D **87** (2013) 081701 [arXiv:1212.5224 [hep-th]].

[11] N. Arkani-Hamed, F. Cachazo and J. Kaplan, JHEP **1009** (2010) 016 [arXiv:0808.1446].

[12] K. Kampf, J. Novotny and J. Trnka, JHEP **1305** (2013) 032 [arXiv:1304.3048 [hep-th]].

[13] K. Kampf, J. Novotny and J. Trnka, *in preparation*

[14] K. Kampf and J. Novotny, JHEP **1410** (2014) 006 [arXiv:1403.6813 [hep-th]].