On finite discrete subgroups of $SU(3)$ and lepton mixing parameters

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Outline

- introduction: lepton mixing
- mixing from flavor symmetry G_f
- overview over $SU(3)$ subgroups
- analysis of patterns from $G_f = \Sigma(n\varphi)$
- comments on patterns from groups of type (C) and (D)
- conclusions

Parametrization of lepton mixing

Parametrization ***(PDG)***

$$U_{PMNS} = \tilde{U}\ \mathrm{diag}(1, e^{i\alpha/2}, e^{i(\beta/2+\delta)})$$

with

$$\tilde{U} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

and $s_{ij} = \sin\theta_{ij}$, $c_{ij} = \cos\theta_{ij}$

Jarlskog invariant J_{CP}

$$\begin{aligned} J_{CP} &= \mathrm{Im}\left[U_{PMNS,11}U^*_{PMNS,13}U^*_{PMNS,31}U_{PMNS,33}\right] \\ &= \frac{1}{8}\sin 2\theta_{12}\sin 2\theta_{23}\sin 2\theta_{13}\cos\theta_{13}\sin\delta \end{aligned}$$

Experimental results on lepton mixing

Latest global fits NH [IH] ***(Capozzi et al. ('13))***

best fit and $1\,\sigma$ error	$3\,\sigma$ range
$\sin^2\theta_{13} = 0.0234[9]^{+0.0022[1]}_{-0.0018[21]}$	$0.0177[8] \leq \sin^2\theta_{13} \leq 0.0297[300]$
$\sin^2\theta_{12} = 0.308^{+0.017}_{-0.017}$	$0.259 \leq \sin^2\theta_{12} \leq 0.359$
$\sin^2\theta_{23} = \begin{cases} 0.425[37]^{+0.029[59]}_{-0.027[9]} \\ [0.531 \leq \sin^2\theta_{23} \leq 0.610] \end{cases}$	$0.357[63] \leq \sin^2\theta_{23} \leq 0.641[59]$
$\delta = 1.39[5]\,\pi^{+0.33[24]\,\pi}_{-0.27[39]\,\pi}$	$0 \leq \delta \leq 2\,\pi$
$\alpha\,,\ \ \beta$	unconstrained

Experimental results on lepton mixing

Latest global fits NH [IH] ***(Capozzi et al. ('13))***

$$||U_{PMNS}|| \approx \begin{pmatrix} 0.82 & 0.55 & 0.15 \\ 0.40[39] & 0.65 & 0.64[5] \\ 0.40[2] & 0.52 & 0.75[4] \end{pmatrix}$$

and no information on Majorana phases

$$\Downarrow$$

Mismatch in lepton flavor space is large!

Origin of lepton mixing

- interpret this mismatch in lepton flavor space as mismatch of residual symmetries G_ν and G_e
- if we want to predict lepton mixing, we have to derive this mismatch
- let us assume that there is a symmetry, broken to G_ν and G_e
- this symmetry is in the following a

finite, discrete, non-abelian symmetry G_f

[Masses do not play a role in this approach.]

Non-trivial breaking of G_f

Idea:

Derivation of the lepton mixing from how G_f is broken
Interpretation as mismatch of embedding of different subgroups G_ν and G_e into G_f

$$G_f$$

$\swarrow$ $\searrow$

neutrinos | charged leptons
G_ν | G_e

Non-trivial breaking of G_f

Idea:

Derivation of the lepton mixing from how G_f is broken
Interpretation as mismatch of embedding of different subgroups G_ν and G_e into G_f

$$G_f$$

$$\swarrow \qquad \searrow$$

neutrinos

$G_\nu = Z_2 \times Z_2$ (Majorana)

$G_\nu = Z_M$ with $M \geq 3$ (Dirac)

charged leptons

$G_e = Z_N$ with $N \geq 3$

Non-trivial breaking of G_f

$$G_f$$

$$\swarrow \qquad \searrow$$

neutrinos

$G_\nu = Z_2 \times Z_2$ (Majorana)

$G_\nu = Z_M$ with $M \geq 3$ (Dirac)

charged leptons

$G_e = Z_N$ with $N \geq 3$

Further requirements

- two/three non-trivial angles $\Rightarrow$ irred. 3-dim. rep. of G_f
- fix angles through G_ν, G_e $\Rightarrow$ 3 families transform diff. under G_ν, G_e

Non-trivial breaking of G_f

- neutrino sector: $Z_2 \times Z_2$ or Z_M, $M \geq 3$, preserved and generated by

$$\Omega_\nu^\dagger \, Z_i \, \Omega_\nu = Z_i^{diag} \; , \quad i = 1, 2$$

or

$$\Omega_\nu^\dagger \, Z \, \Omega_\nu = Z^{diag}$$

- charged lepton sector: Z_N, $N \geq 3$, preserved and generated by

$$\Omega_e^\dagger \, Q_e \, \Omega_e = Q_e^{diag}$$

Non-trivial breaking of G_f

- neutrino sector: $Z_2 \times Z_2$ or Z_M, $M \geq 3$, preserved

$$Z_i^T \, m_\nu \, Z_i = m_\nu \; , \quad i = 1, 2$$

or

$$Z^\dagger \, m_\nu^\dagger m_\nu \, Z = m_\nu^\dagger m_\nu$$

- charged lepton sector: Z_N, $N \geq 3$, preserved

$$Q_e^\dagger \, m_e^\dagger m_e \, Q_e = m_e^\dagger m_e$$

Non-trivial breaking of G_f

- neutrino sector: $Z_2 \times Z_2$ or Z_M, $M \geq 3$, preserved

  $\rightarrow$ neutrino mass matrix m_ν fulfills

  $\Omega_\nu^T \, m_\nu \, \Omega_\nu$ is diagonal

  or $\Omega_\nu^\dagger \, m_\nu^\dagger m_\nu \, \Omega_\nu$ is diagonal

- charged lepton sector: Z_N, $N \geq 3$, preserved

  $\rightarrow$ charged lepton mass matrix m_e fulfills

  $\Omega_e^\dagger \, m_e^\dagger m_e \, \Omega_e$ is diagonal

Non-trivial breaking of G_f

$$U_{PMNS} = \Omega_e^\dagger \Omega_\nu$$

- 3 unphysical phases are removed by $\Omega_e \to \Omega_e K_e$
- neutrino masses are made real and positive through $\Omega_\nu \to \Omega_\nu K_\nu$
- permutations of columns of Ω_e, Ω_ν are possible: $\Omega_{e,\nu} \to \Omega_{e,\nu} P_{e,\nu}$

$$\Downarrow$$

Predictions:

Mixing angles up to exchange of rows/columns

J_{CP} up to sign

Majorana phases undetermined

Discrete subgroups of $SU(3)$

- discrete subgroups of $SU(3)$ can be divided in five categories

  (Miller et al. ('16), Fairbairn et al. ('64), Grimus/Ludl ('11))

- two of them not interesting, since they have no irred. 3-dim. reps.
- groups of type (C) are of the form $(Z_m \times Z_n) \rtimes Z_3$
  these groups can be viewed as "generalizations" of $\Delta(3n^2)$
- groups of type (D) are of the form $(Z_m \times Z_n) \rtimes S_3$
  these groups can be viewed as "generalizations" of $\Delta(6n^2)$
- "exceptional" groups
  $\Sigma(60)(\times Z_3)$, $\Sigma(168)(\times Z_3)$ and $\Sigma(n\varphi)$ with $\varphi = 3$

Groups $\Sigma(n\varphi)$

- groups with $\varphi = 1$: $\Sigma(36)$, $\Sigma(72)$, $\Sigma(216)$ and $\Sigma(360)$;
  these are subgroups of $SU(3)/C$ (C: center)
- groups with $\varphi = 3$: $\Sigma(36 \times 3)$, $\Sigma(72 \times 3)$, $\Sigma(216 \times 3)$ and $\Sigma(360 \times 3)$;
  these are subgroups of $SU(3)$
- the groups $\Sigma(60) \simeq A_5$ and $\Sigma(168) \simeq PSL(2, Z_7)$;
  they are subgroups of $SU(3)/C$ and $SU(3)$

Groups $\Sigma(n\varphi)$

- groups with $\varphi = 1$: $\Sigma(36)$, $\Sigma(72)$, $\Sigma(216)$ and $\Sigma(360)$;
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- groups with $\varphi = 3$: $\Sigma(36 \times 3)$, $\Sigma(72 \times 3)$, $\Sigma(216 \times 3)$ and $\Sigma(360 \times 3)$;
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- the groups $\Sigma(60) \simeq A_5$ and $\Sigma(168) \simeq PSL(2, Z_7)$;
  they are subgroups of $SU(3)/C$ and $SU(3)$

Groups $\Sigma(72 \times 3)$, $\Sigma(216 \times 3)$, $\Sigma(360 \times 3)$ and $\Sigma(168)$ might be interesting, because a version of minimal flavor violation in quark sector could be realized

(Zwicky, Fischbacher ('09))

$\boldsymbol{\Sigma(36 \times 3)}$

- group with 108 elements
- it has four pairs of $\mathbf{3^{(p)}}$ and $(\mathbf{3^{(p)}})^\star$, $\mathrm{p} = 0, 1, 2, 3$
- generators a, v and z ***(Grimus/Ludl ('10))***

$$a^3 = 1\ , \quad v^4 = 1\ , \quad z^3 = 1\ , \quad a v^{-1} z v = 1\ , \quad a v z^{-1} v^{-1} = 1\ , \quad (a z)^3 = 1$$

- generators in $\mathbf{3^{(0)}}$ $(\omega = e^{2\pi i/3})$

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \ , \quad v = \frac{1}{\sqrt{3} i} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \ , \quad z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

$\Sigma(36 \times 3)$

- group with 108 elements
- it has four pairs of $\mathbf{3}^{(\mathbf{p})}$ and $(\mathbf{3}^{(\mathbf{p})})^\star$, $\mathrm{p} = 0, 1, 2, 3$
- generators a, v and z *(Grimus/Ludl ('10))*

$$a^3 = 1 \; , \;\; v^4 = 1 \; , \;\; z^3 = 1 \; , \;\; a v^{-1} z v = 1 \; , \;\; a v z^{-1} v^{-1} = 1 \; , \;\; (a z)^3 = 1$$

- generators in $\mathbf{3}^{(\mathbf{p})}$

$$a \; , \;\; i^{\mathrm{p}} v \; , \;\; z \quad \text{with} \quad \mathrm{p} = 0, 1, 2, 3$$

- elements in $\mathbf{3}^{(\mathbf{p})}$ can be written as

$$g = \omega^o z^\zeta a^\alpha v^\chi \quad \text{with} \quad o, \, \zeta, \, \alpha = 0, 1, 2 \; , \chi = 0, 1, 2, 3$$

$\Sigma(36 \times 3)$

- abelian subgroups

  Z_2, Z_3, Z_4, Z_6, Z_{12} and $Z_3 \times Z_3$

- especially no Klein subgroup $Z_2 \times Z_2$
  $\Rightarrow$ neutrinos have to be Dirac particles

- 30 elements g are represented by matrices with degenerate eigenvalues

- analysis of all combinations G_e and G_ν shows
  - combination $G_e = Z_3$ and $G_\nu = Z_3$ not interesting
  - one pattern for $G_e = Z_3$ and $G_\nu = Z_4$ or $G_\nu = Z_{12}$
  - one pattern for $G_e = Z_4$ or $G_e = Z_{12}$ and $G_\nu = Z_4$ or $G_\nu = Z_{12}$

$\boldsymbol{\Sigma(36 \times 3)}$

- pattern for $G_e = Z_3$ and $G_\nu = Z_4$ or $G_\nu = Z_{12}$

$$||U_{PMNS}|| = \frac{1}{\sqrt{2(3+\sqrt{3})}} \begin{pmatrix} 1+\sqrt{3} & \sqrt{2} & 0 \\ 1 & \sqrt{2+\sqrt{3}} & \sqrt{3+\sqrt{3}} \\ 1 & \sqrt{2+\sqrt{3}} & \sqrt{3+\sqrt{3}} \end{pmatrix}$$

$\mathbf{\Sigma(36 \times 3)}$

- pattern for $G_e = Z_3$ and $G_\nu = Z_4$ or $G_\nu = Z_{12}$

$$||U_{PMNS}|| \approx \begin{pmatrix} 0.888 & 0.460 & 0 \\ 0.325 & 0.628 & 0.707 \\ 0.325 & 0.628 & 0.707 \end{pmatrix}$$

$\Rightarrow \sin^2\theta_{12} \approx 0.211$ and θ_{23} maximal and $\theta_{13} = 0$

- comments
  - fit to data not so good, $\chi^2 \approx 151.5$
  - has been mentioned in analysis of $\Delta(27)$ and CP ***(Nishi ('13))***

$\mathbf{\Sigma(36 \times 3)}$

- pattern for $G_e = Z_4$ or $G_e = Z_{12}$ and $G_\nu = Z_4$ or $G_\nu = Z_{12}$

$$||U_{PMNS}|| = \frac{1}{2\sqrt{2}} \begin{pmatrix} \sqrt{3+\sqrt{3}} & \sqrt{3} & \sqrt{2-\sqrt{3}} \\ \sqrt{2} & \sqrt{3-\sqrt{3}} & \sqrt{3+\sqrt{3}} \\ \sqrt{3-\sqrt{3}} & \sqrt{2+\sqrt{3}} & \sqrt{3} \end{pmatrix}$$

$\Sigma(36 \times 3)$

- pattern for $G_e = Z_4$ or $G_e = Z_{12}$ and $G_\nu = Z_4$ or $G_\nu = Z_{12}$

$$||U_{PMNS}|| \approx \begin{pmatrix} 0.769 & 0.612 & 0.183 \\ 0.500 & 0.398 & 0.769 \\ 0.398 & 0.683 & 0.612 \end{pmatrix}$$

$\Rightarrow \sin^2\theta_{12} \approx 0.388$, $\sin^2\theta_{23} \approx 0.612$ and $\sin^2\theta_{13} \approx 0.033$

- comments
  - better fit to data, $\chi^2 \approx 69.1$
  - $J_{CP} = 0$ although $\theta_{13} \neq 0$
  - second solution with $\sin^2\theta_{23} \approx 0.388$

$\mathbf{\Sigma(72 \times 3)}$

- group with 216 elements
- it has four pairs of $\mathbf{3}^{(\mathbf{p_1},\mathbf{p_2})}$ and $(\mathbf{3}^{(\mathbf{p_1},\mathbf{p_2})})^\star$, $\mathrm{p}_1, \mathrm{p}_2 = 0, 1$
- generators a, v, z and x ***(Grimus/Ludl ('10))***
- abelian subgroups

  Z_2, Z_3, Z_4, Z_6, Z_{12} and $Z_3 \times Z_3$
- 30 elements g reveal degenerate eigenvalues
- analysis of all combinations G_e and G_ν shows
  - only one pattern related to this group
  - this pattern arises from G_e, G_ν being Z_4 or Z_{12}

$\boldsymbol{\Sigma(72 \times 3)}$

- pattern for $G_e = Z_4$ or $G_e = Z_{12}$ and $G_\nu = Z_4$ or $G_\nu = Z_{12}$

$$||U_{PMNS}|| = \frac{1}{2\sqrt{2}} \begin{pmatrix} \sqrt{3+\sqrt{3}} & \sqrt{4-\sqrt{3}} & 1 \\ \sqrt{2} & \sqrt{3+\sqrt{3}} & \sqrt{3-\sqrt{3}} \\ \sqrt{3-\sqrt{3}} & 1 & \sqrt{4+\sqrt{3}} \end{pmatrix}$$

$\Sigma(72 \times 3)$

- pattern for $G_e = Z_4$ or $G_e = Z_{12}$ and $G_\nu = Z_4$ or $G_\nu = Z_{12}$

$$||U_{PMNS}|| \approx \begin{pmatrix} 0.769 & 0.532 & 0.354 \\ 0.500 & 0.769 & 0.398 \\ 0.398 & 0.354 & 0.846 \end{pmatrix}$$

$\Rightarrow \sin^2\theta_{12} \approx 0.324$, $\sin^2\theta_{23} \approx 0.181$ and $\sin^2\theta_{13} = 0.125$

- comments
  - bad fit to data, $\chi^2 > 1000$
  - but interesting: $J_{CP} \neq 0$, $|J_{CP}| = \sqrt{3}/32$

$\mathbf{\Sigma(216 \times 3)}$

- group with 648 elements
- it has three pairs of $\mathbf{3}^{(\mathbf{p})}$ and $(\mathbf{3}^{(\mathbf{p})})^\star$, $\mathrm{p} = 0, 1, 2$ which are faithful
- generators a, v, z and w ***(Grimus/Ludl ('10))***
- abelian subgroups

  Z_2, Z_3, Z_4, Z_6, Z_9, Z_{12}, Z_{18} and $Z_3 \times Z_3$, $Z_3 \times Z_9$
- 102 elements g reveal degenerate eigenvalues

$\boldsymbol{\Sigma(216 \times 3)}$

- analysis of all combinations G_e and G_ν shows
  - only one pattern which fits quite well, $\chi^2 \approx 28.3$
  - patterns with larger χ^2 have $J_{CP} \neq 0$
  - several patterns with $\theta_{12,23}$ OK, but θ_{13} too large
  - pattern with $\chi^2 \approx 10.6$ leads to group with 162 elements

$\mathbf{\Sigma(216 \times 3)}$

- pattern with smallest χ^2: $G_e = Z_{18}$ and $G_\nu = Z_4$ or $G_\nu = Z_{12}$

$$||U_{PMNS}|| = \frac{1}{2\sqrt{2(3+\sqrt{3})}} \begin{pmatrix} \sqrt{2}(2+\sqrt{3}) & \sqrt{2(3+\sqrt{3})} & -1+\sqrt{3} \\ \sqrt{6} & \sqrt{2(3+\sqrt{3})} & 3+\sqrt{3} \\ 2 & 2\sqrt{3+\sqrt{3}} & 2\sqrt{2+\sqrt{3}} \end{pmatrix}$$

$\boldsymbol{\Sigma(216 \times 3)}$

- pattern with smallest χ^2: $G_e = Z_{18}$ and $G_\nu = Z_4$ or $G_\nu = Z_{12}$

$$||U_{PMNS}|| \approx \begin{pmatrix} 0.858 & 0.500 & 0.119 \\ 0.398 & 0.500 & 0.769 \\ 0.325 & 0.707 & 0.628 \end{pmatrix}$$

$\Rightarrow \sin^2\theta_{12} \approx 0.254$, $\sin^2\theta_{23} = 0.600$ and $\sin^2\theta_{13} \approx 0.014$

- comments
  - good fit, $\chi^2 \approx 28.3$, only $\theta_{12,13}$ a bit too small
  - no CP violation, $J_{CP} = 0$
  - second solution with $\sin^2\theta_{23} = 0.400$

$\Sigma(216 \times 3)$

- pattern with $J_{CP} \neq 0$ and good fit for $\theta_{12,23}$: $G_e = Z_4$ or $G_e = Z_{12}$ and $G_\nu = Z_3$

$$||U_{PMNS}|| \approx \begin{pmatrix} 0.804 & 0.525 & 0.279 \\ 0.483 & 0.445 & 0.754 \\ 0.346 & 0.726 & 0.595 \end{pmatrix}$$

$\Rightarrow \sin^2\theta_{12} \approx 0.299$, $\sin^2\theta_{23} \approx 0.616$ and $\sin^2\theta_{13} \approx 0.078$

- comments
  - bad fit of θ_{13}, total χ^2: $\chi^2 \approx 554.6$
  - $|J_{CP}| \approx 0.0417$
  - second solution with $\sin^2\theta_{23} \approx 0.384$

$\boldsymbol{\Sigma(216 \times 3)}$

- pattern with $J_{CP} \neq 0$ and good fit for $\theta_{12,23}$: $G_e = Z_{18}$ and $G_\nu = Z_{18}$

$$||U_{PMNS}|| = \frac{1}{2\sqrt{3}} \begin{pmatrix} \sqrt{7} & \sqrt{3} & \sqrt{2} \\ \sqrt{3} & \sqrt{3} & \sqrt{6} \\ \sqrt{2} & \sqrt{6} & 2 \end{pmatrix}$$

$\mathbf{\Sigma(216 \times 3)}$

- pattern with $J_{CP} \neq 0$ and good fit for $\theta_{12,23}$: $G_e = Z_{18}$ and $G_\nu = Z_{18}$

$$||U_{PMNS}|| \approx \begin{pmatrix} 0.764 & 0.500 & 0.408 \\ 0.500 & 0.500 & 0.707 \\ 0.408 & 0.707 & 0.577 \end{pmatrix}$$

$\Rightarrow \sin^2\theta_{12} = 0.300$, $\sin^2\theta_{23} = 0.600$ and $\sin^2\theta_{13} \approx 0.167$

- comments
  - bad fit of θ_{13}, total $\chi^2 > 3000$
  - $|J_{CP}| \approx 0.0722$
  - second solution with $\sin^2\theta_{23} = 0.400$

$\boldsymbol{\Sigma(216 \times 3)}$

- pattern with $\chi^2 \approx 10.6$, but group $\subset \Sigma(216 \times 3)$: $G_e = Z_3$ & $G_\nu = Z_{18}$

$$||U_{PMNS}|| = \frac{1}{\sqrt{6}} \begin{pmatrix} 2c_{18} & \sqrt{2} & 2s_{18} \\ c_{18} - \sqrt{3}s_{18} & \sqrt{2} & \sqrt{3}c_{18} + s_{18} \\ c_{18} + \sqrt{3}s_{18} & \sqrt{2} & \sqrt{3}c_{18} - s_{18} \end{pmatrix} = ||U_{TB}\, R_{13}(-\pi/18)||$$

and $c_{18} = \cos \pi/18 \approx 0.985$, $s_{18} = \sin \pi/18 \approx 0.174$

$\Sigma(216 \times 3)$

- pattern with $\chi^2 \approx 10.6$, but group $\subset \Sigma(216 \times 3)$: $G_e = Z_3$ & $G_\nu = Z_{18}$

$$||U_{PMNS}|| \approx \begin{pmatrix} 0.804 & 0.577 & 0.142 \\ 0.279 & 0.577 & 0.767 \\ 0.525 & 0.577 & 0.625 \end{pmatrix}$$

$\Rightarrow \sin^2\theta_{12} \approx 0.340$, $\sin^2\theta_{23} \approx 0.601$ and $\sin^2\theta_{13} \approx 0.020$

- comments
  - good fit, only θ_{12} a bit too large ... but no CP violation
  - second solution with $\sin^2\theta_{23} \approx 0.399$
  - has tri-maximal mixing

$\Sigma(216 \times 3)$

- pattern with $\chi^2 \approx 10.6$, but group $\subset \Sigma(216 \times 3)$: $G_e = Z_3$ & $G_\nu = Z_{18}$

$$||U_{PMNS}|| \approx \begin{pmatrix} 0.804 & 0.577 & 0.142 \\ 0.279 & 0.577 & 0.767 \\ 0.525 & 0.577 & 0.625 \end{pmatrix}$$

$\Rightarrow \sin^2\theta_{12} \approx 0.340$, $\sin^2\theta_{23} \approx 0.601$ and $\sin^2\theta_{13} \approx 0.020$

- comments
  - found in literature for other group ***(Holthausen et al. ('12))***;
    very recently, also for group with 162 elements ***(Holthausen/Lim ('13))***
  - structure of patterns also found for groups $\Delta(6n^2)$ ***(de Adelhart Toorop et al. ('11), King et al. ('13))***

$\mathbf{\Sigma(360 \times 3)}$

- group with 1080 elements
- it has two pairs of $\mathbf{3}^{(\mathbf{p})}$ and $(\mathbf{3}^{(\mathbf{p})})^{\star}$, $\mathrm{p} = 1, 2$ which are faithful
- generators a, f, h and q ***(Miller et al. ('16), Fairbairn et al. ('64))***
- abelian subgroups

  Z_2, Z_3, Z_4, Z_5, Z_6, Z_{12}, Z_{15} and $Z_2 \times Z_2$, $Z_3 \times Z_3$, $Z_2 \times Z_6$
- 138 elements g reveal degenerate eigenvalues

$\Sigma(360 \times 3)$

- analysis of all combinations G_e and G_ν shows
  - only one pattern with $\chi^2 < 60$
  - patterns with larger χ^2 have $J_{CP} \neq 0$
  - all patterns for G_e or G_ν being a Klein group have non-zero J_{CP}, but $\chi^2 > 100$ always

$\mathbf{\Sigma(360 \times 3)}$

- pattern with smallest χ^2: $G_e = Z_4$ or $G_e = Z_{12}$ & $G_\nu = Z_4$ or $G_\nu = Z_{12}$

$$||U_{PMNS}|| = \frac{1}{4} \begin{pmatrix} 1+\sqrt{5} & \sqrt{5-\sqrt{3}-\sqrt{5}+\sqrt{15}} & \sqrt{5+\sqrt{3}-\sqrt{5}-\sqrt{15}} \\ \sqrt{5-\sqrt{3}-\sqrt{5}+\sqrt{15}} & \sqrt{8+\sqrt{3}-\sqrt{15}} & \sqrt{3+\sqrt{5}} \\ \sqrt{5+\sqrt{3}-\sqrt{5}-\sqrt{15}} & \sqrt{3+\sqrt{5}} & \sqrt{8-\sqrt{3}+\sqrt{15}} \end{pmatrix}$$

$\Sigma(360 \times 3)$

- pattern with smallest χ^2: $G_e = Z_4$ or $G_e = Z_{12}$ & $G_\nu = Z_4$ or $G_\nu = Z_{12}$

$$||U_{PMNS}|| \approx \begin{pmatrix} 0.809 & 0.554 & 0.197 \\ 0.554 & 0.605 & 0.572 \\ 0.197 & 0.572 & 0.796 \end{pmatrix}$$

$\Rightarrow \sin^2\theta_{12} \approx 0.319$, $\sin^2\theta_{23} \approx 0.341$ and $\sin^2\theta_{13} \approx 0.039$

- comments
  - θ_{13} not accommodated well, total χ^2: $\chi^2 \approx 58.0$
  - no CP violation; pattern with $J_{CP} \neq 0$ has $\chi^2 > 100$
  - second solution with $\sin^2\theta_{23} \approx 0.659$

$\mathbf{\Sigma(360 \times 3)}$

- pattern with smallest χ^2 for $G_e = Z_2 \times Z_2$ & $G_\nu = Z_4$ or $G_\nu = Z_{12}$

$$||U_{PMNS}|| = \frac{1}{4}\begin{pmatrix} 1+\sqrt{5} & \sqrt{5+\sqrt{15}-\sqrt{2(4+\sqrt{15})}} & \sqrt{5+\sqrt{3}-\sqrt{5}(1+\sqrt{3})} \\ 2 & \sqrt{2(3+\sqrt{3})} & \sqrt{2(3-\sqrt{3})} \\ -1+\sqrt{5} & \sqrt{5-\sqrt{3}+\sqrt{5}(1-\sqrt{3})} & \sqrt{5+\sqrt{15}+\sqrt{2(4+\sqrt{15})}} \end{pmatrix}$$

$\Sigma(360 \times 3)$

- pattern with smallest χ^2 for $G_e = Z_2 \times Z_2$ & $G_\nu = Z_4$ or $G_\nu = Z_{12}$

$$||U_{PMNS}|| \approx \begin{pmatrix} 0.809 & 0.554 & 0.197 \\ 0.500 & 0.769 & 0.398 \\ 0.309 & 0.319 & 0.896 \end{pmatrix}$$

$\Rightarrow \sin^2\theta_{12} \approx 0.319$, $\sin^2\theta_{23} \approx 0.165$ and $\sin^2\theta_{13} \approx 0.039$

- comments
  - fit for $\theta_{13,23}$ not good, $\chi^2 \approx 148.1$
  - but $|J_{CP}| \approx 0.0313$
  - note $G_e = Z_2 \times Z_2 \Rightarrow$ Dirac neutrinos
  - for Majorana neutrinos, i.e. $G_\nu = Z_2 \times Z_2$, $\chi^2 > 500$

Some comments on $\Delta(6n^2)$

- series of subgroups of $SU(3)$ with faithful irred. 3-dim. reps.
- isomorphic to $(Z_n \times Z_n) \rtimes S_3$; described with four generators
- generic form of mixing patterns for Majorana neutrinos

$$U_{PMNS} = U_{TB} R_{13}(\theta)$$

and θ depends on n ***(King et al. ('13))***, i.e. mixing angles are of the form

$$\sin^2\theta_{12} = \frac{1}{2+\cos 2\theta} \;, \quad \sin^2\theta_{23} = \frac{1}{2}\left(1 - \frac{\sqrt{3}\,\sin 2\theta}{2+\cos 2\theta}\right) \quad \text{and} \quad \sin^2\theta_{13} = \frac{2}{3}\,\sin^2\theta$$

$$\delta = 0\,,\ \pi$$

Some comments on $\Delta(6n^2)$

- series of subgroups of $SU(3)$ with faithful irred. 3-dim. reps.
- isomorphic to $(Z_n \times Z_n) \rtimes S_3$; described with four generators
- generic form of mixing patterns for Majorana neutrinos

$$U_{PMNS} = U_{TB} R_{13}(\theta)$$

  and θ depends on n ***(King et al. ('13))***

- we conjectured ***(de Adelhart Toorop et al. ('11))***

$$\theta = \frac{\pi}{n} \quad \text{for} \quad G_e = Z_3 \;\; , \; G_\nu = Z_2 \times Z_2$$

$$\theta = \frac{\pi}{3n} \quad \text{for} \quad G_e = Z_3 \;\; , \; G_\nu = Z_2 \times Z_2$$

Some comments on $\boldsymbol{\Delta(6n^2)}$

- series of subgroups of $SU(3)$ with faithful irred. 3-dim. reps.
- isomorphic to $(Z_n \times Z_n) \rtimes S_3$; described with four generators
- generic form of mixing patterns for Majorana neutrinos

$$U_{PMNS} = U_{TB} R_{13}(\theta)$$

  and θ depends on n ***(King et al. ('13))***

- we conjectured ***(de Adelhart Toorop et al. ('11))*** [for Dirac neutrinos]

$$\theta = \frac{\pi}{2n} \quad \text{for} \quad G_e = Z_3 \ , \ G_\nu = Z_{2n}$$
$$\theta = \frac{\pi}{6n} \quad \text{for} \quad G_e = Z_3 \ , \ G_\nu = Z_{2n}$$

Some comments on $\Delta(6n^2)$

- this series of groups contains dihedral groups D_n and relatives
- D_n for certain n can explain Cabibbo angle well

  (Lam ('07), Blum et al. ('07,'09))
- example $n = 8$: $\Delta(384)$ ***(de Adelhart Toorop et al. ('11))***

  subgroups in up and down quark sectors: Z_{16} & $Z_2 \times Z_2$

$$||V_{CKM}|| = \begin{pmatrix} \cos\pi/16 & \sin\pi/16 & 0 \\ \sin\pi/16 & \cos\pi/16 & 0 \\ 0 & 0 & 1 \end{pmatrix} \approx \begin{pmatrix} 0.981 & 0.195 & 0 \\ 0.195 & 0.981 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

[for leptons: $G_e = Z_3$, $G_\nu = Z_2 \times Z_2$: $\theta = \pi/24$

$\sin^2\theta_{12} \approx 0.337$, $\sin^2\theta_{23} \approx 0.424$, $\sin^2\theta_{13} \approx 0.011$]

Some comments on $\boldsymbol{\Delta(3n^2)}$

- series of subgroups of $SU(3)$ with faithful irred. 3-dim. reps.
- isomorphic to $(Z_n \times Z_n) \rtimes Z_3$; described with three generators; thus mostly useful for Dirac neutrinos
- subgroups of $\Delta(6n^2)$
- generic form of mixing patterns: three entries x_1, x_2 and x_3 cyclicly permuted; matrix with two independent parameters
- usually $J_{CP} \neq 0$
- examples are $\Delta(12) \simeq A_4$ and $\Delta(27)$:
  in both cases one gets democratic mixing, i.e. $|x_i| = 1/\sqrt{3}$

Generalizations of $\boldsymbol{\Delta(3n^2)}$ and $\boldsymbol{\Delta(6n^2)}$

- generalizations of $\Delta(3n^2)$: $(Z_m \times Z_n) \rtimes Z_3$;
  also mostly for Dirac neutrinos
- generalizations of $\Delta(6n^2)$: $(Z_m \times Z_n) \rtimes S_3$;
  expectation: results very similar to those for $\Delta(6n^2)$
  examples which have already been discussed are
  - group $(Z_9 \times Z_3) \rtimes S_3$ with 162 elements, see above
  - group $(Z_{18} \times Z_6) \rtimes S_3$ ***(Holthausen et al. ('12,'13))***
    [mixing pattern like for $\Delta(6n^2)$ with $n = 9, 18$]

Conclusions

- lepton mixing is related to breaking of finite, discrete, non-abelian flavor symmetry down to different residual symmetries G_e and G_ν in charged lepton and neutrino sector
- here focus on groups $\Sigma(n\varphi)$ with $\varphi = 3$
- only very few mixing patterns found which match data well
- usually those with small χ^2 imply no CP violation, $J_{CP} = 0$
- those with larger χ^2 instead predict $J_{CP} \neq 0$

Conclusions

- comprehensive study of all patterns from $SU(3)$ subgroups $\Rightarrow$ goal: understand better how results for mixing patterns and group structure are related (why $J_{CP} = 0$?)
- to be seen whether generalizations can(not) lead to new patterns (especially for Dirac neutrinos)
- study of $U(3)$ subgroups

Thank you for your attention.

Back up

$\mathbf{\Sigma(60)}$

- group with 60 elements
- it has two real faithful irred.3-dim. reps. $\mathbf{3}^{(\mathbf{1})}$ and $\mathbf{3}^{(\mathbf{2})}$
- generators s and t ***(Eholzer ('94,'95))***
- abelian subgroups

  Z_2, Z_3, Z_5 and $Z_2 \times Z_2$

$\Sigma(60)$

- analysis of combinations G_e and $G_\nu = Z_2 \times Z_2$ shows
  (de Adelhart Toorop et al. ('11), Lam ('11))
  - only three different patterns, depending on choice of G_e
  - all have $J_{CP} = 0$
  - two lead to $\theta_{13} = 0$ and $\theta_{23} = \pi/4$
    (patterns with golden ratio mixing)

$\mathbf{\Sigma(60)}$

- pattern for $G_e = Z_2 \times Z_2$ & $G_\nu = Z_2 \times Z_2$

$$||U_{PMNS}|| = \frac{1}{2} \begin{pmatrix} \phi & 1 & 1/\phi \\ 1/\phi & \phi & 1 \\ 1 & 1/\phi & \phi \end{pmatrix}$$

with $\phi = (1 + \sqrt{5})/2$

$\Sigma(60)$

- pattern for $G_e = Z_2 \times Z_2$ & $G_\nu = Z_2 \times Z_2$

$$||U_{PMNS}|| \approx \begin{pmatrix} 0.809 & 0.500 & 0.309 \\ 0.309 & 0.809 & 0.500 \\ 0.500 & 0.309 & 0.809 \end{pmatrix}$$

$\Rightarrow \sin^2\theta_{12} \approx 0.276$, $\sin^2\theta_{23} \approx 0.276$ and $\sin^2\theta_{13} \approx 0.095$

- comments
  - fit not good, $\chi^2 \approx 1000$
  - no CP violation
  - second solution with $\sin^2\theta_{23} \approx 0.724$

$\mathbf{\Sigma(60)}$

- example of pattern for $G_e = Z_3$ and $G_\nu = Z_5$

$$||U_{PMNS}|| = \begin{pmatrix} \sqrt{\frac{1}{15}(5+\sqrt{5})} & \sqrt{\frac{1}{15}(5+\sqrt{5})} & \frac{3-\sqrt{5}}{\sqrt{6(5-\sqrt{5})}} \\ \frac{1}{2}\left(1-\sqrt{\frac{1}{15}(5-2\sqrt{5})}\right) & \frac{1}{2}\left(1+\sqrt{\frac{1}{15}(5-2\sqrt{5})}\right) & \sqrt{\frac{1}{15}(5+\sqrt{5})} \\ \frac{1}{2}\left(1+\sqrt{\frac{1}{15}(5-2\sqrt{5})}\right) & \frac{1}{2}\left(1-\sqrt{\frac{1}{15}(5-2\sqrt{5})}\right) & \sqrt{\frac{1}{15}(5+\sqrt{5})} \end{pmatrix}$$

$\mathbf{\Sigma(60)}$

- example of pattern for $G_e = Z_3$ and $G_\nu = Z_5$

$$||U_{PMNS}|| \approx \begin{pmatrix} 0.695 & 0.695 & 0.188 \\ 0.406 & 0.594 & 0.695 \\ 0.594 & 0.406 & 0.695 \end{pmatrix}$$

$\Rightarrow \sin^2\theta_{12} = 0.500$, $\sin^2\theta_{23} = 0.500$ and $\sin^2\theta_{13} \approx 0.035$

- comments
  - two maximal mixing angles
  - no CP violation

$\boldsymbol{\Sigma(168)}$

- group with 168 elements
- it has one pair of faithful complex irred. 3-dim. reps. $\mathbf{3}$ and $\mathbf{3}^\star$
- generators s and t ***(Eholzer ('94,'95))***
- relevant abelian subgroups

  Z_3, Z_4, Z_7 and $Z_2 \times Z_2$

$\mathbf{\Sigma(168)}$

- analysis of combinations G_e and $G_\nu = Z_2 \times Z_2$ shows

  (de Adelhart Toorop et al. ('11))

  - four different patterns, depending on choice of G_e
  - all patterns have $J_{CP} \neq 0$
  - PMNS mixing matrix has several equal elements (up to phase)

$\boldsymbol{\Sigma(168)}$

- pattern for $G_e = Z_2 \times Z_2$ & $G_\nu = Z_2 \times Z_2$

$$||U_{PMNS}|| = \frac{1}{2} \begin{pmatrix} \sqrt{2} & 1 & 1 \\ 1 & \sqrt{2} & 1 \\ 1 & 1 & \sqrt{2} \end{pmatrix}$$

$\Rightarrow \sin^2\theta_{12} = 1/3$, $\sin^2\theta_{23} = 1/3$, $\sin^2\theta_{13} = 1/4$ and $|J_{CP}| \approx 0.083$

- study of patterns with $G_\nu \neq Z_2 \times Z_2$ does not reveal patterns which accommodate data well