The University of Manchester

Numerical Field Integrator For SixTrack

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Aim

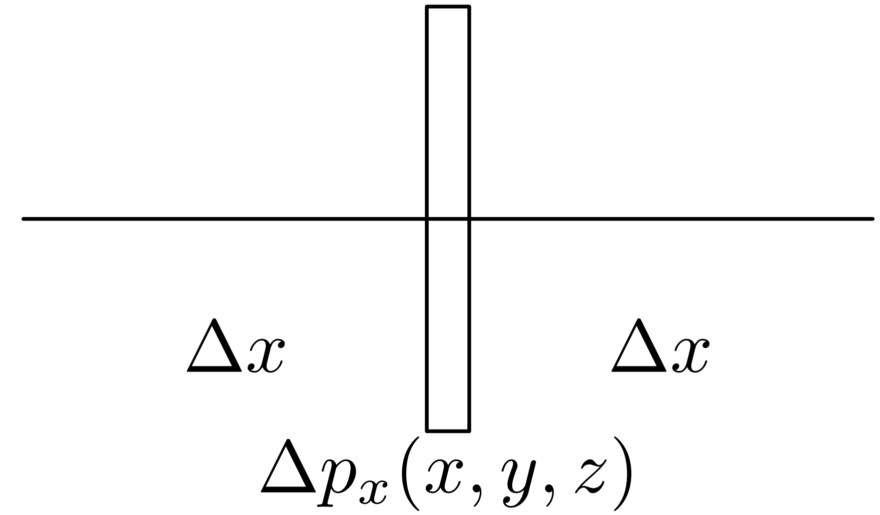
- Describe the implementation of Taylor maps and numerical field integrator in SixTrack.
- Show the properties of these tracking methods for long term tracking, for the case of the crab cavities.

Contents

Integration methods

Model method	Assumptions
RF multipole, thin kick model	•Track rigidity •Single kick at the center of the element.
Symplectic integration, Numerical integration	•Paraxial approximation
Symplectic integration, Taylor map	•Symplectic error has a small impact on tracking •Paraxial approximation

Integration methods
RF multipoles



- In order to compare the magnitude of the representation and symplectic errors of the Taylor maps, the crab cavities are represented by RF multipoles which have symplectic error at machine precision as with the numerical integrator.
- The multipoles represent the integrated kick along fixed transverse position over the crab cavity element.
- Kicks truncated to same order as field expansion (up to n=5).

$$\Delta p_x = \sum_n n \Re \left((b_n + \imath a_n)(x + \imath y)^{n-1} \right) \sin \left(\frac{2\pi f z}{c} + \phi \right)$$

Integration methods
RF multipoles

$$\Delta p_x = \sum_n n\Re\left((b_n + \imath a_n)(x+\imath y)^{n-1}\right)\sin\left(\frac{2\pi f z}{c}+\phi\right)$$

- To match the Taylor maps, Taylor expand multipole kicks.
- Match terms of expansion to get b_n values.
  - e.g $\mathrm{TaylorMap}(p_x(x^3 z)) = 4b_4\frac{2\pi f}{c}x^3 z$
- Not all terms agreed on the same values, all matched based on plane of kick.
- Sensitivity to non rigid trajectory.

Integration methods
Second order explicit integrator

$$H = -\sqrt{(1+\delta)^2 - (p_x - a_x)^2 - (p_y - a_y)^2 - \frac{1}{\gamma_0^2}} - a_s + \delta + p_s$$

$$H = -a_s - (1+\delta) + \frac{1}{2\gamma_0^2}(1+\delta)^{-1} + \delta + \frac{(p_x - a_x)^2}{2(1+\delta)} + \frac{(p_y - a_y)^2}{2(1+\delta)} + p_s$$

- Second order explicit symplectic integrator.
- Uses paraxial approximation of Hamiltonian.
- Includes all orbit/ momentum changes, thick element.

Integration methods
Second order explicit integrator

$$H_1 = \frac{1}{2\gamma^2}(1+\delta)^{-1} + p_s$$

$$H_2 = -a_s$$

$$H_3 = \frac{(p_x - a_x)^2}{2(1+\delta)}$$

$$H_4 = \frac{(p_y - a_y)^2}{2(1+\delta)}$$

- Split Hamiltonian.
- Apply Lie operations upon split Hamiltonian to gain integration scheme.

$$\begin{aligned} e^{:H:\Delta s} = {} & e^{:H_1:\frac{\Delta s}{8}} e^{:H_3:\frac{\Delta s}{4}} e^{:H_1:\frac{\Delta s}{8}} \\ & e^{:H_4:\frac{\Delta s}{2}} \\ & e^{:H_1:\frac{\Delta s}{8}} e^{:H_3:\frac{\Delta s}{4}} e^{:H_1:\frac{\Delta s}{8}} \\ & e^{:H_2:\Delta s} \\ & e^{:H_1:\frac{\Delta s}{8}} e^{:H_3:\frac{\Delta s}{4}} e^{:H_1:\frac{\Delta s}{8}} \\ & e^{:H_4:\frac{\Delta s}{2}} \\ & e^{:H_1:\frac{\Delta s}{8}} e^{:H_3:\frac{\Delta s}{4}} e^{:H_1:\frac{\Delta s}{8}} \end{aligned}$$

Integration methods
Second order explicit integrator

- For mixed operations H3 and H4

$$M_3 = e^{-\frac{\Delta s}{4}:H_3:} = e^{\left(-\frac{\Delta s}{4}:\frac{(p_x - a_x)^2}{2(1+\delta)}:\right)}$$

- By lie algebraic methods can be expanded

$$M_3 = e^{:I_x:} e^{\left(-\frac{\Delta s}{4}:\frac{p_x^2}{2(1+\delta)}:\right)} e^{-:I_x:}$$

- Where the potential is contained in the function

$$I_x = \int_0^x a_x(x', y, z, s) dx'$$

Integration methods
Second order explicit integrator

- Applying this function gives the following transforms

$$e^{:I_x}p_x = p_x - a_x(x, y, z, s)$$

$$e^{:I_x}p_y = p_y - \int_0^x \frac{\partial}{\partial y} a_x(x', y, z, s)dx'$$

$$e^{:I_x}\delta = \delta - \int_0^x \frac{\partial}{\partial z} a_x(x', y, z, s)dx'$$

Integration methods
Taylor map

- Same integration method.
- Use COSY INFINITY to integrate the same vector potential file as goes into SixTrack.
- Integrate in power series to 8th order.
- Can reduce the number of coefficients in Taylor map using weighted ranking (increases symplectic error, reduces precision).

Integration methods
Expression of fields

- RF cavities

PHYSICAL REVIEW SPECIAL TOPICS - ACCELERATORS AND BEAMS **9**, 052001 (2006)

Numerical computation of high-order transfer maps for rf cavities

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Tech-X Corporation, 5621 Arapahoe Avenue, Suite A, Boulder, Colorado 80303, USA
(Received 18 November 2005; published 9 May 2006)

Modern map-based accelerator beam-dynamics codes model magnetic elements so as to include nonlinear effects and realistic fringe fields, but they persist in modeling rf cavities as either energy kicks or linear maps. This work presents a method for including the nonlinear effects of rf cavities in a map-based code.

DOI: 10.1103/PhysRevSTAB.9.052001 PACS numbers: 41.20.Jb, 41.75.−i, 41.85.Ja

- Magnets

PHYSICAL REVIEW SPECIAL TOPICS - ACCELERATORS AND BEAMS **13**, 064001 (2010)

Accurate transfer maps for realistic beam-line elements: Straight elements

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(Received 21 December 2009; published 3 June 2010)

- Expand from the form of a series in x, y and z. e.g. Mathematica

SixTrack Implementation

- SixTrack has two sections for element implementation:
  - FOX 2nd order differential algebra 1 turn map calculation for closed orbit and normal form calculations.
  - Numerical tracking for long term tracking.
- Two different new elements:
  - Taylor maps to arbitary order
  - Numerical integration of vector potential.

SixTrack Implementation

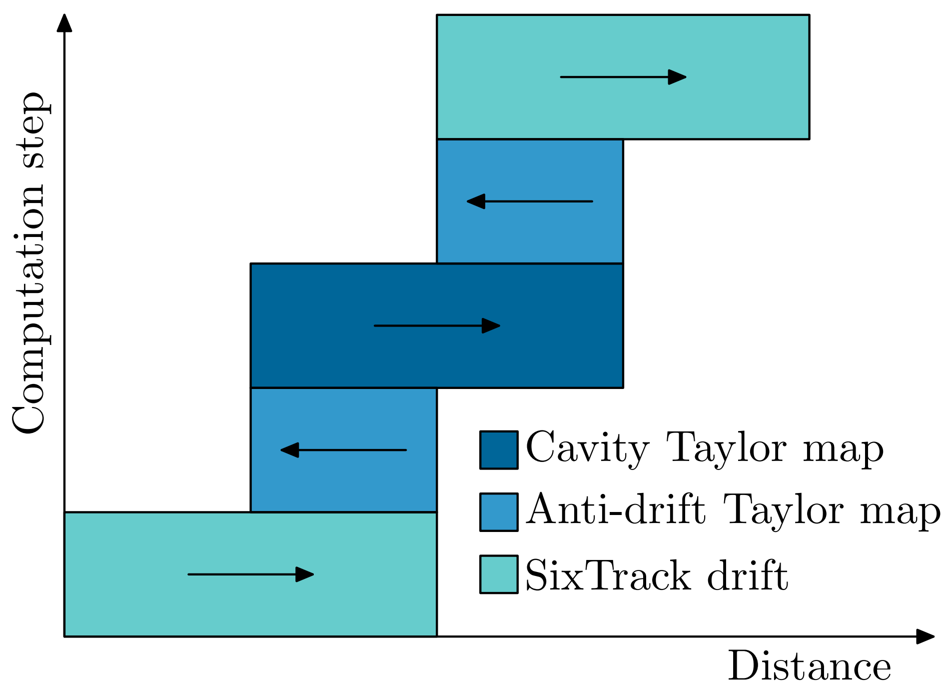
- Read in ascii data files containing vector potential or Taylor map for each element.
- At tracking loop over these arrays to perform integration.
- In order to validate the closed orbit calculation the single element Taylor maps to second order are outputted. (A similar method was used to check the numerical integration.)
- Benchmarked single elements for hardedge solenoid and quadrupole potentials with
  - COSY script
  - Analytical models

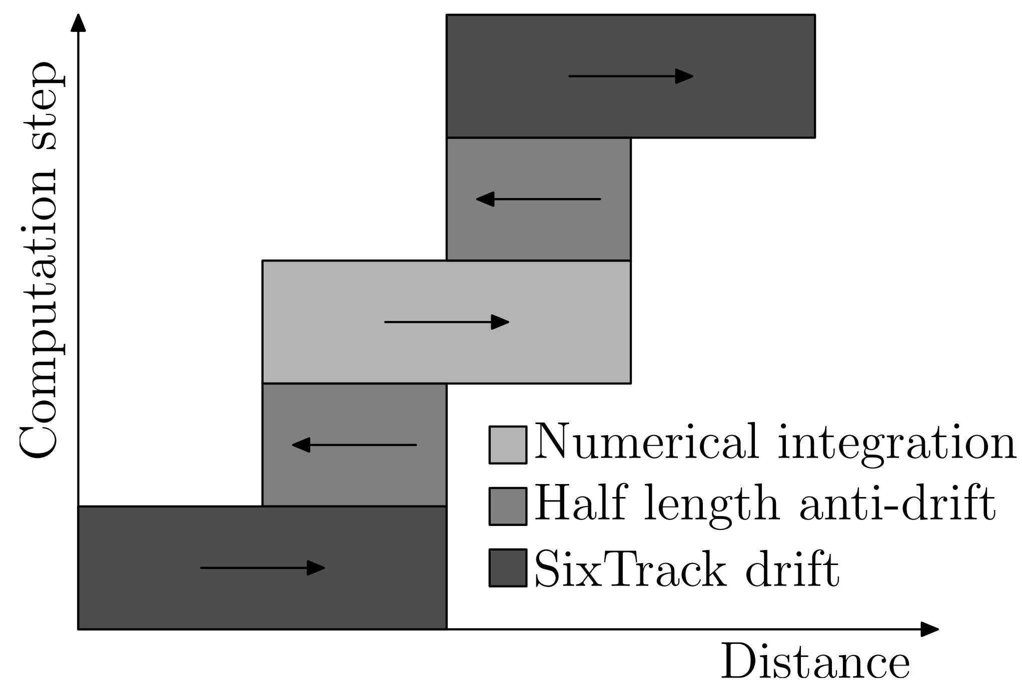
SixTrack Implementation

- Input format of vector potential.
- Vector potential is normalised in SixTrack. In future to allow for calculations during ramping.
- File size for single crab cavity ~ 10,000 lines, up to x^5, y^5, z^3. 70 steps in s.
- Currently in fort.2 the element name corresponds to a file in my public directory, and element type 30 (will change), no other arguments used.

s (m)	x exponent	y exponent	z exponent	Ax	Ay	Az
0. 0.+stepsize . . Length-stepsize Length	Integer	Integer	Integer	Double	Double	Double.

SixTrack Implementation





- In order to keep thin form tracking symplectic anti drifts are applied.
- A conversion from SixTrack coordinates to MADX coordinates occurs at the beginning, and conversion back to SixTrack form at the end of element.
- For the numerical integrator the length is calculated from the range of the vector potential.
- For the Taylor map separate anti drift Taylor maps are produced as the length cannot be found from the Taylor map.

Results
Closed orbit

	x (mm)	x' (mrad)	y (mm)	y' (mrad)	z (mm)	δ
Numerical integration	-2.359427×10^{-3}	-2.730596×10^{-5}	-1.207478×10^{-3}	-1.024538×10^{-5}	5.355347×10^{-3}	4.05421×10^{-6}
Taylor map full	-2.357384×10^{-3}	-2.748526×10^{-5}	-1.210350×10^{-3}	-1.021436×10^{-5}	5.359282×10^{-3}	4.054351×10^{-6}
Taylor map reduced	-2.357384×10^{-3}	-2.74826×10^{-5}	-1.210350×10^{-3}	-1.021436×10^{-5}	5.359283×10^{-3}	4.054351×10^{-6}
RF multipole	-1.829514×10^{-3}	-2.930196×10^{-5}	-3.531746×10^{-4}	-5.207202×10^{-6}	4.829593×10^{-3}	4.054462×10^{-6}

- Closed orbit from same vector potential doesn't agree precisely.
- Large difference between RF multipole and everything else.
- Tracking on the closed orbit with 100,000 turns with the full Taylor map shows uncertainty in closed orbit.

x	9.3E-4 %
x'	1.3E-2 %
y	4.5E-4 %
y'	6.8E-3 %
z	0.12%
δ	4.6E-4 %

Results
Closed orbit

- Error in single element map between COSY and FOX implementation of the integrator.
- These differences lead to the error in closed orbit calculation between numerical integrator and Taylor map.

Linear map COSY

$$\begin{pmatrix} 1. & 0.7 & -5.0451\times10^{-14} & -2.1262\times10^{-16} & 1.7069\times10^{-6} & 4.9596\times10^{-8} \\ 1.2848\times10^{-12} & 1. & 5.8419\times10^{-16} & 5.0861\times10^{-14} & 4.9625\times10^{-6} & 3.6024\times10^{-14} \\ -5.0451\times10^{-14} & -2.1265\times10^{-16} & 1. & 0.7 & 1.0292\times10^{-11} & 2.7207\times10^{-13} \\ 5.8419\times10^{-16} & 5.0861\times10^{-14} & -8.3870\times10^{-13} & 1. & 2.9899\times10^{-11} & 2.2293\times10^{-19} \\ 3.4868\times10^{-14} & 4.9596\times10^{-8} & 2.0422\times10^{-19} & 2.7207\times10^{-13} & 1. & 1.4285\times10^{-8} \\ 4.9625\times10^{-6} & 1.7668\times10^{-6} & 2.9899\times10^{-11} & 1.0636\times10^{-11} & 1.9095\times10^{-12} & 1. \end{pmatrix}$$

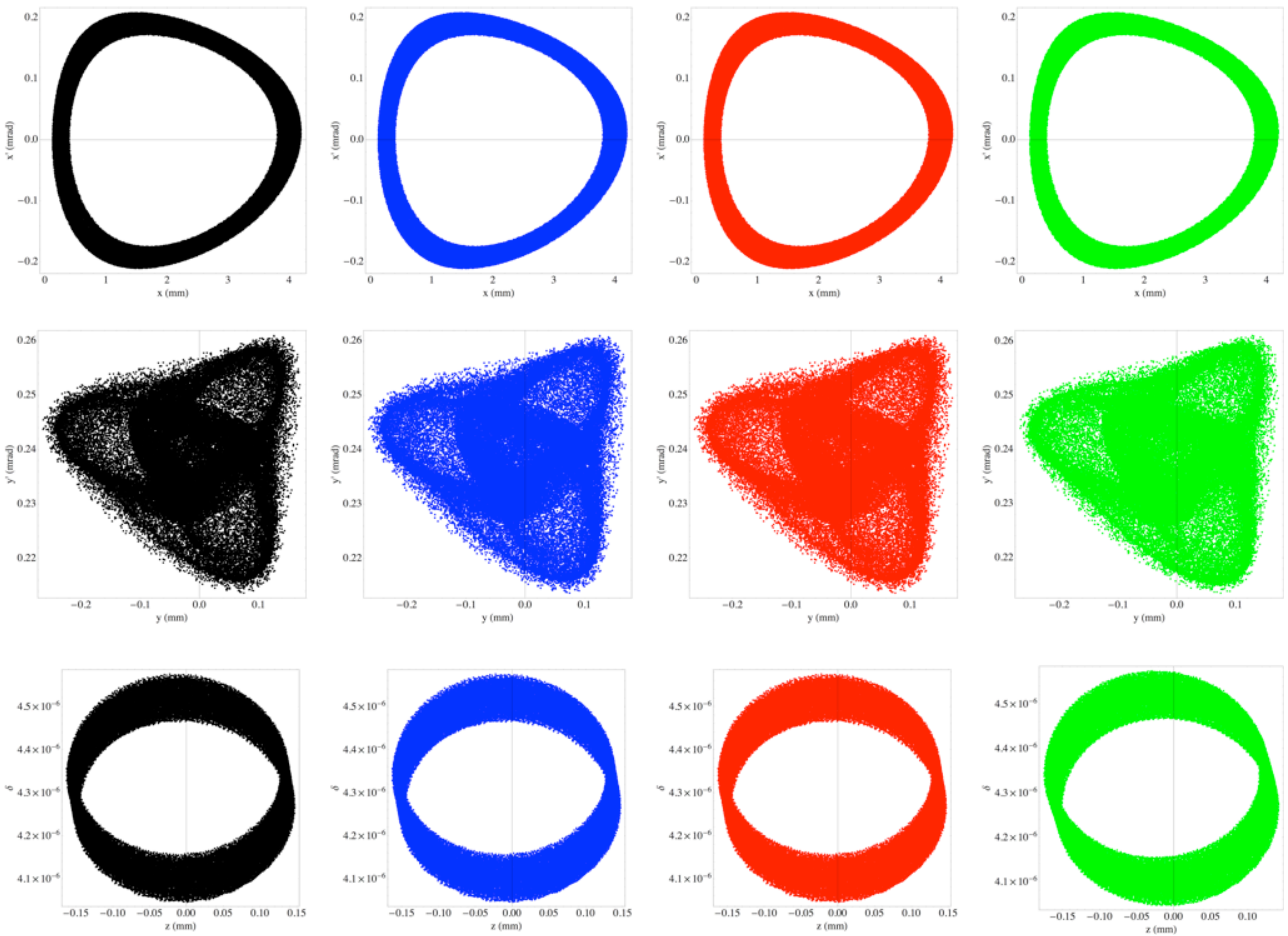
% difference with FOX

$$\begin{pmatrix} 0. & 0. & 1.07204\times10^{-2} & 1.58102\times10^{1} & 0. & 2.62115\times10^{-9} \\ 2.41265\times10^{-8} & 0. & 1.38823\times10^{-6} & 1.5385\times10^{-1} & 0. & 9.54905\times10^{-5} \\ 1.07204\times10^{-2} & 1.58084\times10^{1} & 0. & 0. & 2.08886\times10^{-7} & 2.64075\times10^{-3} \\ 1.38652\times10^{-6} & 1.5385\times10^{-1} & 2.66531\times10^{-2} & 0. & 2.50173\times10^{-7} & 1.73732 \\ 0. & 0. & 9.79312\times10^{-10} & 2.64002\times10^{-3} & 0. & 0. \\ 0. & 0. & 9.69922\times10^{-9} & 2.79963\times10^{-3} & 1.38547\times10^{-2} & 0. \end{pmatrix}$$

Results
Long term tracking

- A fixed initial position is set in fort.13 file and used for each case so not to include the closed orbit differences in the errors.
- Fixed position is at max. horizontal amplitude of 6 + the closed orbit determined by the numerical integration.
- A BPM is placed at IP2 and recorded for 100,000 turns. (Only have data to 70,000 for numerical integration)
- Close to 3rd order resonance increasing sensitivity to cavity model.

Results



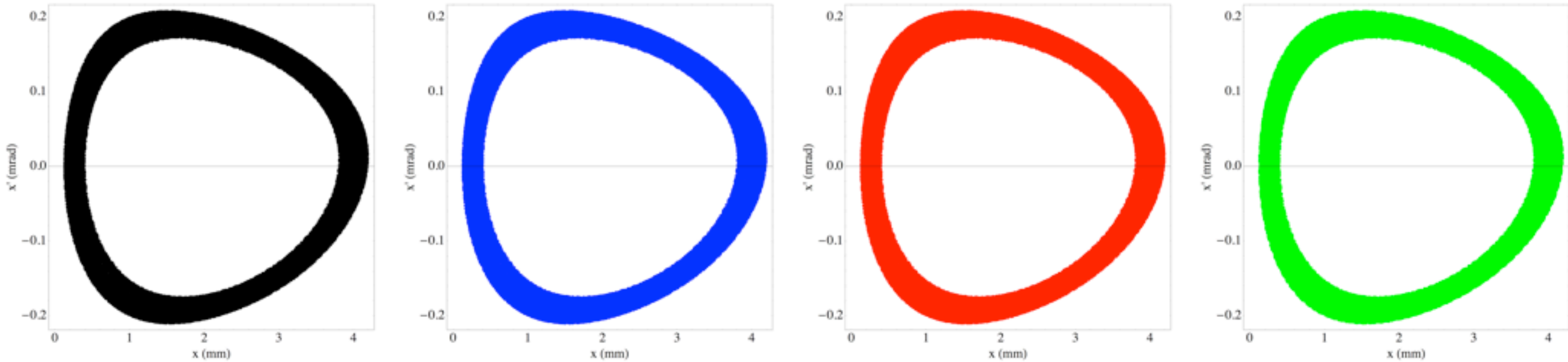
Taylor map full

Taylor map reduced

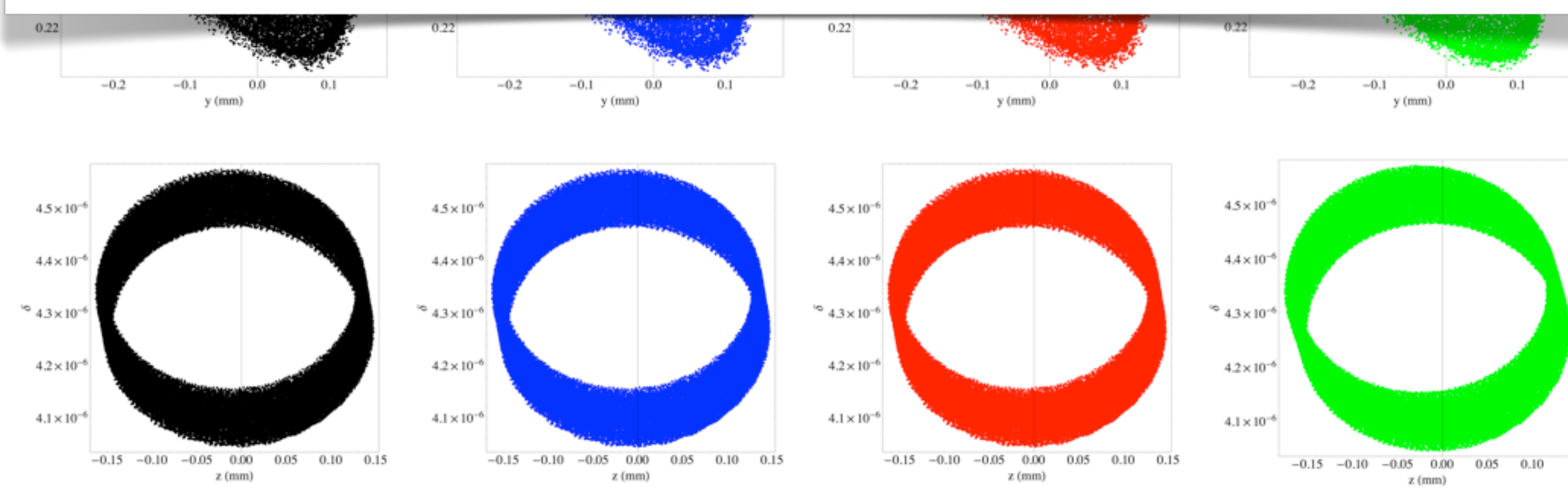
Numerical integration

RF multipole

Results



Look the same...



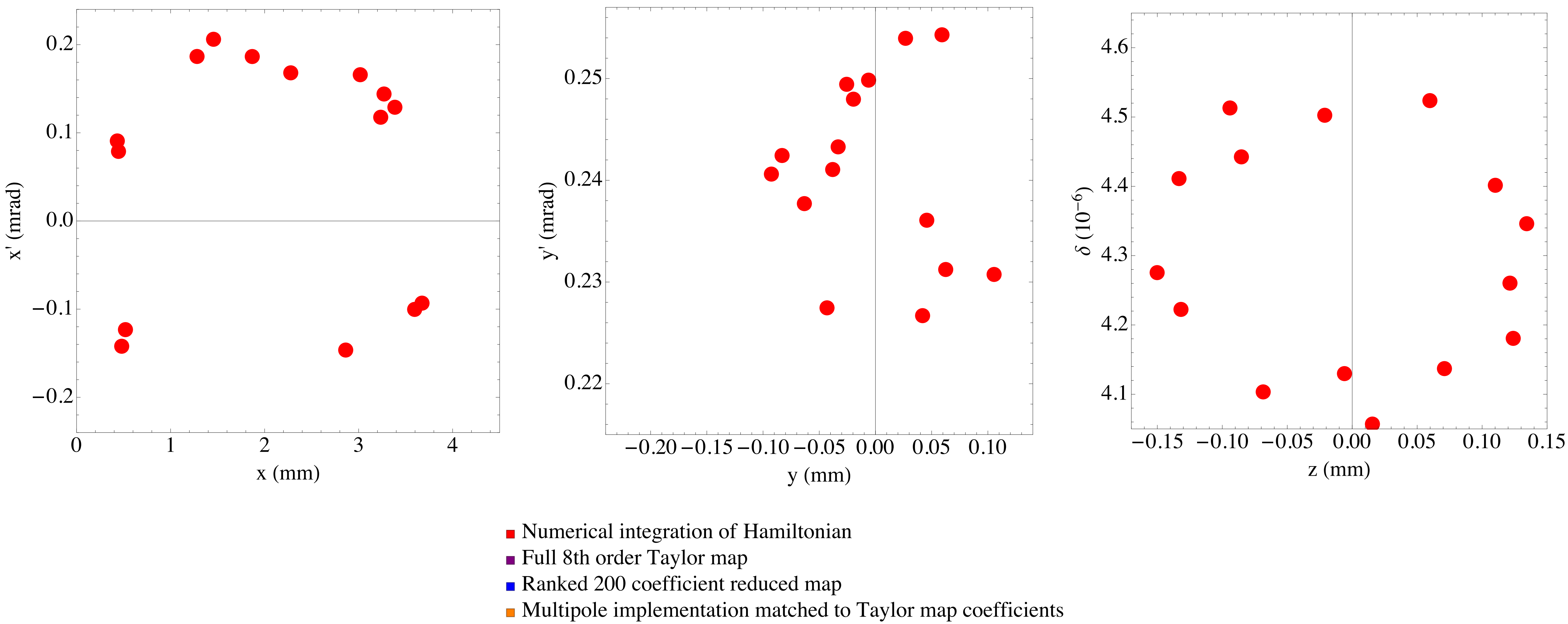
Taylor map full

Taylor map reduced

Numerical integration

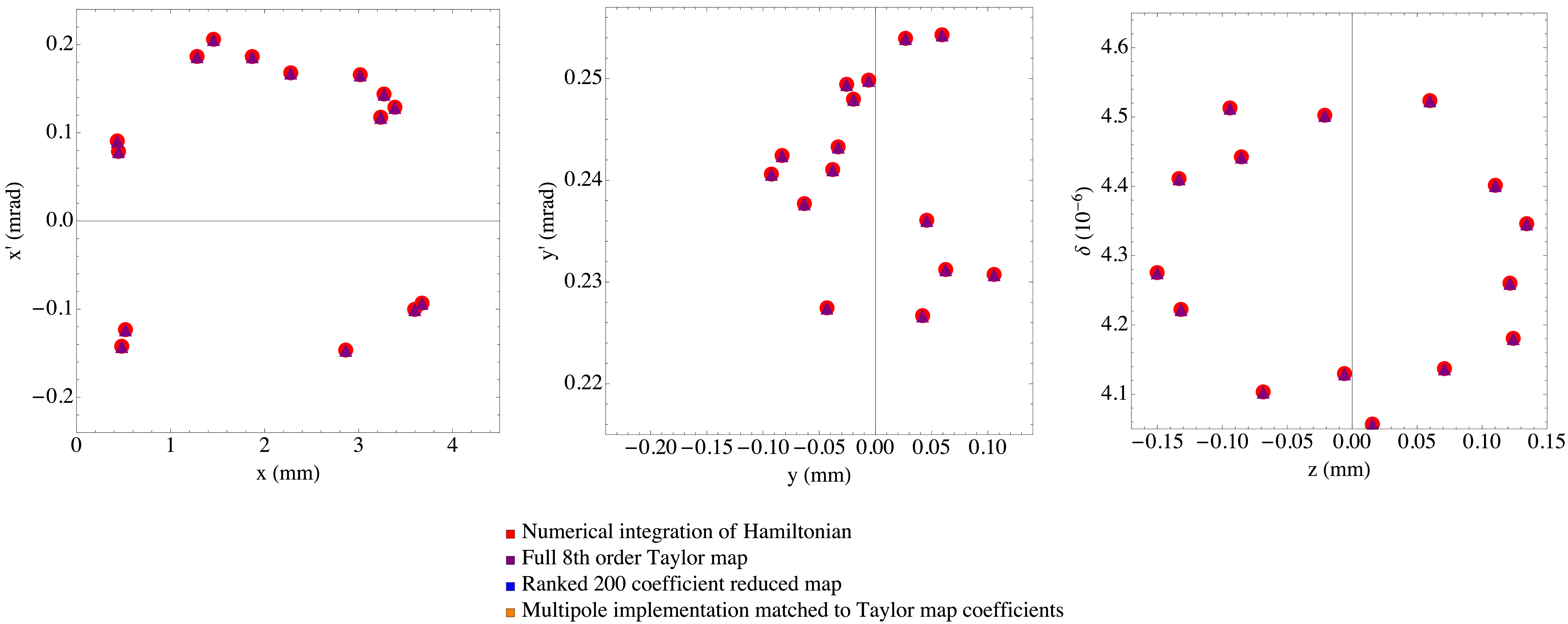
RF multipole

Results



- Plotted every 5000 turns.
- **Large difference between multipoles and numerical integration.**

Results



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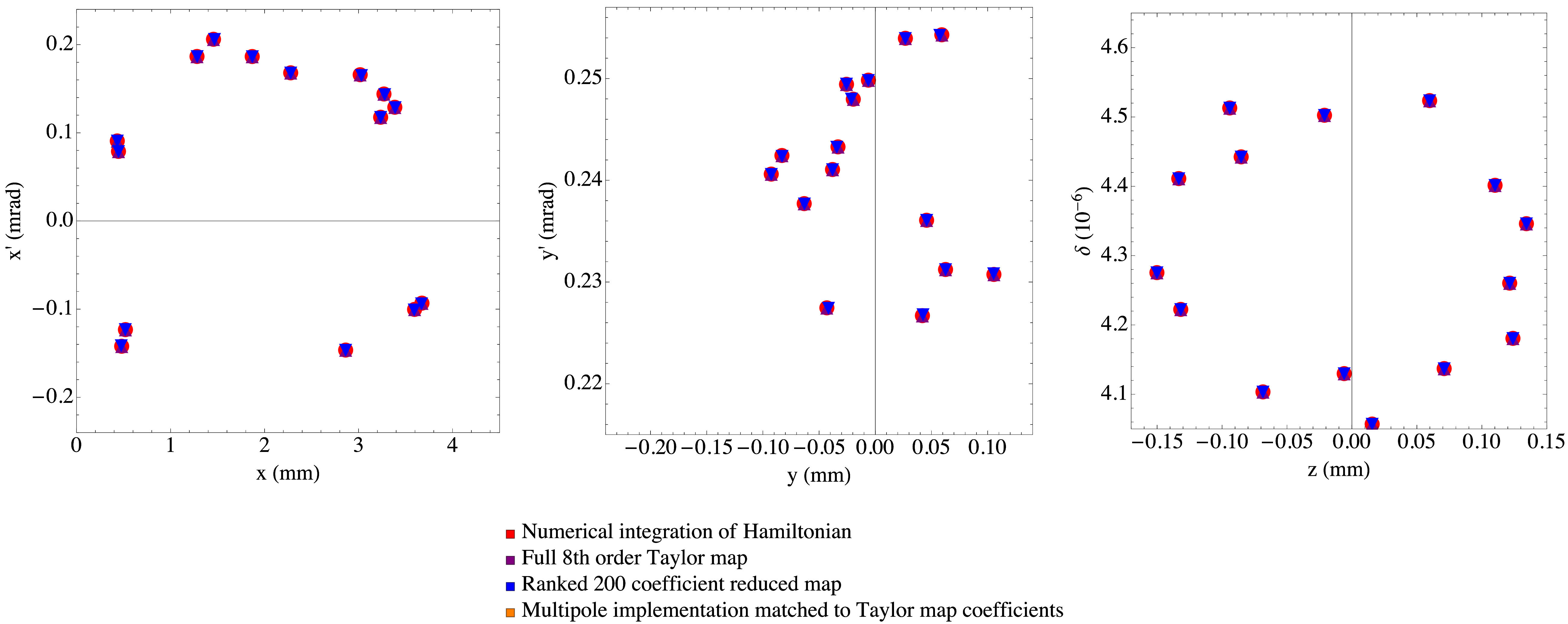

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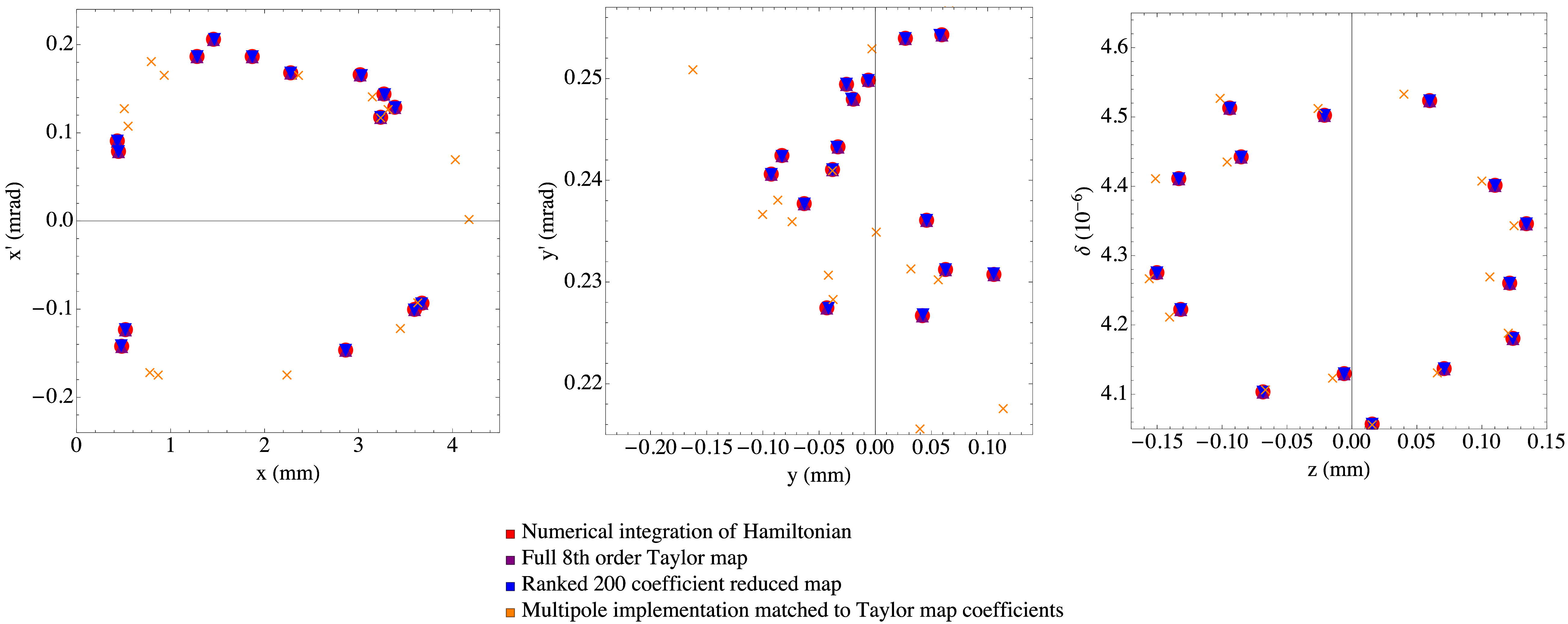

- Plotted every 5000 turns.
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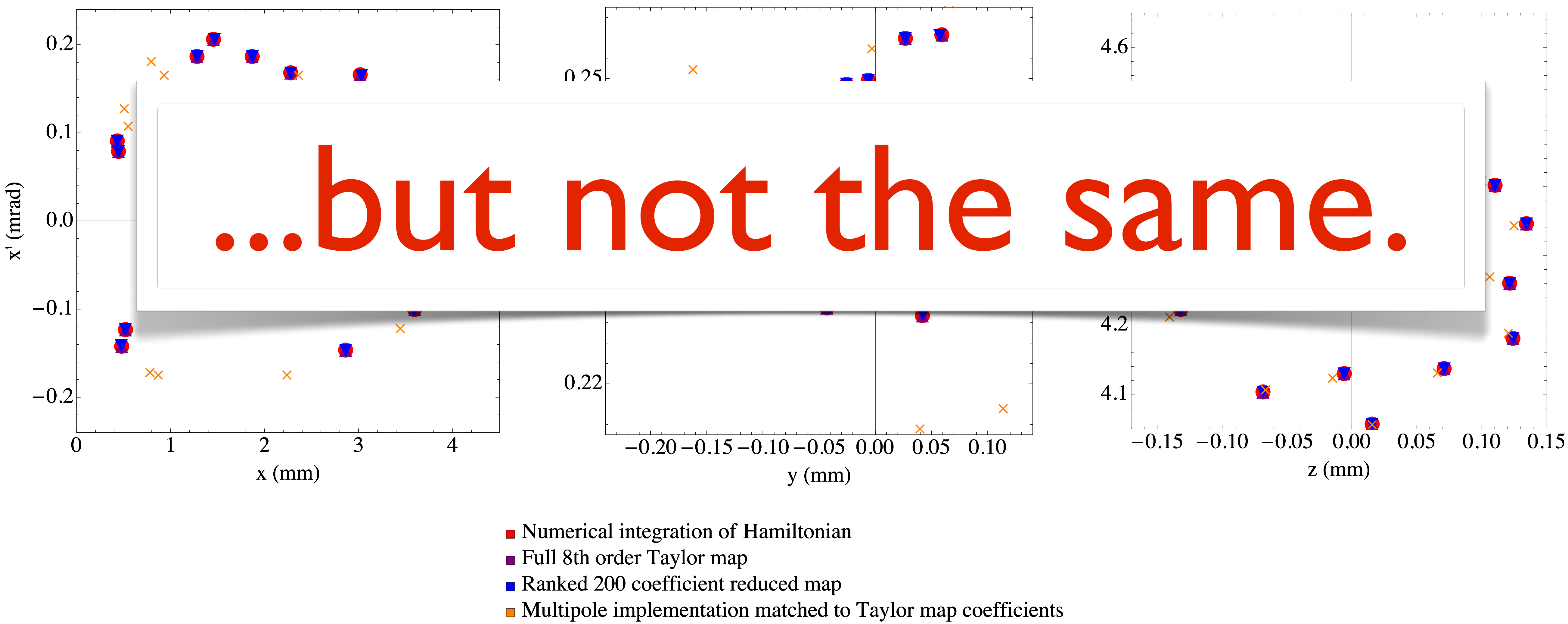
Results

...but not the same.



- Plotted every 5000 turns.
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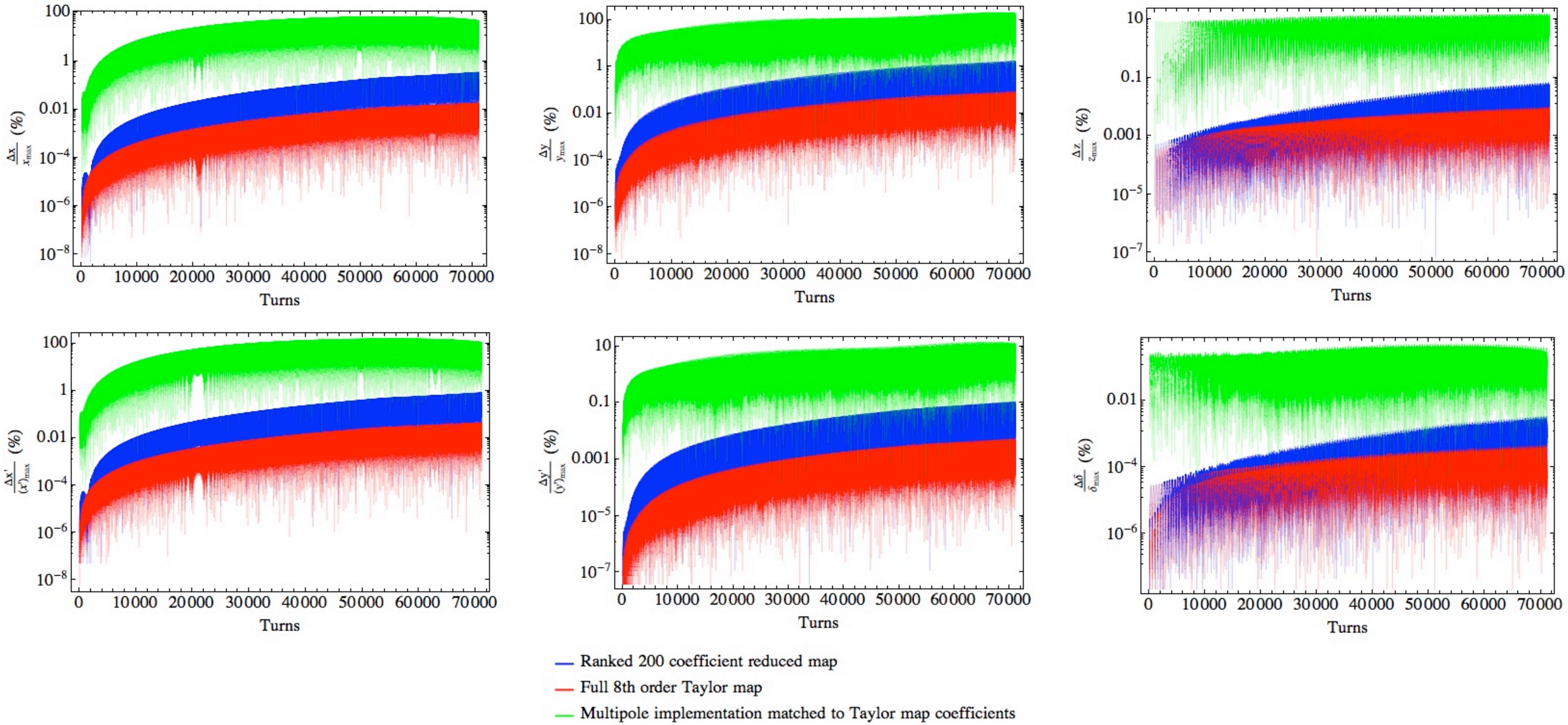
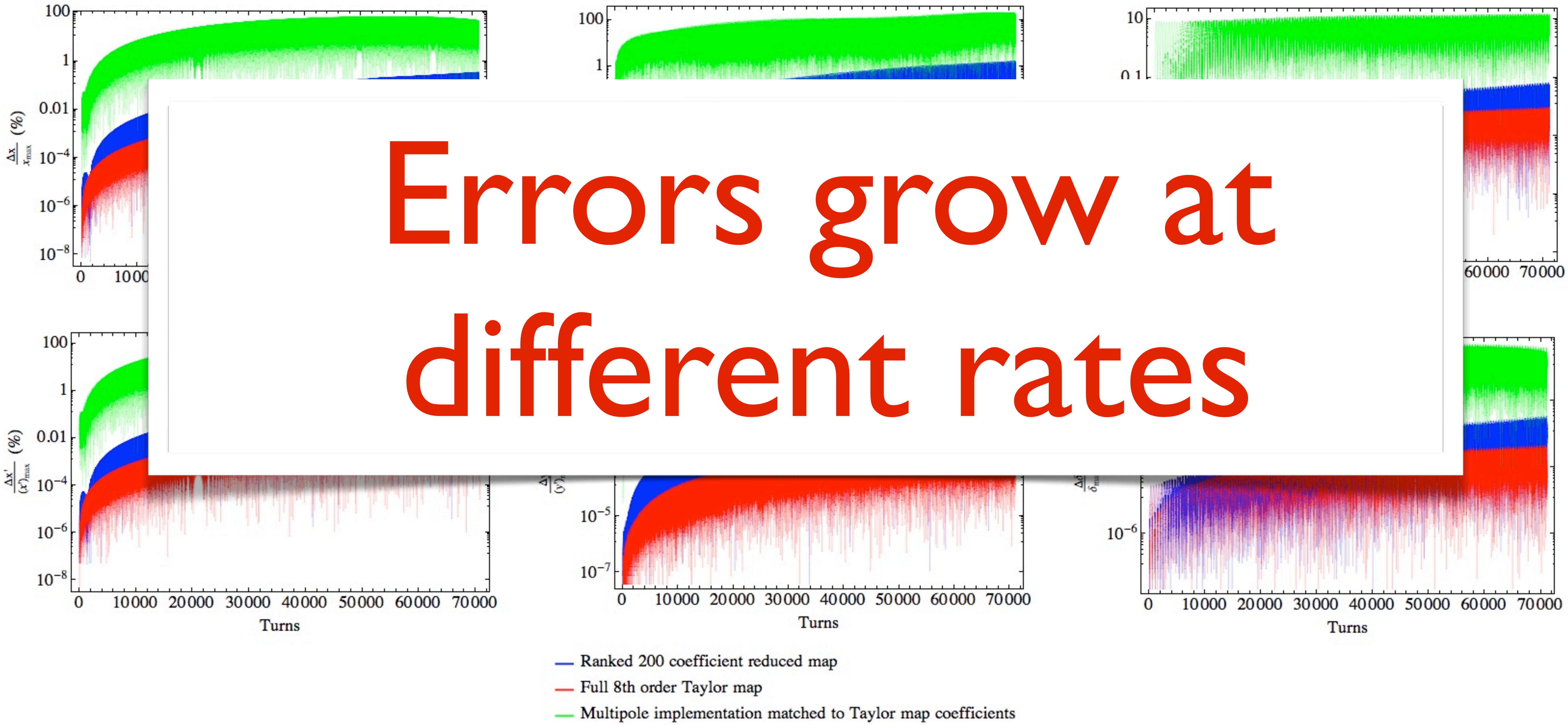
Results



- Error scaled with maximum value.
- Difference with numerical integration.

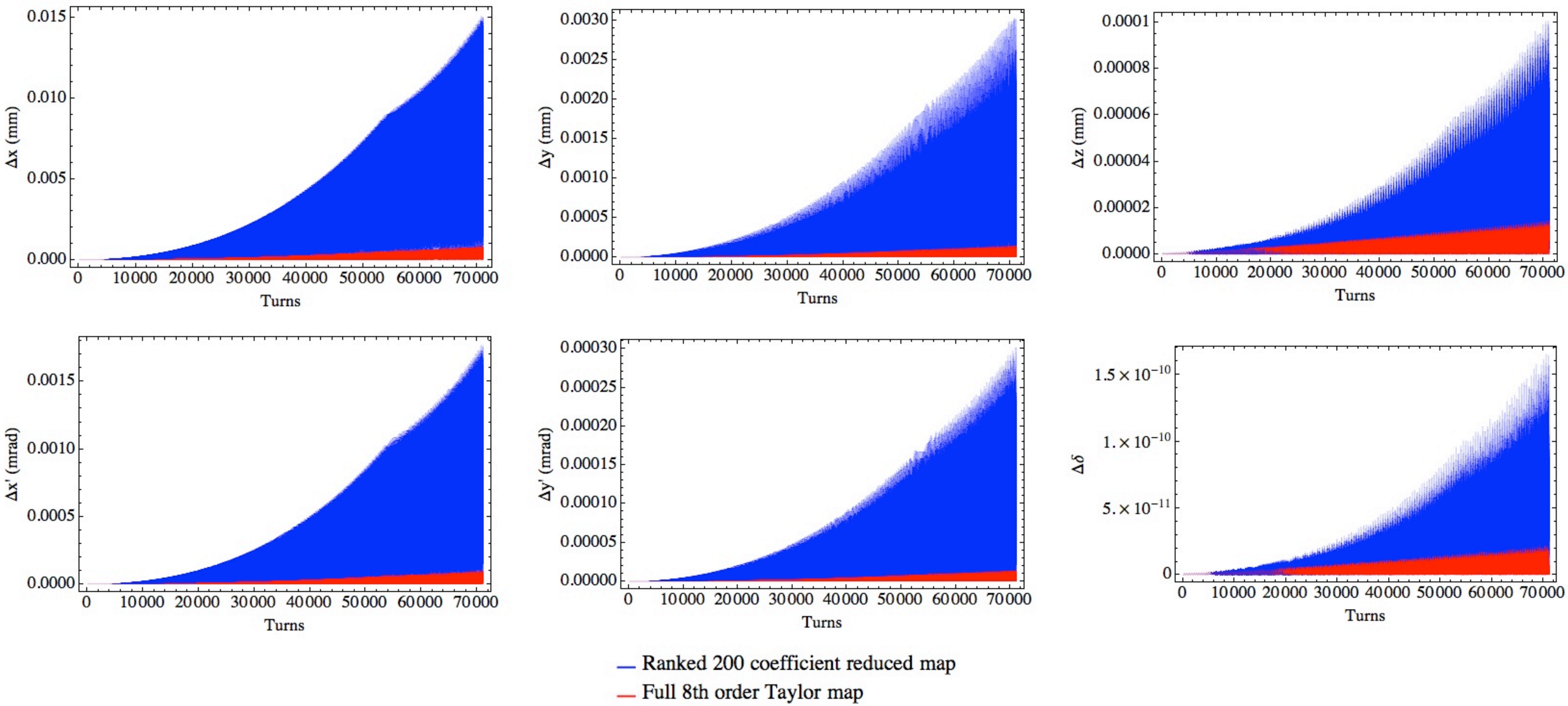
Results



Errors grow at different rates

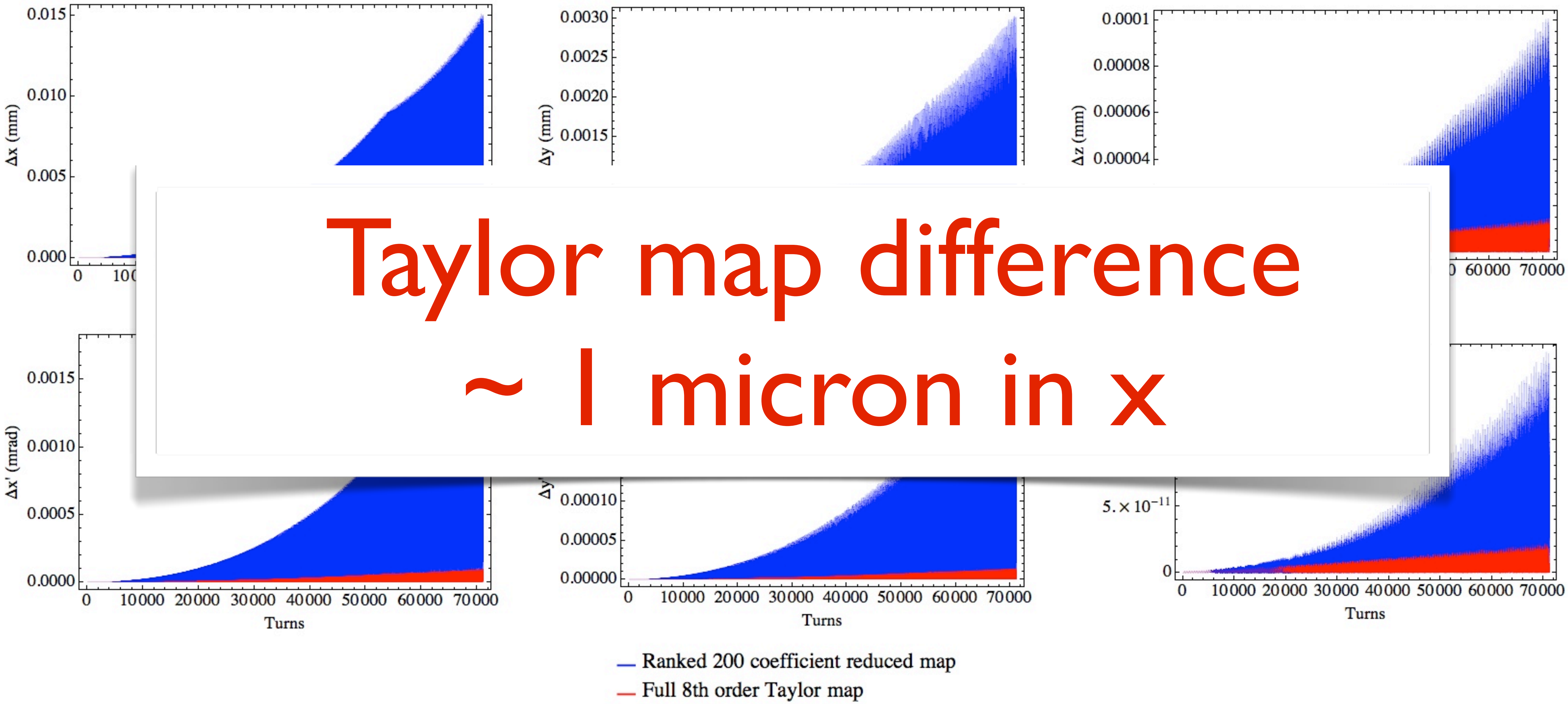
- Error scaled with maximum value.
- Difference with numerical integration.

Results



- Difference with numerical integration.
- Almost exponential growth with number of turns in error.
- Full map the error for this boundary is almost linear.
- Reduced Taylor map order of magnitude greater error.

Results



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Summary

Method	Symplectic to machine precision	Tracking time (s/turn)	Tracking error (after 70,000 turns) in x,y (mm)
Numerical integration	✔	2.32	-
Full Taylor map (3003 coefficients)	✖	5.34E-02	6.44E-05
Reduced Taylor map (200 coefficients)	✖	3.88E-03	1.23E-03
RF multipole	✔	2.51E-03	1.02

- Significant tracking difference between thin model and thick integrated models.
- Payoff between error in tracking and time to track.

Proposal

- Collaboration on Inner Triplet using these methods considering the possibility of the use of full Taylor maps using numerical integrator for validation in SixTrack.
  - Including trip to CERN.
- Prepare with Riccardo my SixTrack implementations for release/ update SixTrack manual.