Probing Conformal Field Theories

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Overview of work done in collaboration with
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Probing a CFT

- **Consider a heavy probe coupled to a CFT, in some rep. of the gauge group.**
- **It may be coupled to additional fields.**
- **Its world-line is prescribed. It defines a line operator (Wilson line,....).**
- **What are the fields it creates? Energy radiated? Momentum fluctuations?.....**

An example: Maxwell theory

Static particle

$$< \mathcal{L}(\vec{x}) > = q^2 \frac{1}{|\vec{x}|^4}$$

Coulomb

$$\mathcal{L} \sim F^2 \sim E^2 - \cancel{B^2}$$

Accelerated Particle

$$P = \frac{2}{3} q^2 \; a^\mu a_\mu$$

Larmor

In today's talk....

Probes will be infinitely heavy.

We will only consider probes in the vacuum state of the CFT.

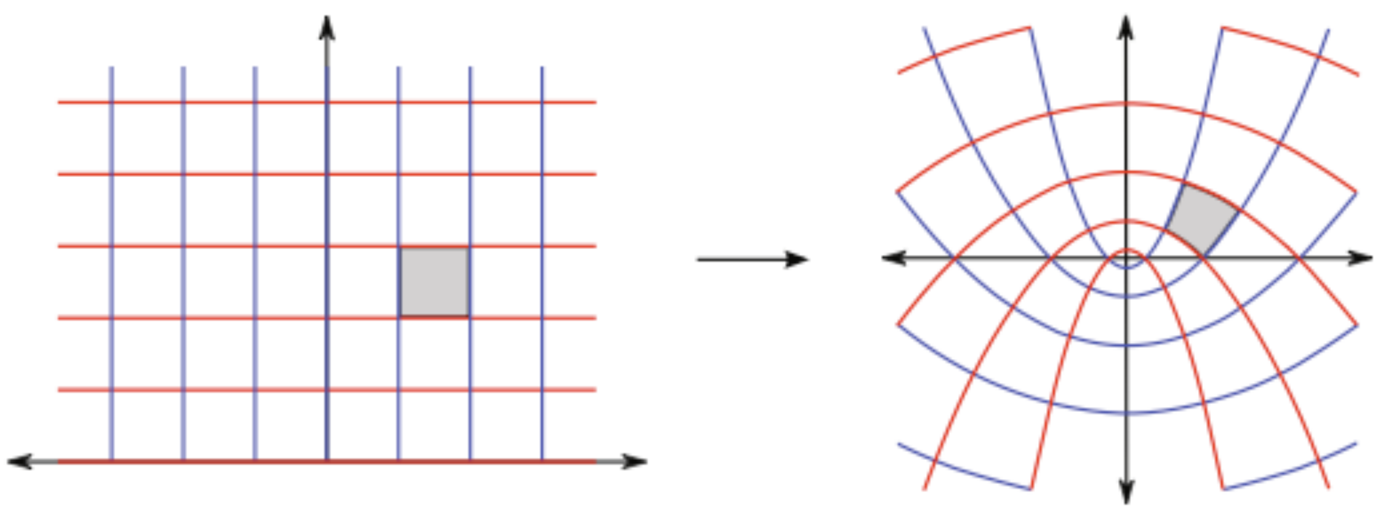
Plan of the Talk

- **External Probes in CFTs.**
- **Computing Bremsstrahlung functions.**
  - **AdS/CFT**
  - **Localization**
- **An application.**
- **Outlook.**

Conformal Group

Conformal Transformation

$$g_{\mu\nu}(x) \rightarrow \Lambda(x) g_{\mu\nu}(x)$$



d>2 **SO(2,d): Poincaré + dilatations + special conformal transformations.**

CFTs and local operators

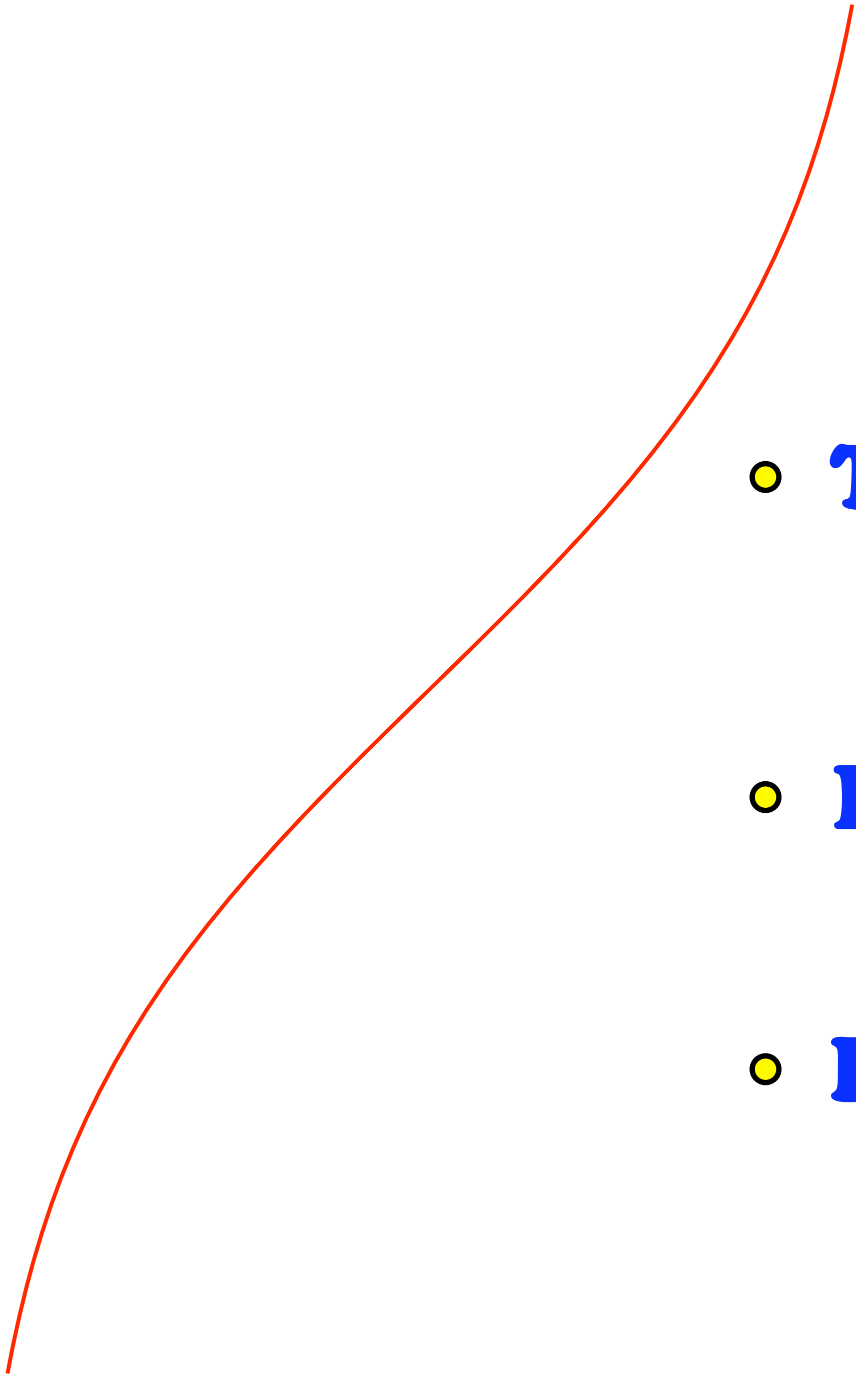
- **Conformal symmetry gives constraints on correlation functions, but it doesn't fix all coefficients.**

$$< \mathcal{O}_i(x)\mathcal{O}_j(0) >= \frac{\delta_{ij}}{(x)^{2\Delta_i(\lambda,N)}}$$

$$< \mathcal{O}_i(x_i)\mathcal{O}_j(x_j)\mathcal{O}_k(x_k) >= \frac{c_{ijk}(\lambda,N)}{(x_{ij})^{2\alpha_{ijk}}(x_{ik})^{2\alpha_{ikj}}(x_{jk})^{2\alpha_{jki}}}$$

- **We need additional tools to compute coefficients.**

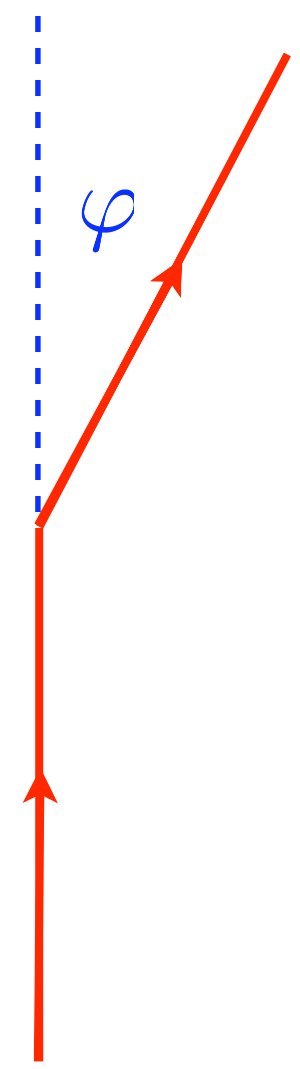
Let’s add external probes… (aka line operators)



- **Timelike curve C**
- **Electrically/magnetically charged**
- **Representation R of G**

→ **Wilson operator, ‘t Hooft operator,....**

Cusped Wilson loop: Radiation

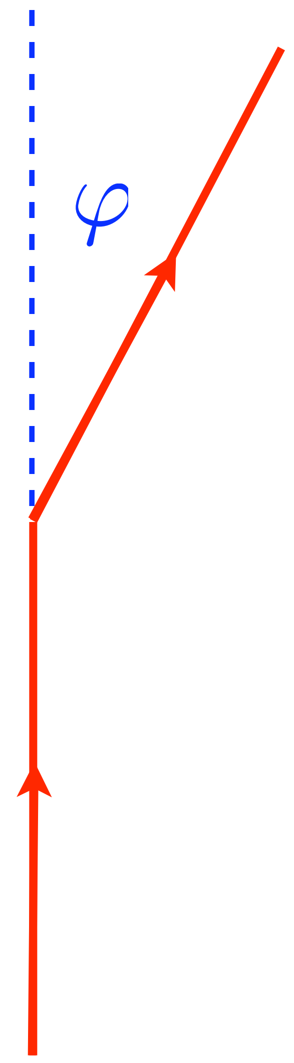


J.J. Thomson

Bremsstrahlung function

$$\Delta E = 2\pi B(\lambda, N) \int dt\ (\dot{v})^2$$

Cusped Wilson loop

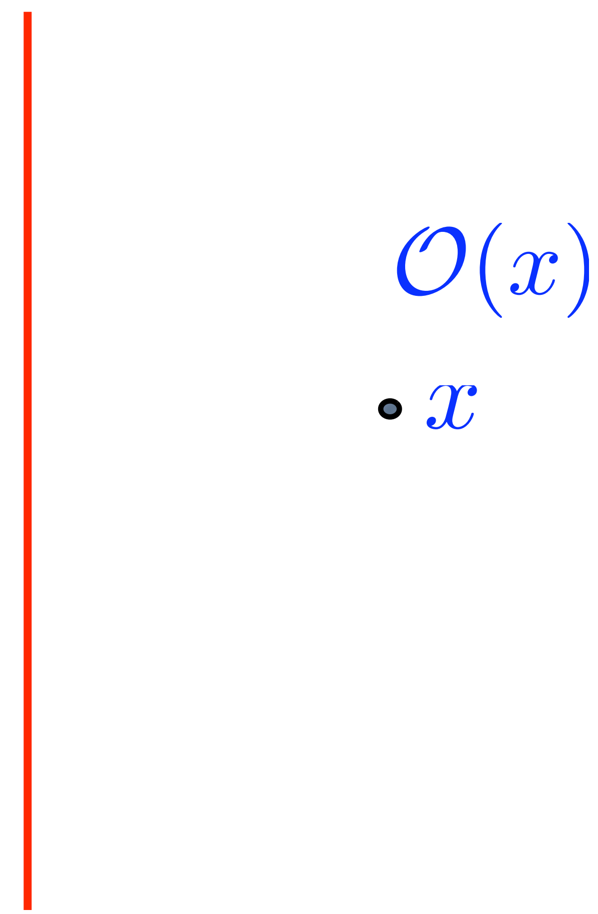


Cusp anomalous dimension

Polyakov 80

$$< W > \sim e^{-\,\Gamma_{cusp}(\varphi,\lambda)\; Log\frac{L}{\epsilon}}$$

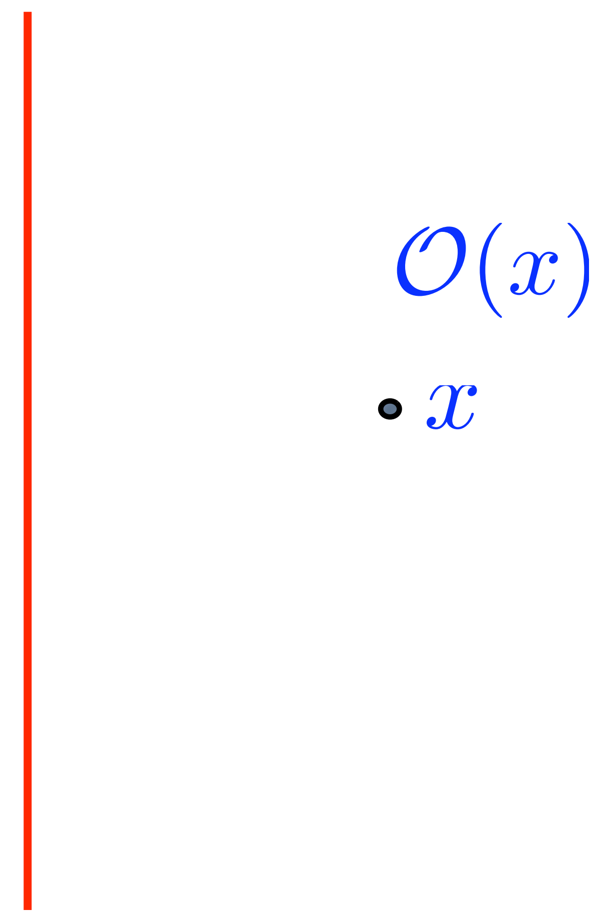
Line operators and local operators



$$< \mathcal{O}(x) >_W \equiv \frac{< \mathcal{O}(x) W >}{< W >}$$

Kapustin 05

Line operators and local operators

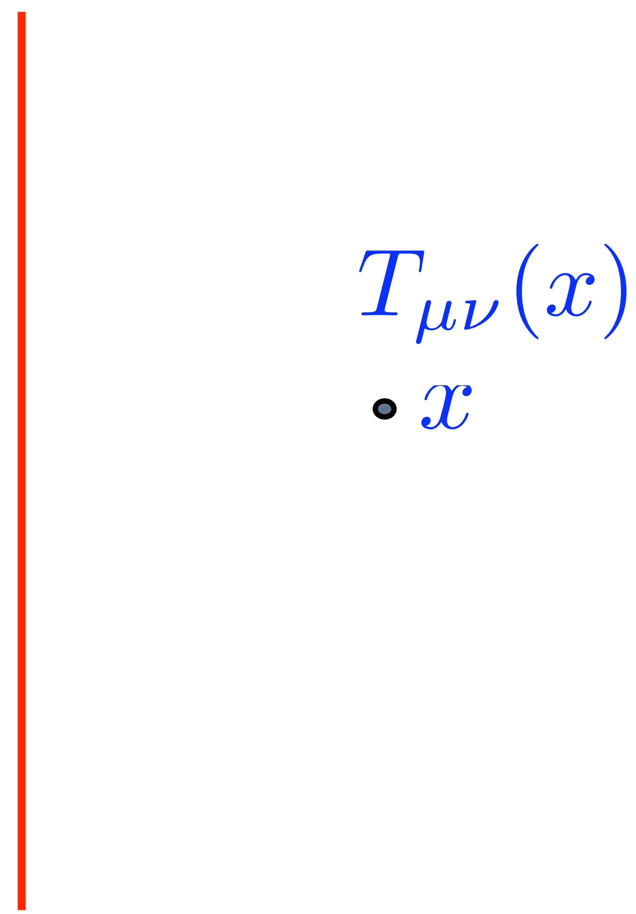


Conformal symmetry fixes $< \mathcal{O} >_W$ **up to a coefficient.**

$< \mathcal{O}(x_1)\mathcal{O}(x_2) >_W$ **is no longer fixed.**

Buchbinder, Tseytlin 12

Line operators and local operators



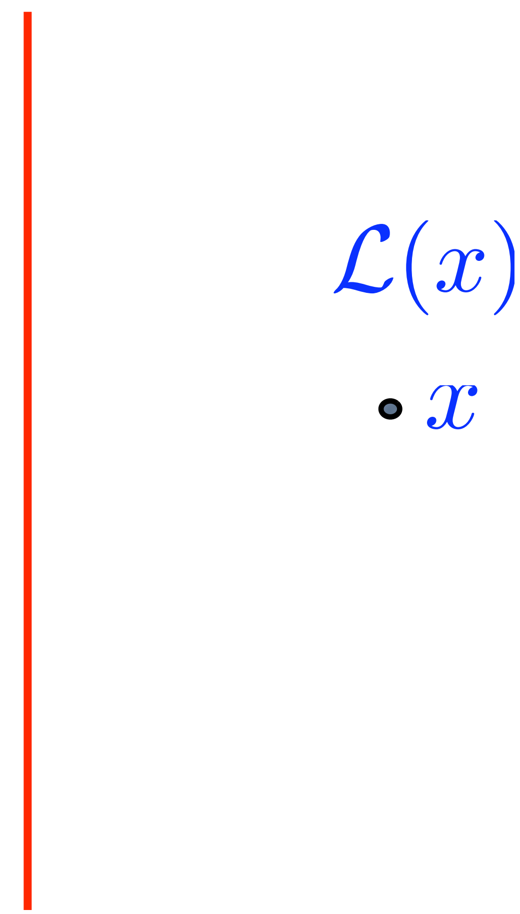
For a generic CFT

$$< T_{00}(x) >_W = h(\lambda, N)\,\frac{1}{|\vec{x}|^4}$$

$$< T_{ij}(x) >_W = h(\lambda, N)\;\frac{-\delta_{ij} + 2\frac{x_i x_j}{|\vec{x}|^2}}{|\vec{x}|^4}$$

Kapustin 05

Line operators and local operators



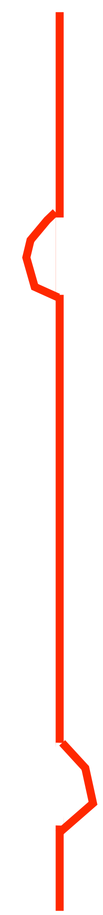
For a CFT with $\mathcal{L}$

$$< \mathcal{L}(x) >_W = f(\lambda, N) \, \frac{1}{|\vec{x}|^4}$$

World-line Operators: Displacement Operators

Polyakov, Rychkov 00
Semenoff, Young 04
Drukker, Kawamoto 06

$\mathbb{D}_j(t_2)$ $\delta x^j(t_2)$



$\mathbb{D}_i(t_1)$ $\delta x^i(t_1)$

$$\Delta(\mathbb{D}_i) = 2$$

$$<< \mathbb{D}_i(t_1)\mathbb{D}_j(t_2) >> = \tilde{\gamma}(\lambda, N) \frac{\delta_{ij}}{(t_1 - t_2)^4}$$

For any line operator in any 4d CFT, we have defined:

$$< W >\sim e^{-\,\Gamma_{cusp}(\varphi,\lambda)\,Log\frac{L}{\epsilon}}$$

$$\Delta E = 2\pi B(\lambda, N)\int dt\ (\dot{v})^2$$

$$<< \mathbb{D}_i(t_1)\mathbb{D}_j(t_2) >>= \ \tilde{\gamma}(\lambda, N)\ \frac{\delta_{ij}}{(t_1 - t_2)^4}$$

$$< T_{00}(x) >_W = h(\lambda, N)\,\frac{1}{|\vec{x}|^4}$$

$$< \mathcal{L}(x) >_W = f(\lambda, N)\ \frac{1}{|\vec{x}|^4}$$

These coefficients are actually not independent...

Expand the cusp anomalous dimension at small angles,

$$\Gamma(\varphi) = \Gamma(\lambda, N)\,\varphi^2 + \mathcal{O}(\varphi^4) + \ldots$$

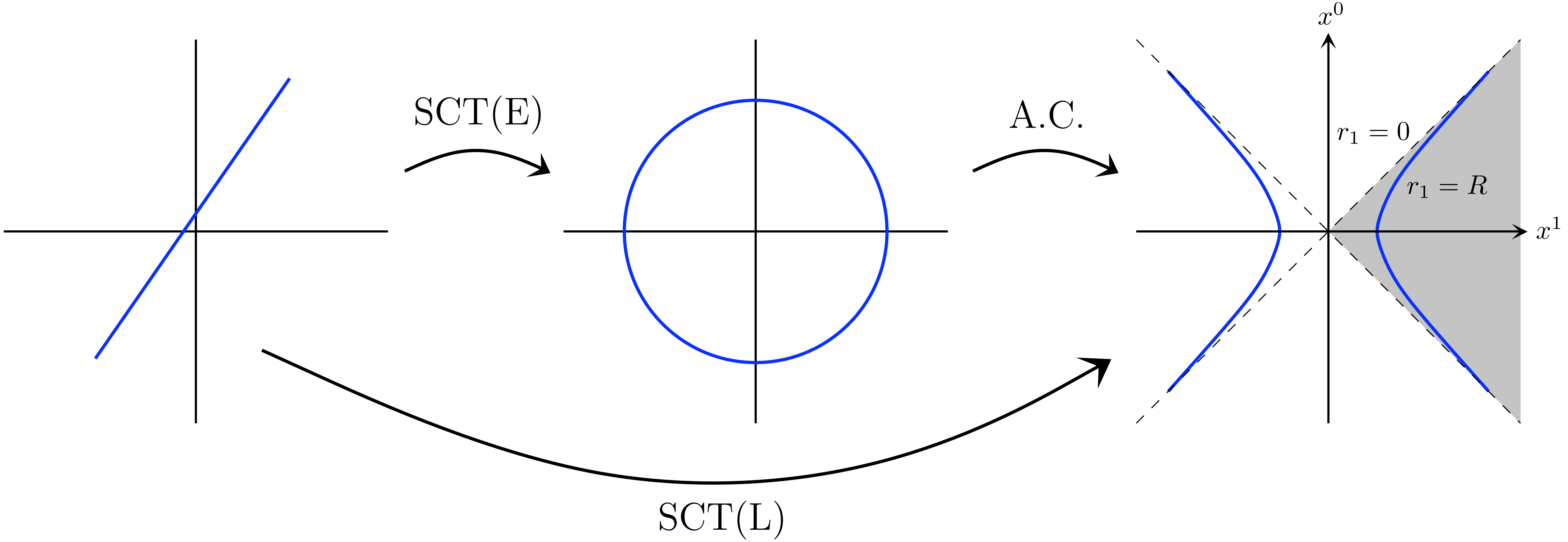
Then, for any line operator and any 4d CFT,

$$\Gamma = \frac{\tilde{\gamma}}{12} = B$$

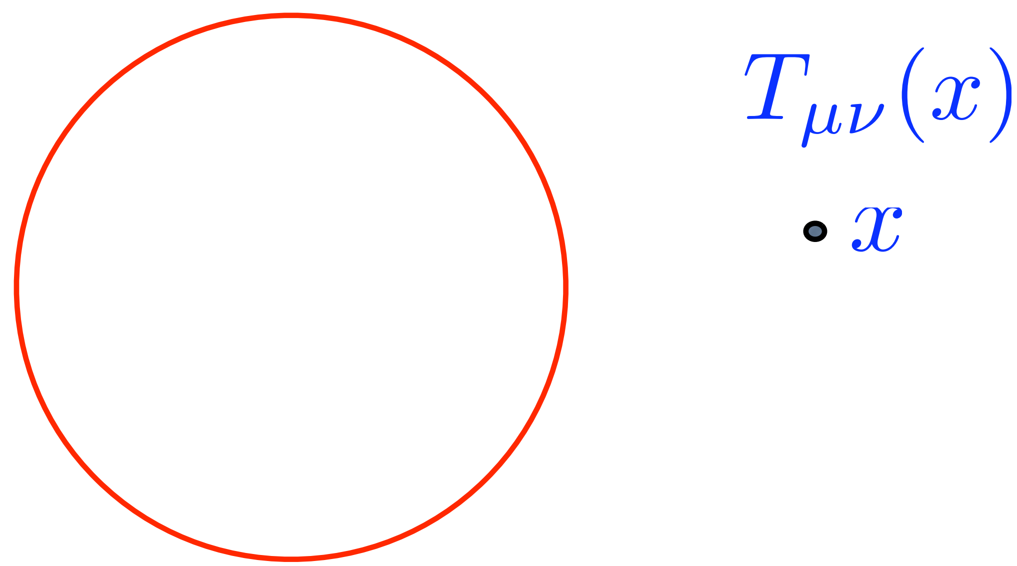
Correa, Henn, Maldacena, Sever 12

But wait!, there is more...

Hyperbolic Wilson line: accelerated probe



Hyperbolic Wilson line: accelerated probe

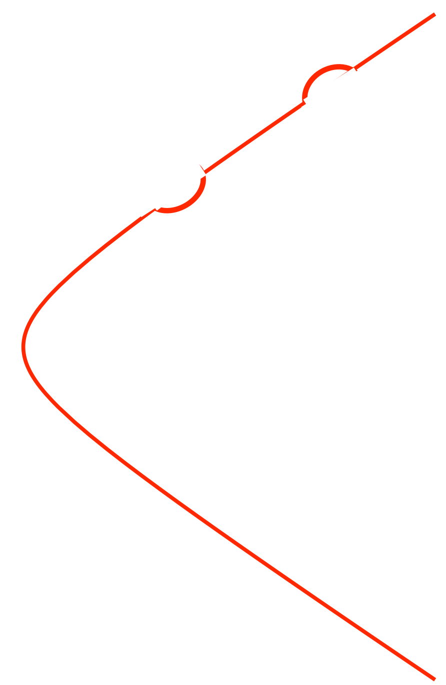


An alternative way to compute energy loss by radiation

$$P = 2\pi\, B(\lambda, N)\, a^\mu a_\mu$$

BF, Garolera, Lewkowycz 12

Hyperbolic Wilson line: accelerated probe



BF, Garolera, Torrents 13

$$<< \mathbb{D}_i(\tau)\mathbb{D}_j(0) >>= 12\ B(\lambda, N)\ \frac{\delta_{ij}}{16R^4 \sinh^4(\frac{\tau}{2R})}$$

$$\kappa = \lim_{w\to 0} \int d\tau e^{iw\tau} << \mathbb{D}_i(\tau)\mathbb{D}_j(0) >>= 16\pi^3 B(\lambda, N)\, T^3$$

Unruh temperature

Very pretty.... but can we actually compute $B(\lambda, N)$ for any probe in any CFT?

Yes! 1/2-BPS probe coupled to $\mathcal{N}=4$ SU(N) SYM.

Computing the Bremsstrahlung function

- Conformal symmetry gives constraints on correlation functions, but it doesn't fix all coefficients.

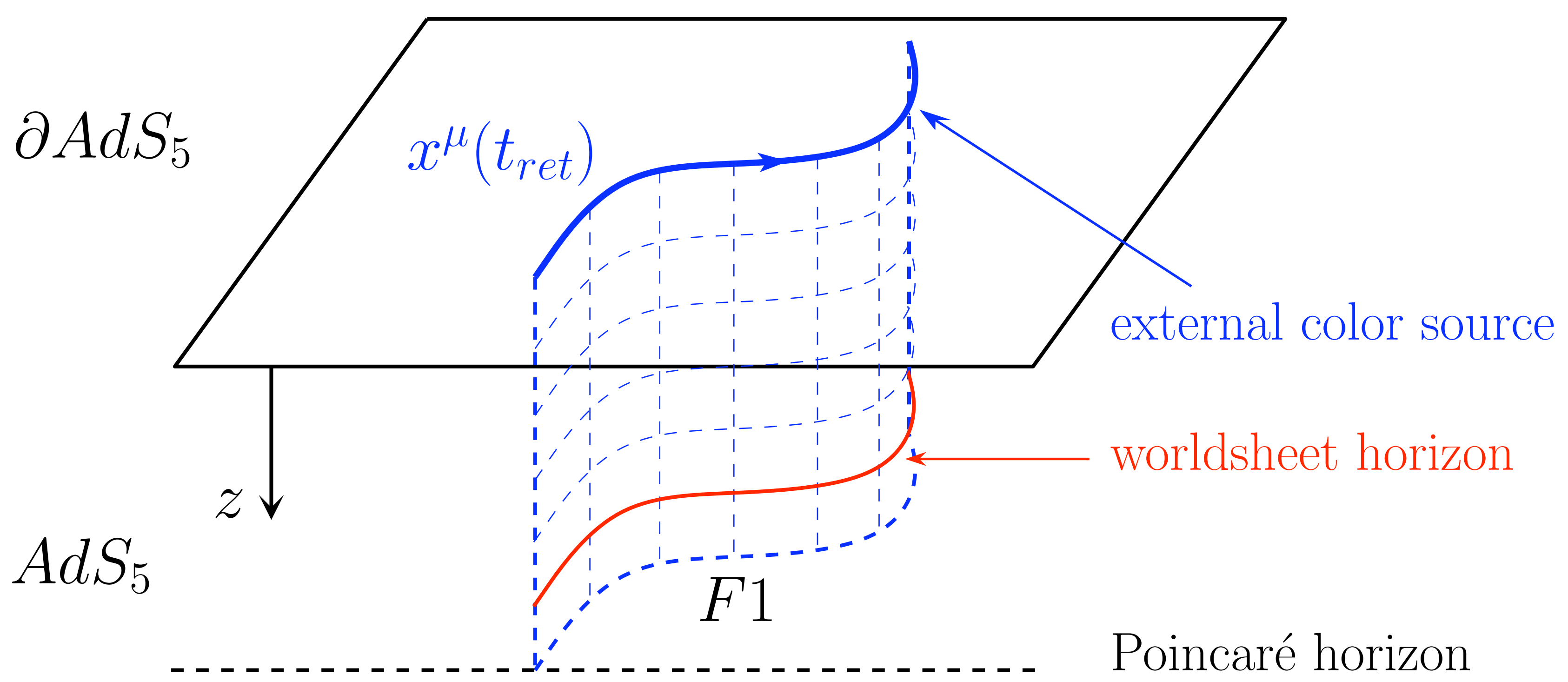
- We need additional tools to compute coefficients:

  - Pert. Theory (finite N, small λ)
  - → AdS/CFT (large N, large λ)
  - Integrability (large N, finite λ)
  - → Localization (finite N, finite λ)

External Probes in AdS/CFT

External Probes in AdS/CFT

Consider a particle in the fundamental representation. Its dual is a fundamental string, reaching the boundary of AdS at the particle word-line.



External Probes in AdS/CFT

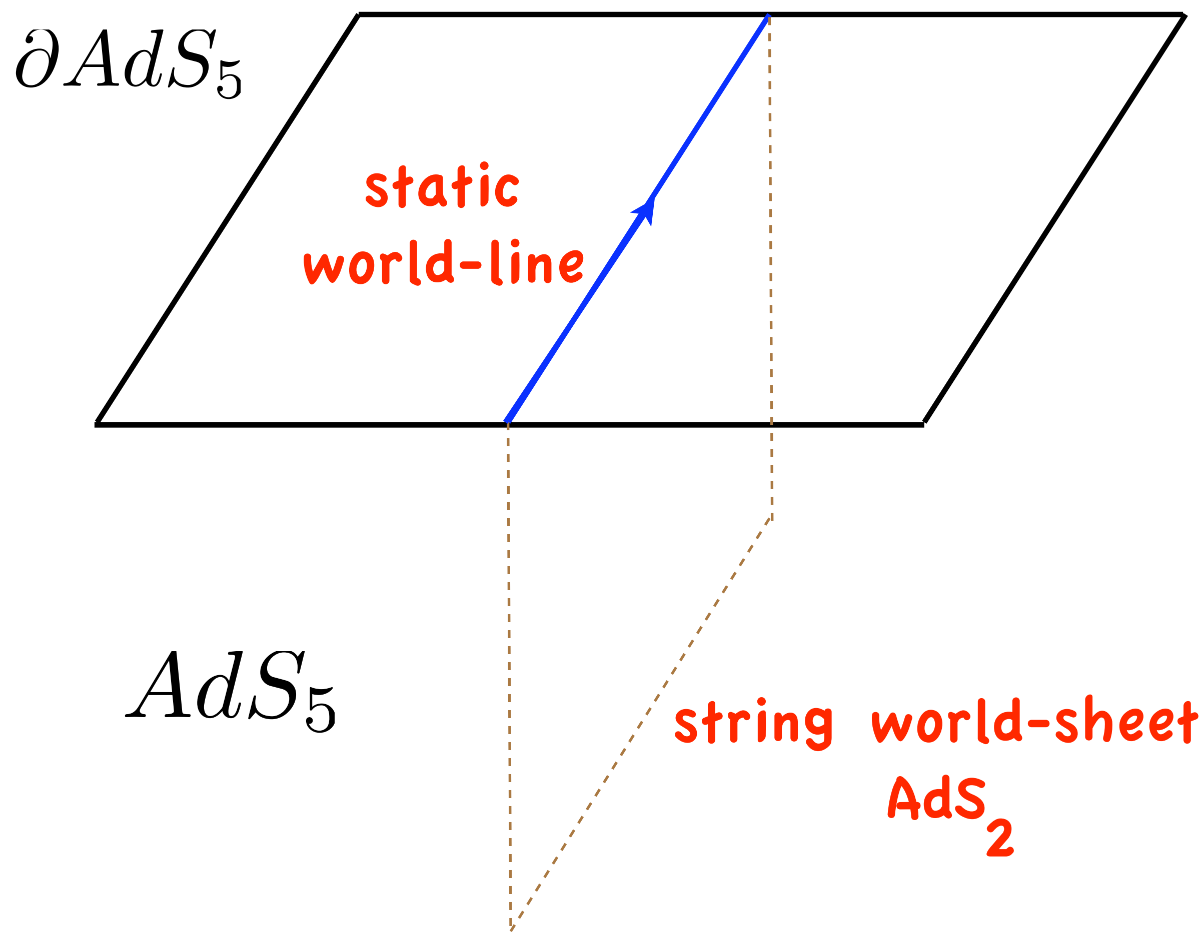
In the absence of other scales, the effective charge is

$$e^2_{\Box} \sim \sqrt{\lambda}$$

It signals screening of the charge at strong coupling.

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-|g|} = -\frac{\sqrt{\lambda}}{2\pi L^2} \int d^2\sigma \sqrt{-|g|}$$

First Example: static particle



$$\mathcal{L} = \frac{1}{2g_{YM}^2} \mathrm{Tr}\ \left(F^2 + [X_I, X_J][X^I, X^J] + \text{fermions} \right)$$

?

$$< \mathcal{L}(\vec{x}) > = f(\lambda, N) \frac{1}{|\vec{x}|^4}$$

First Example: static particle

The Lagrangian density is dual to the dilaton.

The string backreaction perturbs the AdS background.

$$\phi(x) = \int d^5x' G(x, x') J(x')$$

The dilaton was constant in the unperturbed solution its linearized perturbation decouples.

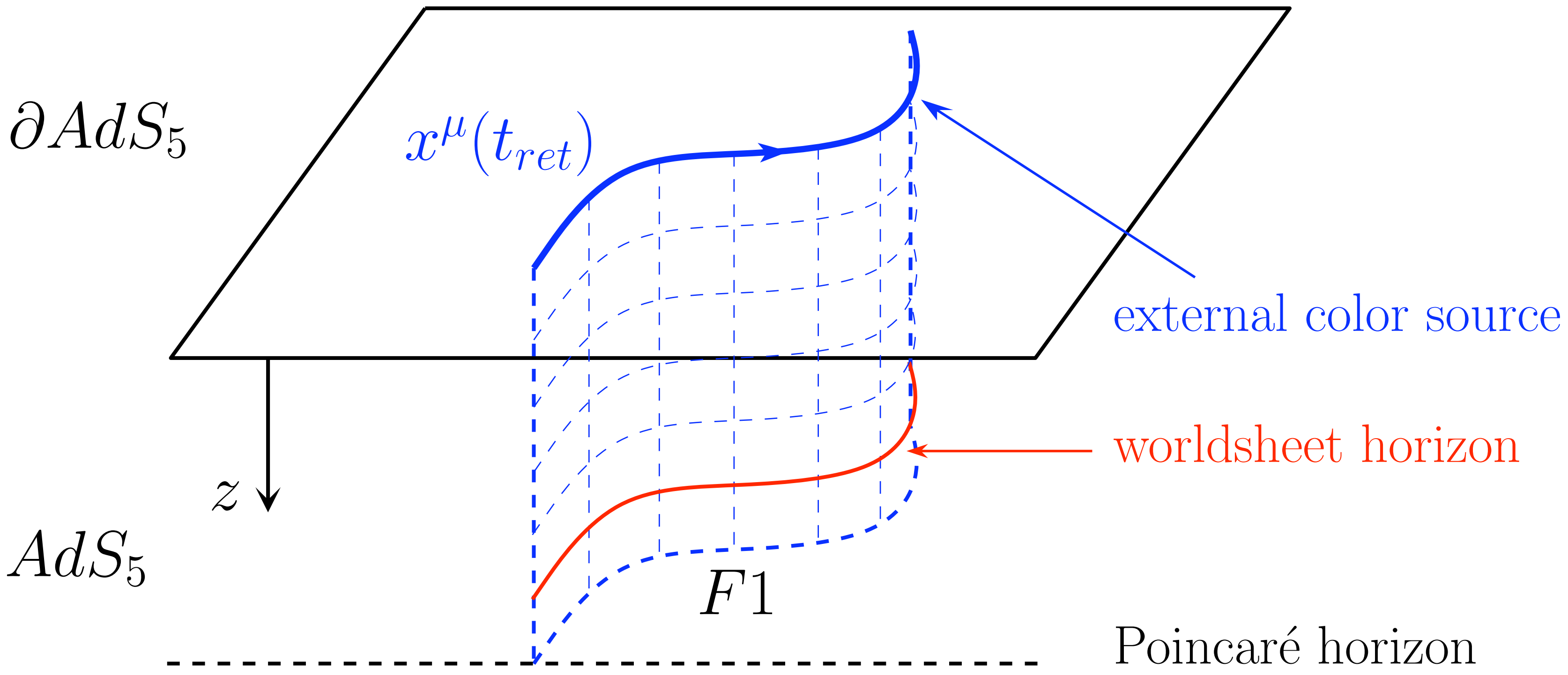
$$< \mathcal{L}(\vec{x}) > = \frac{1}{16\pi^2} \frac{\sqrt{\lambda}}{|\vec{x}|^4}$$

Danielsson, Keski-Vakkuri, Kruczenski
Callan, Güijosa

Second Example: accelerated particle

Mikhailov found the fundamental string dual to a particle following an arbitrary timelike trajectory.

Mikhailov 03

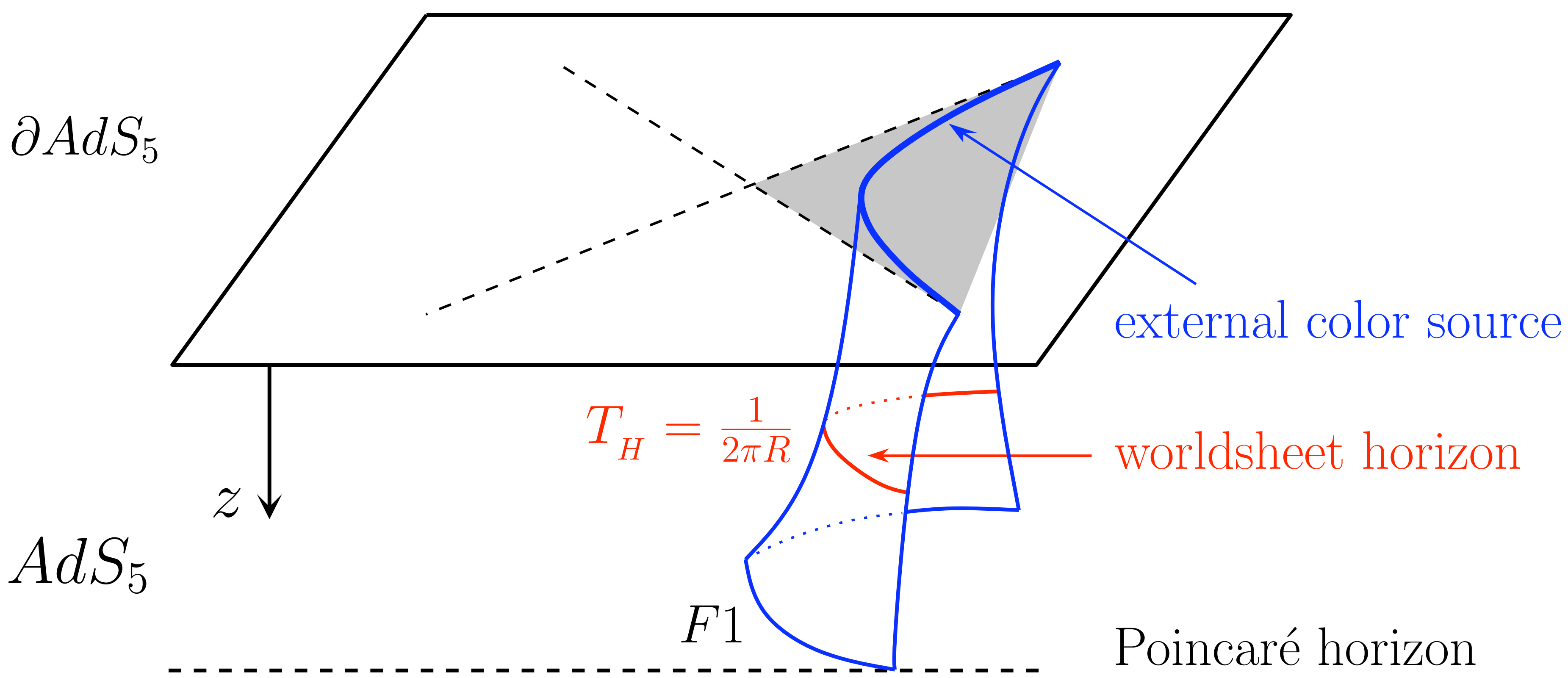


$$P = \frac{\sqrt{\lambda}}{2\pi} a^\mu a_\mu$$

Mikhailov 03

Second Example: accelerated particle

The world-sheet horizon splits the gluonic cloud into a Coulombic and a radiative part.



$$E = \int d\sigma \, \mathcal{E}$$

External Probes in AdS/CFT

Static Particle

$$< \mathcal{L}(\vec{x}) > = \frac{1}{16\pi^2} \frac{\sqrt{\lambda}}{|\vec{x}|^4}$$

Danielsson *et al.*
Callan, Güijosa 98

Accelerated particle

$$P = \frac{\sqrt{\lambda}}{2\pi} a^\mu a_\mu$$

Mikhailov 03

$$\kappa = 4\pi \sqrt{\lambda} \; T^3$$

Xiao 08

Circular Wilson loop

$$\ln \; < W_{\bigcirc} > = \sqrt{\lambda}$$

Berenstein *et al.* 98

$q\bar{q}$ Potential

$$V_{q\bar{q}} = -\frac{4\pi^2}{\Gamma^4\left(\frac{1}{4}\right)} \frac{\sqrt{\lambda}}{L}$$

Rey,Yee
Maldacena 98

External Probes in AdS/CFT

All these computations yield

$$B(\lambda, N) = \frac{\sqrt{\lambda}}{4\pi^2}$$

The $\sqrt{\lambda}$ in these results appears from evaluating classical string solutions to the NG action. There are two types of corrections:

$1/\sqrt{\lambda}$ **world-sheet fluctuations.**

Forste, Ghoshal, Theisen 99
Drukker, Gross, Tseytlin 00
....
Buchbinder, Tseytlin 13

$1/N$ **higher genus world-sheets.**

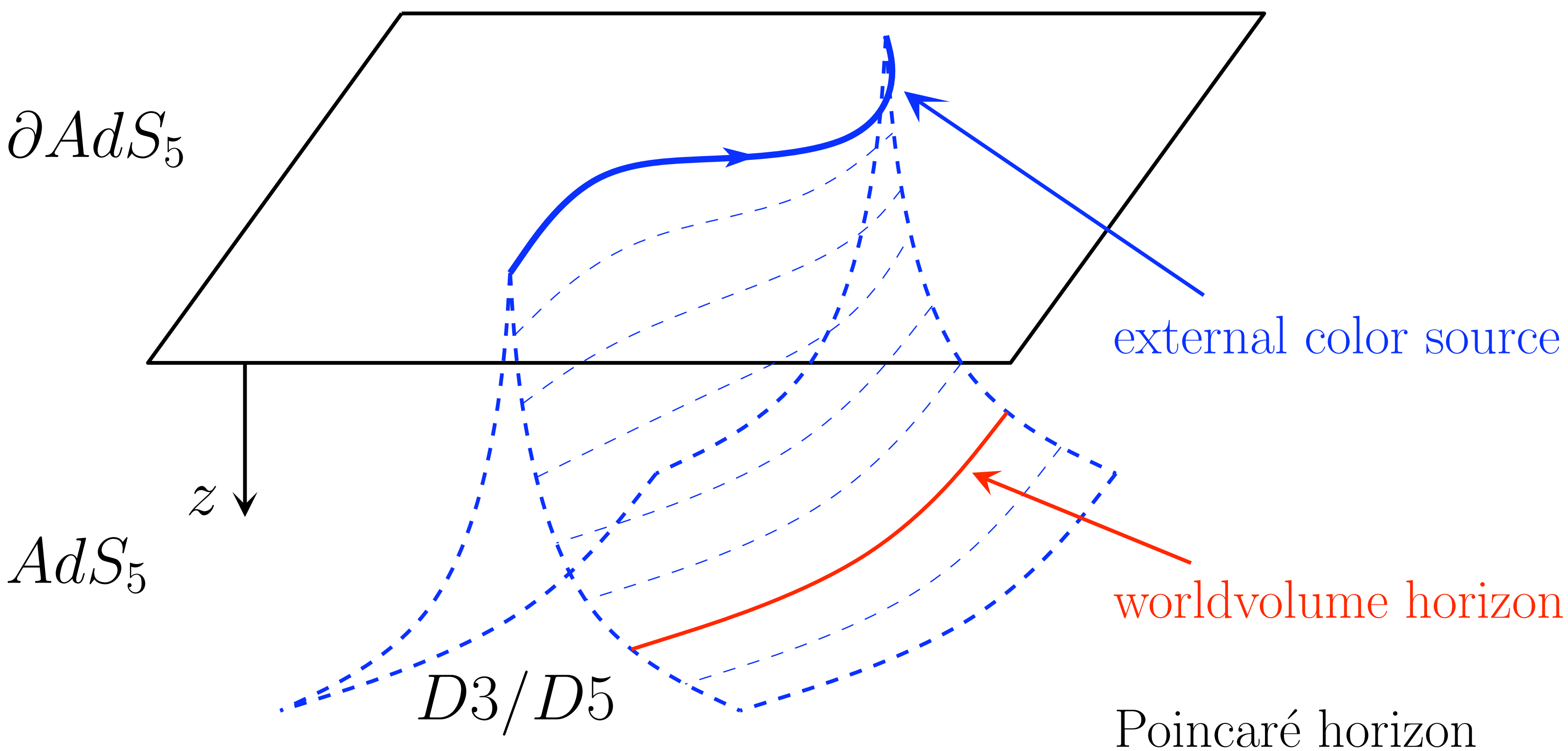
1/N Corrections with AdS/CFT

Probes in higher rank representations

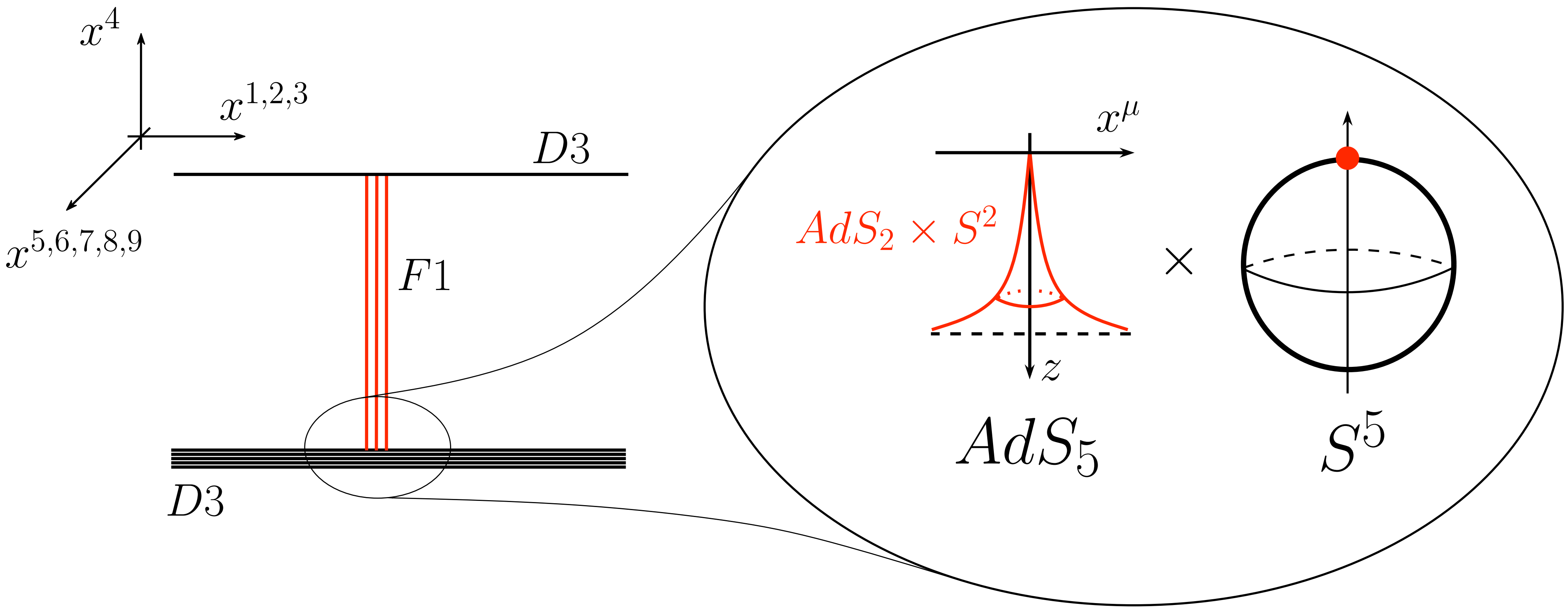
D3, k-units of flux ⟺ **k-symmetric rep.**

D5, k-units of flux ⟺ **k-antisymmetric rep.**

Hartnoll, Prem Kumar 06
Yamaguchi
Gomis, Passerini



First Example: static particle



Drukker, BF 05

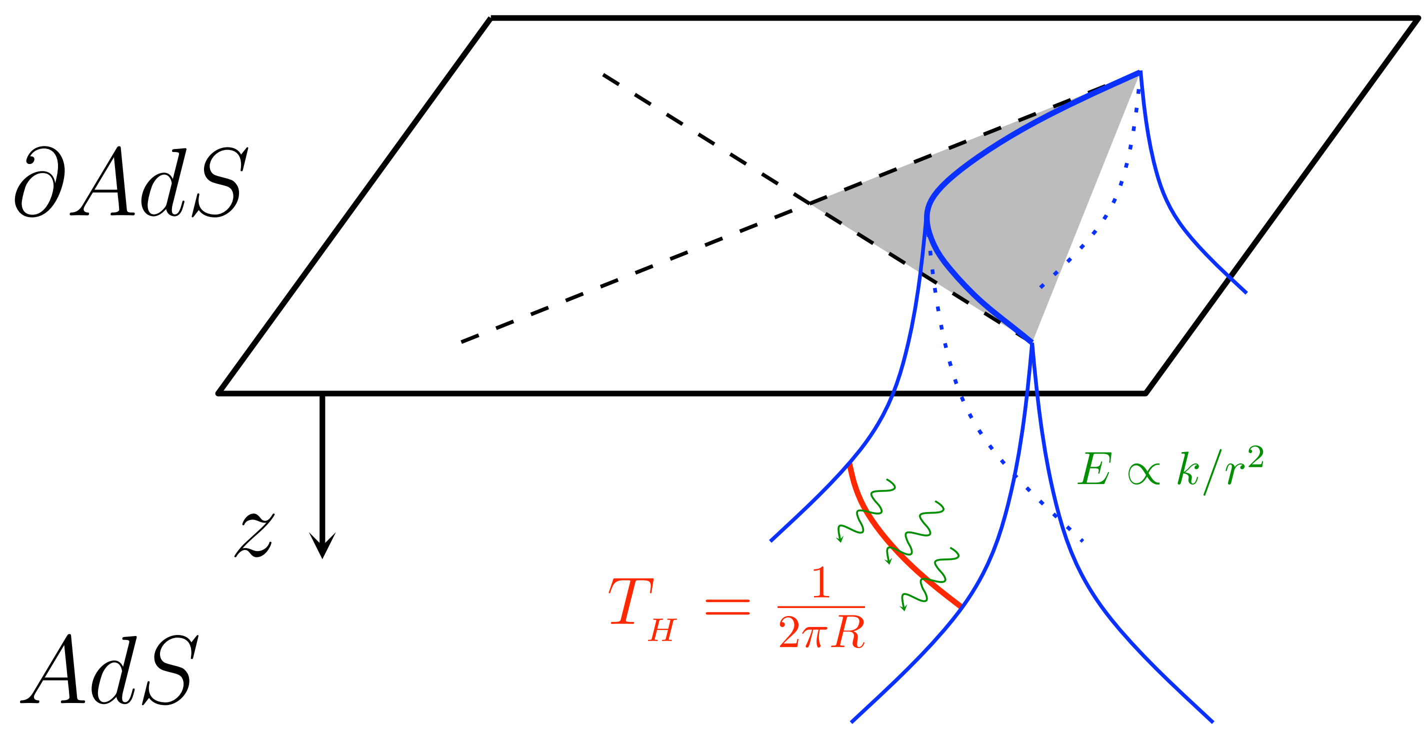
Again, the D3-brane backreacts on the AdS solution. From the linearized perturbations for the dilaton,

$$< \mathcal{L}(\vec{x}) >_{S_k} = \frac{k\sqrt{\lambda}\sqrt{1+\frac{k^2\lambda}{16N^2}}}{16\pi^2} \, \frac{1}{|\vec{x}|^4}$$

BF, Garolera, Lewkowycz 12

Second Example: accelerated particle

We can use a hyperbolic D3-brane to evaluate the energy loss by radiation in hyperbolic motion.



$$P = \frac{k\sqrt{\lambda}}{2\pi}\sqrt{1+\frac{k^2\lambda}{16N^2}}\; a^\mu a_\mu$$

BF, Garolera 11

Probes in k-symmetric representation

Static Particle

$$< \mathcal{L}(\vec{x}) >_{S_k} = \frac{k\sqrt{\lambda}}{16\pi^2}\sqrt{1+\frac{k^2\lambda}{16N^2}}\,\frac{1}{|\vec{x}|^4}$$

BF, Garolera, Lewkowycz 12

Accelerated Particle

$$P = \frac{k\sqrt{\lambda}}{2\pi}\sqrt{1+\frac{k^2\lambda}{16N^2}}\;a^\mu a_\mu$$

BF, Garolera 11

$$\kappa = 4\pi k\sqrt{\lambda}\sqrt{1+\frac{k^2\lambda}{16N^2}}\;T^3$$

BF, Garolera, Torrents 13

Circular Wilson loop

$$\ln < W(\bigcirc) > = \frac{k\sqrt{\lambda}}{2}\sqrt{1+\frac{k^2\lambda}{16N^2}} + 2N\sinh^{-1}\frac{k\sqrt{\lambda}}{4N}$$

Drukker, BF 05

Probes in k-symmetric representation

All these computations yield

$$B(\lambda, N) = \frac{k\sqrt{\lambda}}{4\pi^2}\sqrt{1+\frac{k^2\lambda}{16N^2}}$$

A priori, not justified to trust this result for k=1.

$$\underbrace{\frac{N^2}{\lambda^2} >> \overbrace{k}^{}}_{\text{probe approx.}}\overbrace{>> \frac{N}{\lambda^{3/4}}}^{\text{SUGRA approx.}}$$

Probes in k-antisymmetric representation

Universal result:

String world-sheet $\Sigma \hookrightarrow M$ (M Ricci flat: AdS, S-AdS,...)

↓

D5 world-volume $\Sigma \times S^4 \hookrightarrow M \times S^5$ Hartnoll 06

This amounts to $$\sqrt{\lambda} \to \frac{2N}{3\pi} \sin^3 \theta_k \sqrt{\lambda}$$

where $$\sin \theta_k \cos \theta_k - \theta_k = \pi(\frac{k}{N} - 1)$$

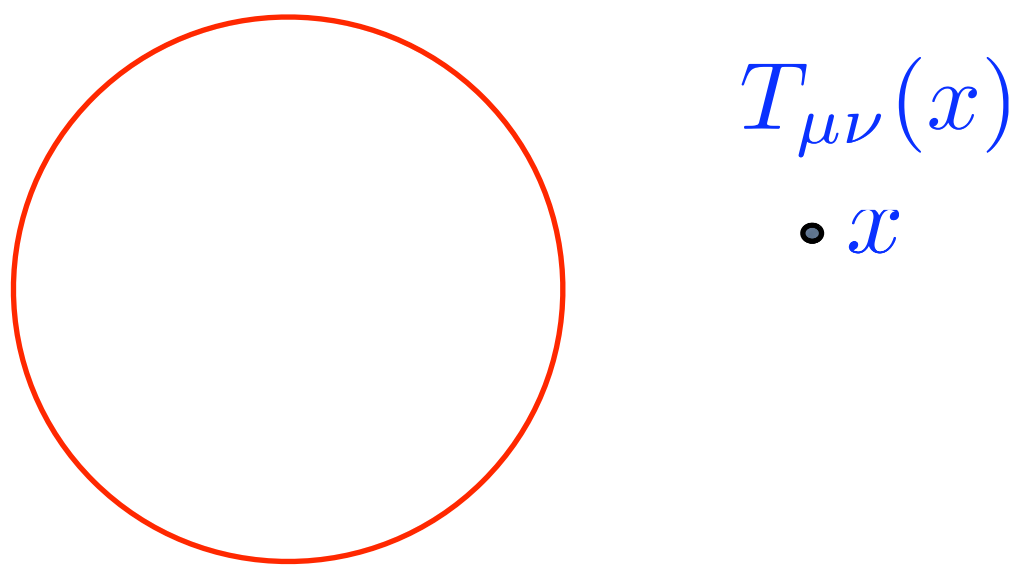
Valid even at finite temperature ! (e.g. drag force)

Exact Results for External Probes

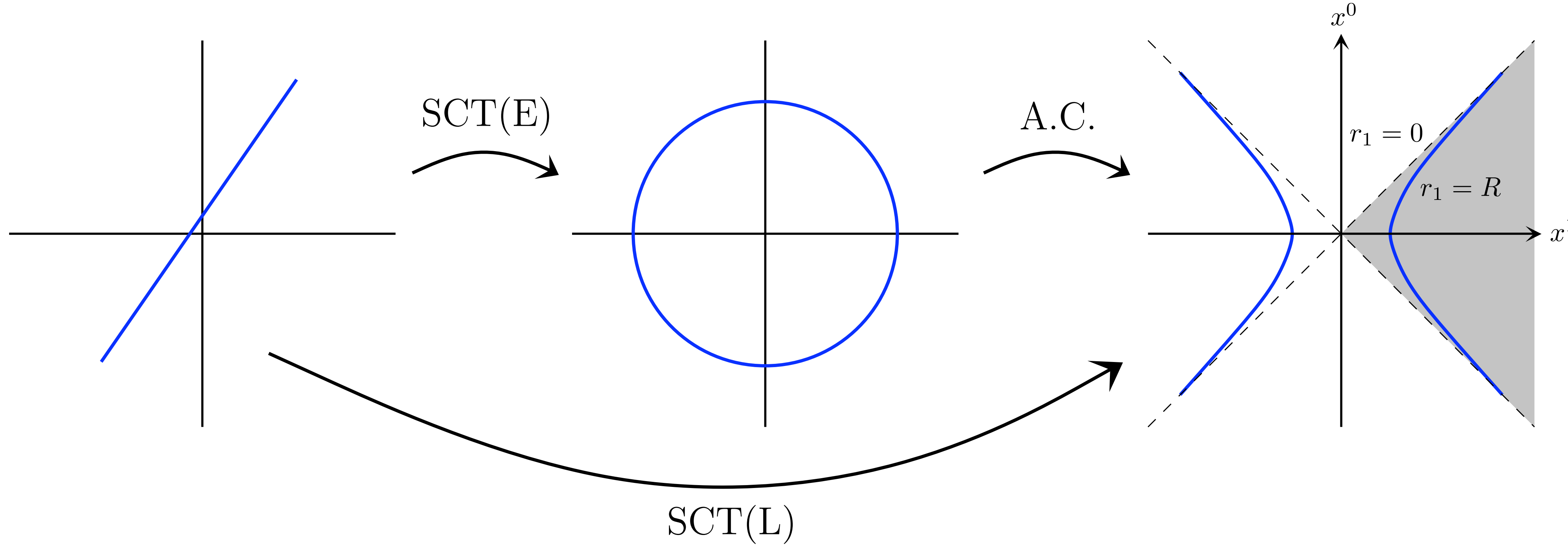
An exact Bremsstrahlung function

We will derive the Bremsstrahlung function for an electric 1/2 BPS probe in the fundamental rep. of N=4 U(N) SYM.

Our strategy: compute $< T_{\mu\nu} >_W = \frac{< T_{\mu\nu}(x) W_{\bigcirc} >}{< W_{\bigcirc} >}$



Start with <W>. Recall the Special Conformal Transformation,



$$< W_| > = 1 \qquad < W_\bigcirc > \neq 1$$

Conformal Anomaly !

The anomaly is localized at a point in space-time ⇒

It is perturbatively captured by a matrix model.

$$< W_{\bigcirc} >= \frac{1}{N} L^1_{N-1}\left(-\frac{\lambda}{4N}\right) e^{\frac{\lambda}{8N}}$$

Erickson, Semenoff, Zarembo 00

Drukker, Gross 00

Using localization techniques, Pestun proved the result to be correct, and exact.

Pestun 07

What about $< T_{\mu\nu}(x) W_{\bigcirc} >$ **?**

In $\mathcal{N}=4$ **SYM, the Lagrangian density and the stress energy tensor belong to a short multiplet, the supercurrent multiplet.**

$$\mathcal{O}_2 = \mathrm{Tr}\,\left(\Phi^{\{I}\Phi^{J\}}\right) \qquad T_{\mu\nu} \sim Q^2\bar{Q}^2\mathcal{O}_2 \qquad \mathcal{L} \sim Q^4\mathcal{O}_2$$

$< W_{\bigcirc}\mathcal{O}_2(x) >$ **is computed with a normal matrix model**

Okuyama, Semenoff 06

$$< W(\bigcirc)_{\square}\mathcal{O}_2 >= \frac{\sqrt{2}\lambda}{4N^3}\left[L^2_{N-1}\left(-\frac{\lambda}{4N}\right) + L^2_{N-2}\left(-\frac{\lambda}{4N}\right)\right] e^{\frac{\lambda}{8N}}$$

and finally we arrive at the Bremsstrahlung function for an electric 1/2 BPS probe in the fundamental rep. of N=4 U(N) SYM,

$$B_{U(N)}(\lambda, N) = \frac{\lambda}{16\pi^2 N} \frac{L_{N-1}^2\left(-\frac{\lambda}{4N}\right) + L_{N-2}^2\left(-\frac{\lambda}{4N}\right)}{L_{N-1}^1\left(-\frac{\lambda}{4N}\right)}$$

It is a rational function (why?)

Equivalently,

$$B = \frac{1}{2\pi^2}\lambda\partial_\lambda \log \; < W_{\bigcirc} >$$

Correa, Henn, Maldacena, Sever 12

B function for higher rank reps.

Strategy: Compute <W> exactly and use

BF, Torrents
work in progress

Define $$g = \frac{\lambda}{4N} \quad , \qquad A_{ij}(g) = L_i^{j-i}(-g)e^{g/2}$$

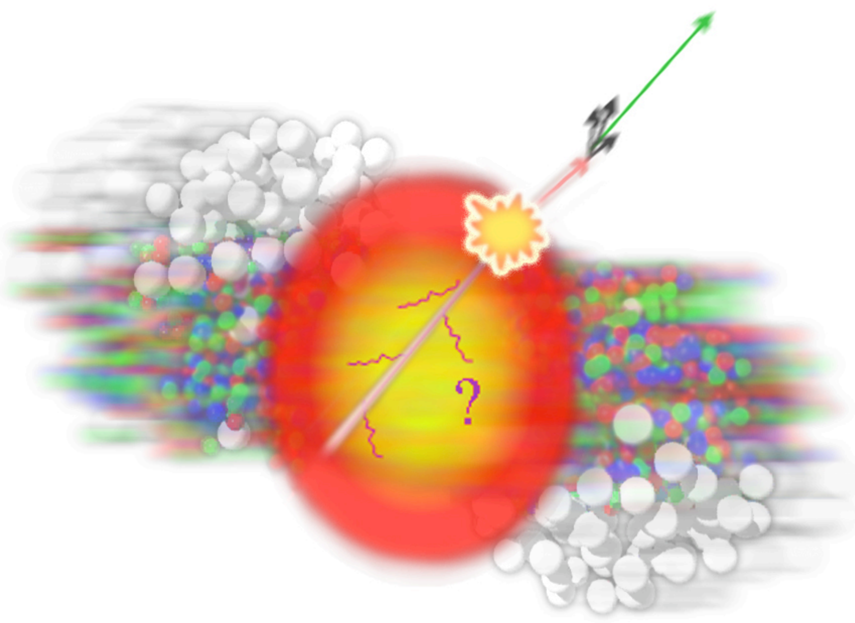
We find

$$< F_A(t) >= \sum_t t^{N-k} < W_{A_k}(g) >= |t + A(g)|$$

$$< W_{\mathcal{A}_k}(g) >= e^{\frac{kg}{2}} \sum_{j=0}^{k(N-k)} d_j \frac{g^j}{j!} \qquad d_j \in \mathbb{N}$$

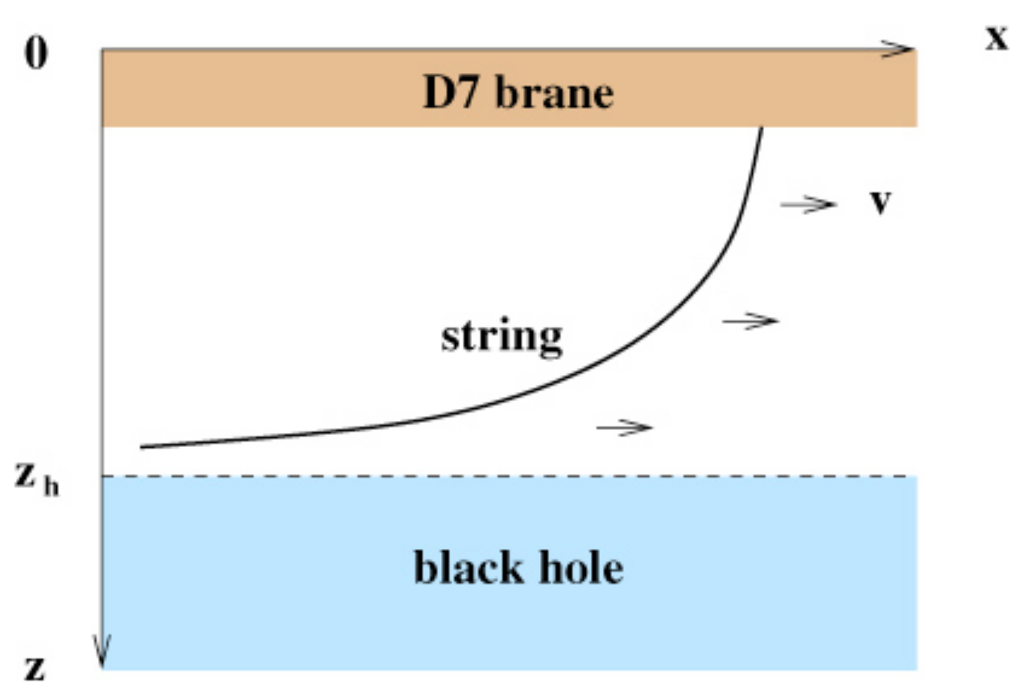
An application:
A benchmark for transport coefficients

Momentum broadening in Q.G.P.



$$\kappa = g(\lambda, N) T^3$$

Modelling QGP by N=4 SYM : trailing string



Herzog *et al.* **06**
Casalderrey-Solana, Teaney **06**
Gubser **06**

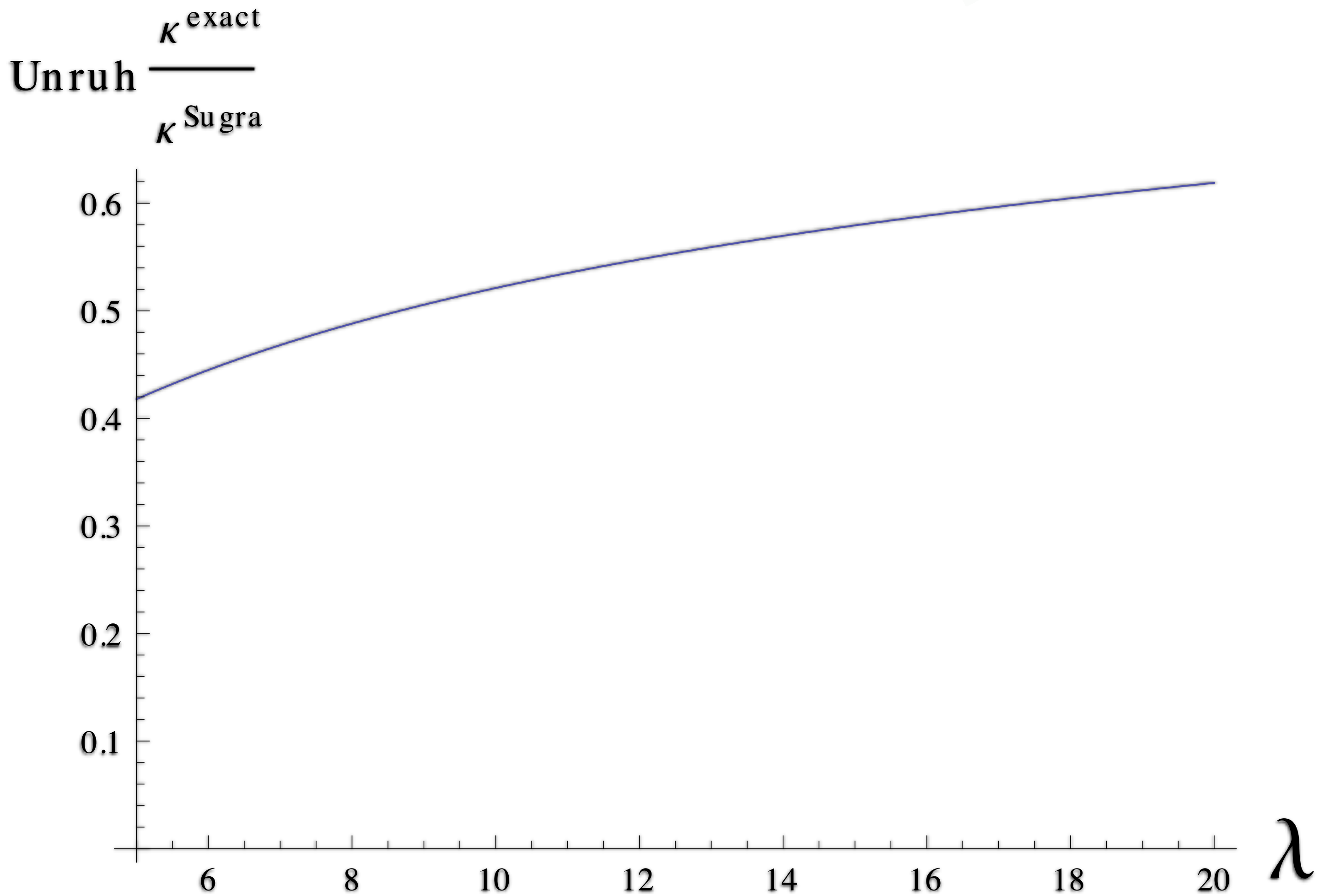
SUGRA ⟶ $\kappa = \pi \sqrt{\lambda} T^3$

SUGRA vs. exact results

SU(3), finite λ $$\kappa = 4\pi \frac{\lambda}{18} \frac{\lambda^2 + 144\lambda + 3456}{\lambda^2 + 72\lambda + 864} T^3$$

Large N, large λ $$\kappa = 4\pi\sqrt{\lambda}\, T^3$$

Unruh temperature



QGP-modelling range

Roughly, in this range

$$\kappa_{\text{Unruh}}^{\text{SUGRA}} \approx 2\, \kappa_{\text{Unruh}}^{\text{Exact}}$$

IF the same were true for κ_{thermal}

$$D_{\text{thermal}}^{\text{Exact}} \approx 2\, D_{\text{thermal}}^{\text{SUGRA}}$$

push towards QGP value ...

Conclusions and Outlook

The Bremsstrahlung function, the small angle limit of the cusp anomalous dimension, determines many properties of heavy probes coupled to CFTs.

Thanks to localization, the Bremsstrahlung functions of probes in various reps. of N=4 SU(N) SYM can be determined exactly via matrix model computations.

In the regime of validity of SUGRA, these results reduce to functions of $\sqrt{\lambda}/N$, and D-brane probe computations capture them precisely.

Conclusions and Outlook

- **Compute the full cusp anomalous dimension.**
- **Compute $B(\lambda, N)$ for other probes/CFTs.**
- **Finite mass ? (OK with localization).**

Adding Flavor

Karch, Katz 02

Schwinger effect and critical electric field

$$E_c = \frac{2\pi m^2}{\sqrt{\lambda}}$$

Semenoff, Zarembo 11

- **Beyond the vacuum state... Finite μ , T ?**