

 1

Regulation Theory

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Power Converters 2014
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 Introduction 2

Power converters:
Convert electric energy from one form to another that is optimally suited for user loads

⇒ Regulation is an important part of the design and construction of any power converter

Course objective:

- ✓ Recall common continuous-time control techniques
- ✓ Present digital control: Associated tools and more specifically the Z-transform, concept of discrete-time model, main methods to synthesize digital controllers, choice of the sampling frequency
- ✓ This course does not aim to describe control theory in a systematic and exhaustive way

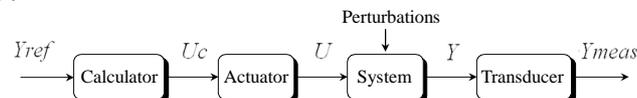
It will be limited to the control of single-input single-output linear time-invariant systems
Non-linear control theory will not be presented here

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- K. Aström ; B. Wittenmark, “Computer Controlled Systems: Theory and Design”, Prentice Hall, Englewood Cliffs, 1984, 1990
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- P. De Larminat, “Automatique: Commande des systèmes linéaires”, 2ème édition, Hermès, Paris, 1996
- A. Besançon-Voda ; S. Gentil, “Régulateurs PID analogiques et numériques”, Techniques de l’Ingénieur, R7416, Traité Informatique industrielle, 1999
- G. Alengrin, “Méthodes de synthèse de correcteurs numériques”, Techniques de l’Ingénieur, R7420, Traité Informatique industrielle, 1996
- F. Bordry, “Regulation theory: Review and digital regulation”, CERN Accelerator School and CLRC Daresbury Laboratory: Specialized CAS Course on Power Converters, Warrington, UK, 2004, pp. 275-298

- **Control theory**

- Interdisciplinary branch of engineering and mathematics that deals with the behavior of dynamical systems with inputs
- Usual objective: Provide the input(s) to a system to obtain the desired effect on its output(s)



If the system is a single-input single-output linear and time-invariant system, then its input and output are related by a differential equation with constant coefficients:

$$a_n \cdot \frac{d^n}{dt^n} y + \dots + a_0 \cdot y = b_0 \cdot u + \dots + b_m \cdot \frac{d^m}{dt^m} u \quad m \leq n \quad n = \text{system order}$$

= Time domain representation



Basic recalls

5

Taking the Laplace transform of both sides and assuming zero initial conditions:

$$a_n \cdot s^n Y + \dots + a_0 \cdot Y = b_0 \cdot U + \dots + b_m \cdot s^m U$$

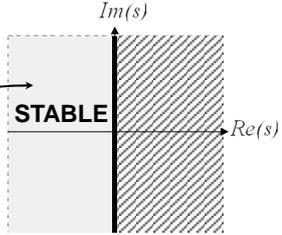
where s is the Laplace operator

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{\sum_{i=0}^m b_i \cdot s^i}{\sum_{j=0}^n a_j \cdot s^j} = \frac{Num(s)}{Den(s)} = H(s) \quad \text{H(s) = system transfer function}$$

= Frequency domain representation

NB :

- poles = roots of the polynomial $Den(s)$
- zeros = roots of the polynomial $Num(s)$
- Stability condition:
determined by the location of the poles of $H(s)$
- if $H(s)$ behaves like K/s^α for $s \rightarrow 0$: α = system class
 $K = H(0)$ = static gain
- System with delay:
 $u(t) \Rightarrow u(t - t_0)$ Reflects the fact that the input will act with a delay of t_0
 $H(s) \Rightarrow H(s) \cdot e^{-s \cdot t_0}$ = Laplace time shifting property



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Basic recalls

6

- **Why use feedback control ?**

What if we design a controller that equals the plant inverse:
 $C = H^{-1}$ (Feedforward correction)
 Then in theory (assuming a unit gain for the actuator TF): $Y = Y_{ref}$
 Impossible in practice because of:

- Uncertainty or variation of the model parameters
- Random disturbances



A feedback correction is needed

Advantages:

- Guaranteed performance even with model uncertainties
- Reduced sensitivity to parameter variations
- Disturbance rejection
- Stabilization of unstable open-loop systems

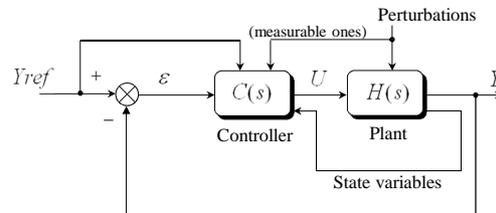
Drawback (contrary to feedforward correction):
 Reaction after the error has arisen \rightarrow Slowness

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- **Typical control structure**

= Combination of feedback and feedforward control



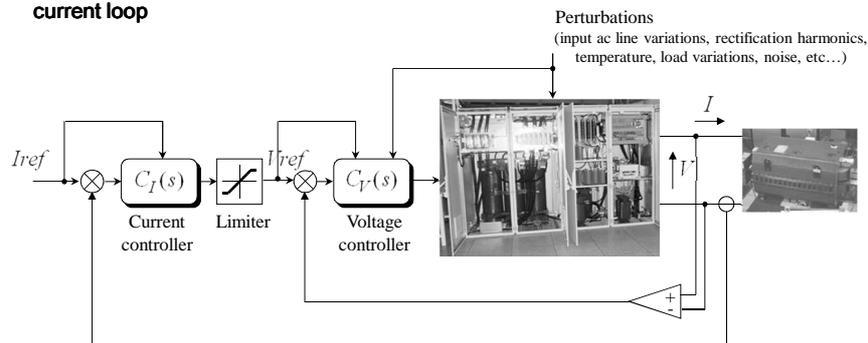
Closed control loop performance criteria:

- Static & dynamic behavior
- Stability & robustness

⇒ Translate into constraints on the frequency response of the compensated system in open-loop (= Controller + Plant)

- **Typical control structure for current-regulated power supplies**

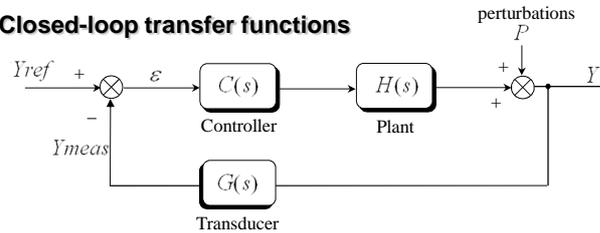
= Cascade structure: Nested connection of a fast inner voltage loop and a slower outer current loop



- **Fast inner voltage loop:** Acts as an active filter to reject the output voltage ripple and the output voltage fluctuations due to float of input mains + simplifies the design of the current loop

- **Outer current loop:** Ensures the overall stability of the power supply + provides an inherent over-voltage limitation

- Open-loop / Closed-loop transfer functions



Open-loop: $OL(s) = \frac{Y_{meas}(s)}{\varepsilon(s)} = C(s) \cdot H(s) \cdot G(s)$

Open-loop bandwidth: Interval of pulsataces for which $|C(j\omega) \cdot H(j\omega) \cdot G(j\omega)| > 1$

Closed-loop: $CL(s) = \frac{Y(s)}{Y_{ref}(s)} = \frac{C(s) \cdot H(s)}{1 + C(s) \cdot H(s) \cdot G(s)}$

Error vs. input & perturbation: $\varepsilon(s) = \varepsilon_{Y_{ref}}(s) + \varepsilon_P(s)$

$$\frac{\varepsilon_{Y_{ref}}(s)}{Y_{ref}(s)} = \frac{1}{1 + C(s) \cdot H(s) \cdot G(s)}$$

$$\frac{\varepsilon_P(s)}{P(s)} = -\frac{G(s)}{1 + C(s) \cdot H(s) \cdot G(s)}$$

NB: Defining transfer functions in Matlab → Function 'tf'

- Precision of closed-loop systems

- Static error

- Precision versus the input

Final value theorem: $\lim_{t \rightarrow \infty} \varepsilon_{Y_{ref}}(t) = \lim_{s \rightarrow 0} s \cdot \varepsilon_{Y_{ref}}(s) = \lim_{s \rightarrow 0} \frac{s}{1 + C(s) \cdot H(s) \cdot G(s)} \cdot Y_{ref}(s)$

=> To achieve zero steady-state error, we require

- at least 1 integrator (pole @ $s=0$) in the open-loop TF for a step input ($Y_{ref}(s) = K/s$)

- at least 2 integrators in the open-loop TF for a ramp input ($Y_{ref}(s) = K/s^2$)

- ...

- Sinusoidal input: $K \cdot \sin(\omega_0 \cdot t)$

At steady state, $\varepsilon_{Y_{ref}}$ is a harmonic signal which module $|\varepsilon_{Y_{ref}}|$ is such that

$$\frac{|\varepsilon_{Y_{ref}}|}{K} = \left| \frac{1}{1 + C(s) \cdot H(s) \cdot G(s)} \right|_{s=j\omega_0} \Rightarrow \text{if } \omega_0 \text{ is inside the OL bandwidth: } \frac{|\varepsilon_{Y_{ref}}|}{K} \approx \left| \frac{1}{C(s) \cdot H(s) \cdot G(s)} \right|_{s=j\omega_0}$$

Error amplitude inversely proportional to OL gain @ ω_0

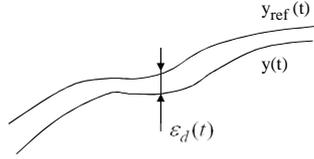
- Perturbation rejection

$$\lim_{t \rightarrow \infty} \varepsilon_P(t) = \lim_{s \rightarrow 0} s \cdot \varepsilon_P(s) = \lim_{s \rightarrow 0} \frac{-s \cdot G(s)}{1 + C(s) \cdot H(s) \cdot G(s)} \cdot P(s)$$

To reject disturbances of class N → at least N integrators in $C(s) \cdot H(s)$

- Precision of closed-loop systems**

- Dynamic error**



Pb: Limit ϵ_d to $\epsilon_{d\max}$

Assumption: Velocity v and acceleration γ of the input are limited

The input signal is then defined by the following constraints: $v < v_{\max}$ $\gamma < \gamma_{\max}$

It can be demonstrated that

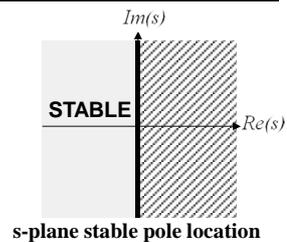
$$\epsilon_d < \epsilon_{d\max} \Rightarrow |OL(s)|_{s=j\frac{\gamma_{\max}}{v_{\max}}} > \frac{v_{\max}^2}{\gamma_{\max} \cdot \epsilon_{d\max}}$$

- Stability & robustness of closed-loop systems**

$$CL(s) = \frac{Y(s)}{Y_{ref}(s)} = \frac{C(s) \cdot H(s)}{1 + C(s) \cdot H(s) \cdot G(s)}$$

Closed-loop stability \Leftrightarrow

Real part of the closed-loop TF poles (= Roots of the characteristic equation $1 + C(s) \cdot H(s) \cdot G(s) = 0$) < 0



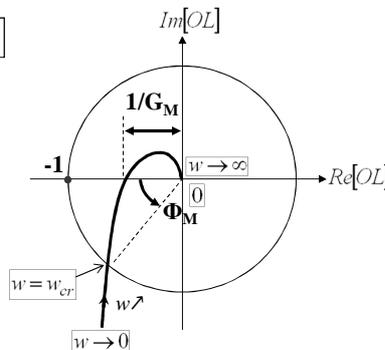
Phase Margin Φ_M : $\Phi_M = 180^\circ + \arg [OL(j \cdot w_{cr})]$

where w_{cr} is such that $|OL(j \cdot w_{cr})| = 1$

Gain margin G_M : $G_M = \frac{1}{|OL(j \cdot w_\pi)|}$

where w_π is such that $\arg [OL(j \cdot w_\pi)] = -180^\circ$

Typically: $30^\circ < \Phi_M < 60^\circ$, $G_M > 6\text{dB}$



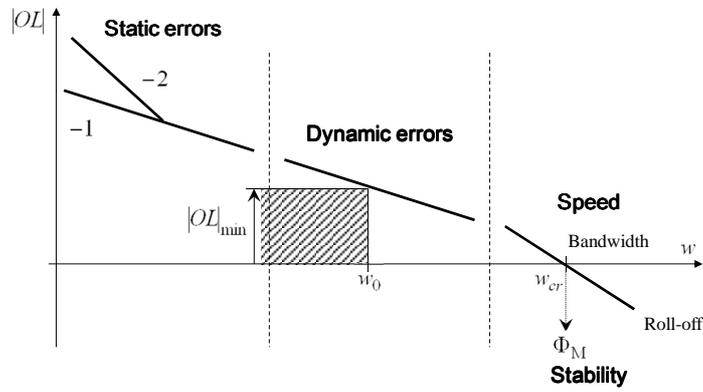


Analysis of closed-loop systems

13

- **Summary**

Open-loop TF



Contradiction between precision and stability \Rightarrow Tradeoff

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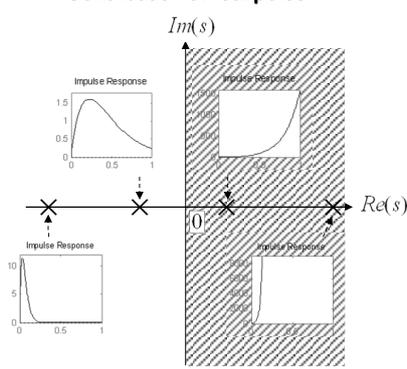


Analysis of closed-loop systems

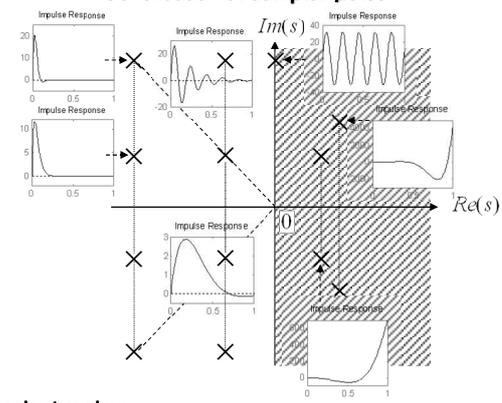
14

- **Influence of the poles on the transient behavior**

Contribution of real poles



Contribution of complex poles



NB: Poles farther to the left \rightarrow Faster transient regime
 \Rightarrow The poles closest to the imaginary axis are the ones that tend to dominate the response since their contribution takes a longer time to die out: Called dominant poles if the ratio of their real part to the one of any other poles $<$ typically 1/5
 Enables to simplify the TF by keeping the dominant pole(s) (and the static gain unchanged)

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- Particular case: 2nd order systems with complex conjugate poles

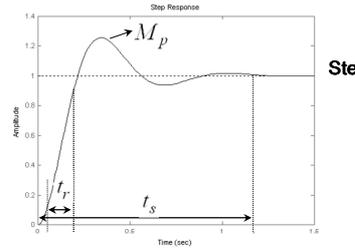
A common strategy for controller design consists to derive its parameters from a pole placement such that the closed-loop behaves like a 1st order or a 2nd order system

2nd order Σ : The design specifications imply constraints on

the dominant poles $p_1, \bar{p}_1 \Rightarrow$ on the cut-off frequency

ω_n and the damping ratio ζ of the TF CL_{des}

$$CL_{des}(s) = \frac{\omega_n^2}{s^2 + 2 \cdot \zeta \cdot \omega_n \cdot s + \omega_n^2}$$

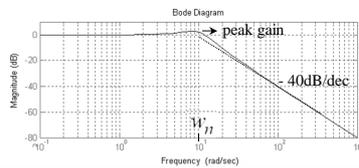


Step input response

- Rise time (10% \rightarrow 90%): $t_r \approx (2.6 \cdot \zeta^2 - 0.45 \cdot \zeta + 1.2) / \omega_n$
- Peak overshoot: $M_p = e^{-\pi \zeta / \sqrt{1 - \zeta^2}}$
- Settling time (to 1%): $t_s \approx 4.6 / \zeta \cdot \omega_n$

$$t_r, M_p, t_s \Rightarrow \zeta, \omega_n$$

ζ	M_p in %	Φ_M in degrees
0,1	73	11
0,3	37	33
0,5	16	52
0,7	4,6	65
0,9	0,15	73



NB: For 2nd order systems, a good phase margin guarantees a good gain margin

- Particular case: 2nd order systems

Influence of a zero

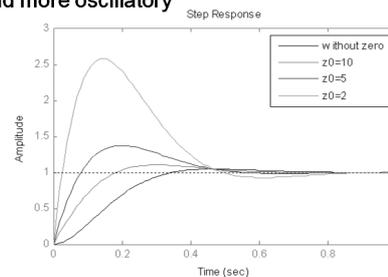
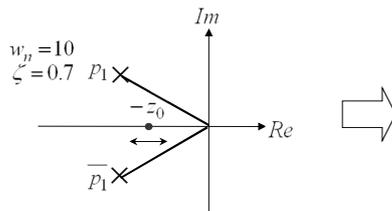
$$CL(s) = K \cdot \frac{s + z_0}{(s - p_1)(s - \bar{p}_1)} \quad \text{where } z_0 \in \mathbb{R}_-^* \text{ and } K = p_1 \cdot \bar{p}_1 / z_0 \quad (\rightarrow \text{unit static gain})$$

The unit step response of the above TF can be written as:

$$Y(s) = \frac{K}{s} \cdot \frac{s + z_0}{(s - p_1)(s - \bar{p}_1)} = \frac{K \cdot z_0}{s \cdot (s - p_1)(s - \bar{p}_1)} + \frac{K \cdot s}{s \cdot (s - p_1)(s - \bar{p}_1)}$$

$$\Rightarrow y(t) = y_{2^{nd} \text{ order}}(t) + \frac{1}{z_0} \cdot \frac{d}{dt} y_{2^{nd} \text{ order}}(t)$$

The additional zero makes the system faster and more oscillatory
= more prominent effect as z_0 decreases



Controller design process:

1. Specification of the desired closed control loop performance => Tradeoff (cf. above)
 - = Linked to plant dynamics & power availability of the actuator during transients (Prevent actuator saturation \Rightarrow Loss of controllability)
2. Choice of the controller type and its design method
3. Modelling of the plant to be controlled => Transfer function, state-space equations

To get the plant dynamic model:

1. Use physic laws to derive the differential equations used to represent it mathematically

Power converters:

- Construct equivalent averaged circuit model
- Determine large-signal averaged model
- Perturb and linearize about quiescent operating point to obtain small-signal averaged model
- Simplify the transfer function (by keeping the dominant poles)

If hysteretic control is to be used rather than PWM: Model this non-linear element using the 1st order harmonic approximation method \rightarrow Complex equivalent gain

2. Other method: Given generic model structure, estimate parameters from experimental data (= plant model identification)

Proportional-integral-derivative (PID) controller

- By far the most widely used control algorithm
- Involves only 3 separate constant parameters to tune the control loop
 - \rightarrow Simple and intuitive (many controllers do not even use derivative action = PI)
 - \rightarrow Well-suited for systems exhibiting dominant 1st or 2nd order behavior, for which the desired performance of the CL compared to the OL response of the Σ is not too demanding
 - \rightarrow For systems with higher order dominant dynamics, or systems including high delay or several oscillation modes, the PID is no longer adequate and a more complex regulator (with more parameters) has to be used

PID algorithm:

$$u(t) = K_p \cdot \varepsilon(t) + K_i \int_0^t \varepsilon(t) dt + K_d \cdot \frac{d}{dt} \varepsilon(t)$$

Controller TF standard form:

$$C_{PID}(s) = K_p + \frac{K_i}{s} + \frac{K_d \cdot s}{1 + \frac{K_d}{N \cdot K_p} \cdot s} \quad 10 \leq N_{typ} \leq 20$$

NB: Pure derivative amplifies noise \Rightarrow LP filter

Effects of PID tuning on closed control loop

Param. change	Rise time	Overshoot	Settling time	Steady-state error	System stability
Increase K_p	Decrease	Increase	Small change	Decrease	Degrade
Increase K_i	Decrease	Increase	Increase	Eliminate	Degrade
Increase K_d	Small change	Decrease	Decrease	No effect	Improve if K_d small



Continuous-time controller synthesis

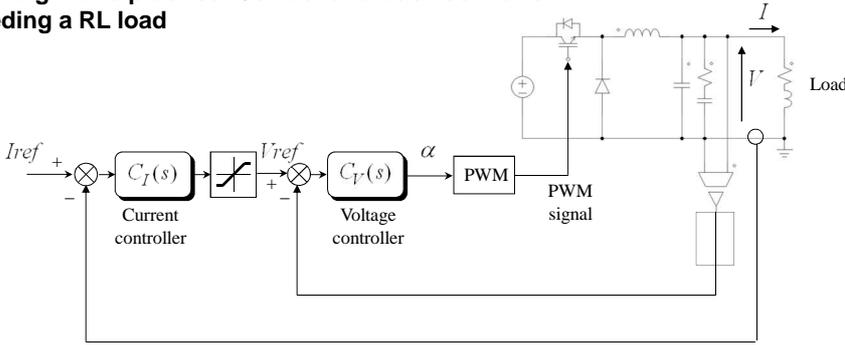
19

PID synthesis methods

A number of alternative approaches for PID tuning are available:

- Heuristic PID tuning procedures: Ziegler-Nichols, Cohen-Coon, ...
- Graphical methods: Loop shaping, root locus, ...
- Pole placement
- Minimization of integral type criterion
- ...

Putting it into practice: Control of a Buck converter feeding a RL load



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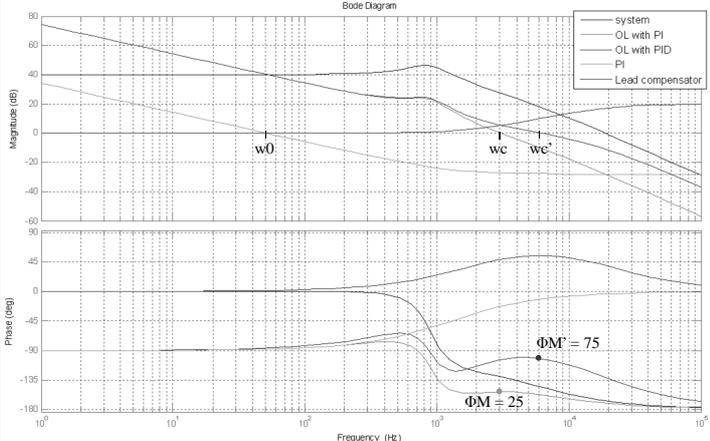


Continuous-time controller synthesis

20

Design of the voltage loop controller using the loop shaping method

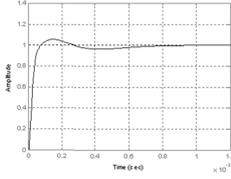
Loop shaping is one of the primary methodologies used for designing classical controllers such as PIDs => The controller structure and gains are selected such that the magnitude of the OL frequency response has particular characteristics - or a particular shape



Assume the following specs:

- 1/ Zero static error
- 2/ Dynamic precision: $|OL| > 40\text{dB} @ w_0$
- 3/ Bandwidth: $\{0, w_c\}$
- 4/ $\Phi_M \geq 50^\circ$

A PI controller is first tried for $C_V(s)$ but it leads to insufficient phase margin
 → A lead compensator is then added to correct Φ_M



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Design of the current loop controller using pole placement

Open current loop transfer function:

$$OL_T(s) = C_I(s) \cdot CL_T(s) \cdot \frac{b_0}{1 + a_1 \cdot s} \quad \text{where } b_0 = 1/R_{load} \quad a_1 = L_{load}/R_{load}$$

With a PI controller for $C_i(s)$ and after Σ simplification by keeping the dominant pole:

$$OL_T(s) \approx k_p \cdot \left(1 + \frac{k_i}{s}\right) \cdot \frac{b_0}{1 + a_1 \cdot s}$$

Setting $k_I = 1/a_1$ (pole cancellation) the closed loop can be written as

$$CL_T(s) = \frac{1}{1 + (a_1/k_p \cdot b_0) \cdot s}$$

Then choosing $k_p = a_1 \cdot \omega_n / b_0$ with $\omega_n = 2.2/t_r$, the CL behaves like a 1st order system with a rise time equal to t_r .

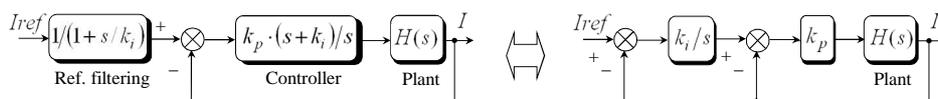
This pole cancellation method requires a good knowledge of the process. If a_1 is likely to vary ($L_{load} = f(I)$) and especially when the pole $-1/a_1$ is close to the origin, the following pole placement gives better results (Cf. next slide): $k_p = (a_1 \cdot 2 \cdot \xi \cdot \omega_n - 1) / b_0$ $k_i = a_1 \cdot \omega_n^2 / (k_p \cdot b_0)$

The CL then behaves like a 2nd order system which characteristic eq. = $s^2/\omega_n^2 + (2 \cdot \xi/\omega_n) \cdot s + 1$

Pb: This controller setting gives rise to a zero $-k_i$ in the CL TF which may affect the transient response => Solution to cancel this zero = filter the reference

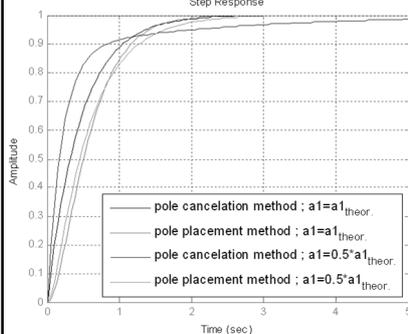
Design of the current loop controller using pole placement

Solution to cancel the zero of the CL TF:



IP controller

Results after zero cancellation, for $t_r = 1$, $\xi = 1$ (aperiodic behavior), and $-1/a_1 = -1/2$



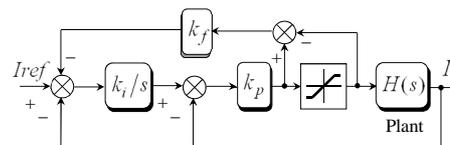
Integral windup issue:

When the output of the current or voltage controller reaches its saturation value, the integral part of the controller gets 'overcharged'

At the end of the saturated mode of operation, a negative error is needed to remove the accumulated positive error, which may give large transients

=> Anti-windup mechanism

Example:



Benefits and consequences of using digital control over analog

- Development of digital technology over the past 2 decades
=> Improvement in performance, cost and usability
 - Increasing demands for higher performance and monitoring capabilities
- } ⇒ Growing use of digital control in power converters

Benefits of using digital control:

- Performance enhancement, as digital control allows more complex regulation schemes (Ex: Non linear, predictive, adaptive control strategies, ...)
- Improved flexibility
- Better reliability and reproducibility (no ageing effects, thermal drifts, ...)
- Better noise immunity
- Provides system monitoring and archiving capability
- More compact and lightweight
- Implementation of human-machine interface & external communication requires some kind of embedded processor
- ...

One major issue: Time delays introduced to do computations of control algorithm in the processor

Other drawbacks = Aliasing, quantization errors, limit cycling, software bugs, ...

NB: Interesting alternative = Mix of analog and digital

Z transform = Major mathematical tool for analysis in such topics as digital control and digital signal processing

- **Reminder:** The Laplace transform $X(s)$ of a continuous-time causal signal $x(t)$ is given by

$$X(s) = \int_{0^-}^{+\infty} x(t) \cdot e^{-st} \cdot dt \quad s = \sigma + j \cdot \omega \quad \text{such that } X(s) \text{ converges}$$

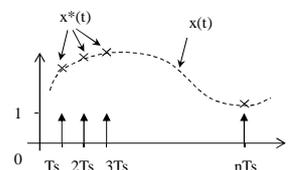
- **Case of discrete-time causal signals:**

$$x^*(t) = \sum_{k=0}^{+\infty} x(k) \cdot \delta(t - k \cdot Ts) \quad x(k) = x(t) \Big|_{t=k \cdot Ts}$$

{x(k)}, k ∈ Z: Sequence of sampled values (= 0 ∀ k < 0)

Ts: Sampling period (assumed constant)

δ: Dirac delta function



- ⇒ **The Laplace transform $X^*(s)$ of a discrete-time signal $x^*(t)$ is given by**

$$X^*(s) = \sum_{k=0}^{+\infty} x(k) \cdot e^{-s \cdot k \cdot Ts} \quad (1)$$

Not a polynomial form...

With the change of variable $z = e^{sTs}$ in eq. (1), we derive the following expression
= definition of the Z-transform:

$$X(z) = \sum_{k=0}^{+\infty} x(k) \cdot z^{-k} \quad \forall z \in \mathbb{C} \text{ for which } X(z) \text{ converges}$$

=> Takes the form of a polynomial of the complex variable z

The Z-transform is the discrete-time counter-part of the Laplace transform
 => Essential tool for the analysis and design of discrete-time systems

Interpretation of the variable z^{-1}

From Laplace time shifting property, we know that e^{-sTs} is time delay by T s second
 Therefore $z^{-1} = e^{-sTs}$ corresponds to unit sample period delay

- **Properties of Z-transforms**

- **Linearity** $Z[\lambda \cdot x(k) + \mu \cdot y(k)] = \lambda \cdot X(z) + \mu \cdot Y(z)$

- **Shifting property** $Z[x(k-n)] = z^{-n} \cdot X(z)$

- **Convolution** $Z[x(k) * y(k)] = Z\left[\sum_{n=-\infty}^{+\infty} x(n) \cdot y(k-n)\right] = X(z) \cdot Y(z)$

- **Multiply by k property** $Z[k \cdot x(k)] = -z \cdot \frac{d}{dz}(X(z))$

- **Final value** $\lim_{k \rightarrow \infty} x(k) = \lim_{z \rightarrow 1} (z-1) \cdot X(z)$

• **Examples of Z-transforms**

- **Discrete impulse** $x(k) = \delta(k) = \begin{cases} 1 & k=0 \\ 0 & k \neq 0 \end{cases}$

$$X(z) = \sum_{k=0}^{\infty} x(k) \cdot z^{-k} = x(0) + x(1) \cdot z^{-1} + x(2) \cdot z^{-2} + \dots = x(0)$$

$$\Rightarrow X(z) = 1$$

- **Discrete step** $x(k) = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$

$$X(z) = x(0) + x(1) \cdot z^{-1} + x(2) \cdot z^{-2} + \dots = 1 + z^{-1} + z^{-2} + \dots$$

$$\Rightarrow X(z) = \frac{1}{1-z^{-1}} \quad |z| > 1$$

- **Discrete ramp** $x(k) = \begin{cases} k & k \geq 0 \\ 0 & k < 0 \end{cases}$

$$X(z) = -z \cdot \frac{d}{dz} \left(\frac{1}{1-z^{-1}} \right) \Rightarrow X(z) = \frac{z^{-1}}{(1-z^{-1})^2} \quad |z| > 1$$

• **Z-transform Table**

Time function	Laplace Transform	Discrete Time function	Z transform
$\delta(t)$	1	$\delta(nT)$	1
$u(t)$	$\frac{1}{s}$	$u(nT)$	$\frac{z}{z-1}$
t	$\frac{1}{s^2}$	nT	$\frac{zT}{(z-1)^2}$
$\frac{t^2}{2}$	$\frac{1}{s^3}$	$\frac{(nT)^2}{2}$	$\frac{z(z+1)T^2}{2(z-1)^3}$
e^{-at}	$\frac{1}{s+a}$	e^{-anT}	$\frac{z}{z-e^{-aT}}$
$t e^{-at}$	$\frac{1}{(s+a)^2}$	$nT e^{-anT}$	$\frac{zT e^{-aT}}{(z-e^{-aT})^2}$
$a^{t/T}$	$\frac{1}{s - (1/T) \ln(a)}$	a^n	$\frac{z}{z-a} \quad (a > 0)$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$	$\sin(\omega nT)$	$\frac{z \sin(\omega T)}{z^2 - 2z \cos(\omega T) + 1}$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$	$\cos(\omega nT)$	$\frac{z^2 - z \cos(\omega T)}{z^2 - 2z \cos(\omega T) + 1}$

X(s) → X(z) ?

- **Case of signals having only simple poles**

$$X(s) = \sum_{i=1}^N \frac{A_i}{s - s_i} \quad \Rightarrow x(t) = \sum_{i=1}^N A_i \cdot e^{s_i t} \quad t \geq 0$$

By sampling $x(t)$, we obtain the following discrete sequence

$$x(k) = \sum_{i=1}^N A_i \cdot e^{s_i k T_s} \quad k \geq 0$$

From line 5 of the Z-transform table: $X(z) = \sum_{i=1}^N \frac{A_i \cdot z}{z - e^{s_i T_s}}$

$$\Rightarrow X(s) = \sum_{i=1}^N \frac{A_i}{s - s_i} \xrightarrow{Z} X(z) = \sum_{i=1}^N \frac{A_i}{1 - e^{s_i T_s} \cdot z^{-1}} \quad (2)$$

⇒ A pole s_i in $X(s)$ yields a pole $z_i = e^{s_i T_s}$ in $X(z)$

$$s_i \xrightarrow{Z} z_i = e^{s_i T_s}$$

- **General case**

$$X(z) = \sum_{s_i = \text{poles of } X(s)} \text{Residues} \left\{ X(s) \cdot \frac{1}{1 - e^{s_i T_s} \cdot z^{-1}} \right\}_{s=s_i} \quad (3)$$

Calculation of the residue at the pole s_j of multiplicity m :

$$\text{Residue} \left\{ X(s) \cdot \frac{1}{1 - e^{s_i T_s} \cdot z^{-1}} \right\}_{s=s_j} = \frac{1}{(m-1)!} \cdot \lim_{s \rightarrow s_j} \frac{d^{m-1}}{ds^{m-1}} \left[(s - s_j)^m \cdot X(s) \cdot \frac{1}{1 - e^{s_i T_s} \cdot z^{-1}} \right]$$

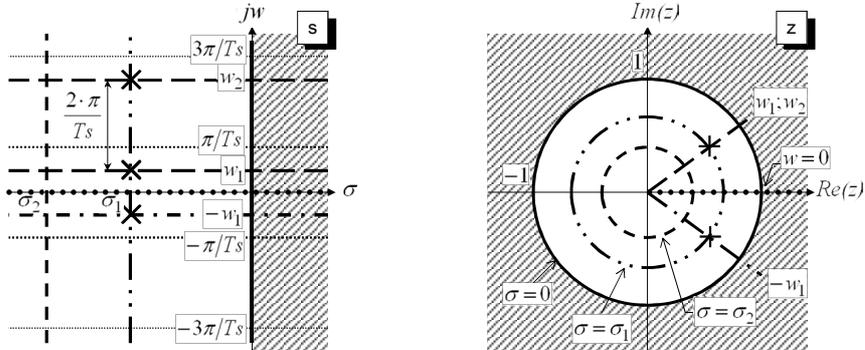
For a simple pole ($m = 1$):

$$\text{Residue} \left\{ X(s) \cdot \frac{1}{1 - e^{s_i T_s} \cdot z^{-1}} \right\}_{s=s_j} = \lim_{s \rightarrow s_j} (s - s_j) \cdot X(s) \cdot \frac{1}{1 - e^{s_i T_s} \cdot z^{-1}}$$

An example of calculation will be given further on in this document

• Mapping from s-plane to z-plane

Since $z_i = e^{s_i T_s} = e^{\sigma_i T_s} \cdot e^{j\omega_i T_s}$ we can map the s-plane to the z-plane as below:

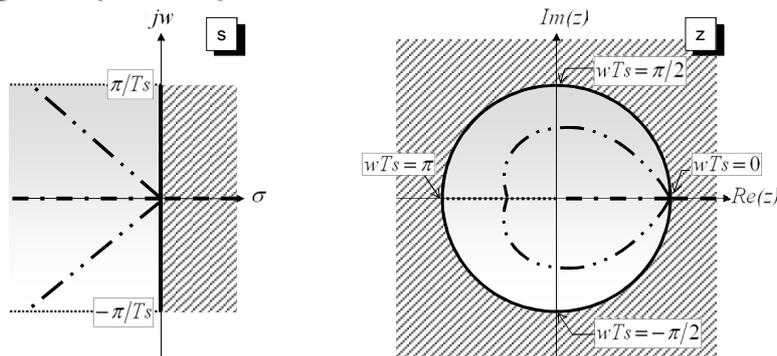


NB: 2 poles in the s-plane which imaginary part differ by $2\pi/T_s$ map to the same pole in the z-plane

Bijective mapping between both planes $\Rightarrow \text{Im}(X(s)) \in [-\pi/T_s; +\pi/T_s]$

$$\Rightarrow \frac{\pi}{T_s} > \max_i \{ |\text{Im}(s_i)| \}$$

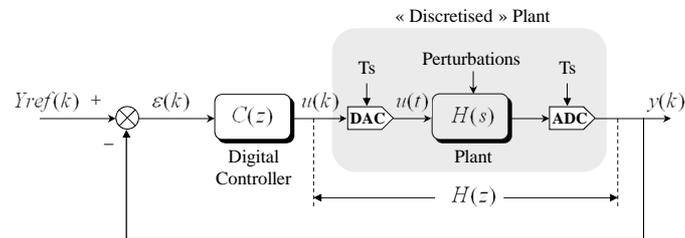
• Mapping from s-plane to z-plane



Legend:

—	$s = j\omega$	$ z = 1$
- - - -	$s = \sigma \geq 0$	$z = r, r \geq 1$
- . . . -	$s = \sigma \leq 0$	$z = r, 0 \leq r \leq 1$
- : : -	$\begin{cases} s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \\ \zeta = \text{Constant}, \omega_n \text{ varies} \end{cases}$	Logarithmic spiral
.....	$s = \sigma \pm j\pi/T_s$	$z = -r$

- Modelling of the plant to be controlled



Model of the DAC = Zero-order hold (ZOH)

→ Converts $u(k)$ to $u(t)$ by holding each sample value for one sample interval

$$u(t) = u(k), \quad k \cdot Ts \leq t \leq (k+1) \cdot Ts$$

=> Delay introduced by the ZOH = $Ts/2$

The Laplace transform transfer function of the ZOH is

$$H_{ZOH}(s) = \frac{1 - e^{-sTs}}{s}$$

$$\Rightarrow H(z) = (1 - z^{-1}) \cdot Z \left[\frac{H(s)}{s} \right]$$

If $H(s)$ has poles $s = s_i$, then $H(z)$ has poles $z = e^{s_i Ts}$. But the zeros are unrelated

Calculation of $Z \left[\frac{H(s)}{s} \right]$:

- Partial fraction decomposition + use z-transform table

- If $\frac{H(s)}{s}$ has only simple poles, use Eq. (2):

$$\frac{H(s)}{s} = \frac{A_1}{s} + \sum_{i=2}^N \frac{A_i}{s - s_i} \quad \xrightarrow{Z} \quad X(z) = \frac{A_1}{1 - z^{-1}} + \sum_{i=2}^N \frac{A_i}{1 - e^{s_i Ts} \cdot z^{-1}}$$

- Use Eq. (3): $H(z) = \sum_{s_i = \text{poles of } H(s)} \text{Residues} \left\{ \frac{H(s)}{s} \cdot \frac{1}{1 - e^{s_i Ts} \cdot z^{-1}} \right\}_{s=s_i}$

- $H(z) \rightarrow$ Ask Matlab: Function 'c2d'

Syntax

`sysd = c2d(sys,Ts)`

Description

`sysd = c2d(sys,Ts)` discretizes the continuous-time LTI model `sys` using zero-order hold on the inputs and a sample time of `Ts` seconds.

Let $Y(z) = Z[y(k)]$; $Y_{ref}(z) = Z[y_{ref}(k)]$; $E(z) = Z[\varepsilon(k)]$; $U(z) = Z[u(k)]$

• **Open-loop transfer function:** $\frac{Y(z)}{E(z)} = C(z) \cdot H(z)$

• **Closed-loop transfer function:** $\frac{Y(z)}{Y_{ref}(z)} = \frac{C(z) \cdot H(z)}{1 + C(z) \cdot H(z)}$

• **Controller algorithm:**

Transfer function of the digital controller

$$C(z) = \frac{U(z)}{E(z)} = \frac{b_0 + b_1 \cdot z^{-1} + \dots + b_p \cdot z^{-p}}{1 + a_1 \cdot z^{-1} + \dots + a_n \cdot z^{-n}}$$

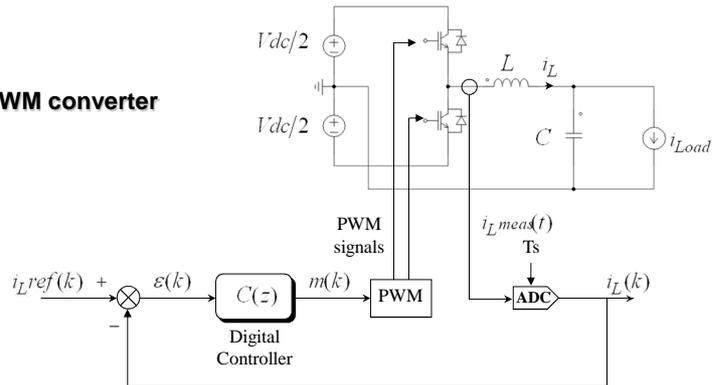
$$\Rightarrow (1 + a_1 \cdot z^{-1} + \dots + a_n \cdot z^{-n}) \cdot U(z) = (b_0 + b_1 \cdot z^{-1} + \dots + b_p \cdot z^{-p}) \cdot E(z)$$

Hence $u(k) + a_1 \cdot u(k-1) + \dots + a_n \cdot u(k-n) = b_0 \cdot \varepsilon(k) + b_1 \cdot \varepsilon(k-1) + \dots + b_p \cdot \varepsilon(k-p)$
($z^{-n} = z^{-n} \cdot X(z)$)

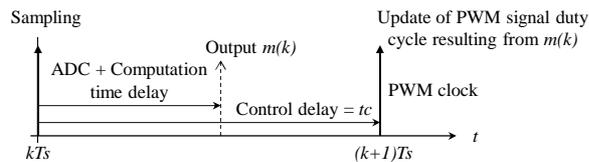
$$\Rightarrow u(k) = b_0 \cdot \varepsilon(k) + b_1 \cdot \varepsilon(k-1) + \dots + b_p \cdot \varepsilon(k-p) - a_1 \cdot u(k-1) - \dots - a_n \cdot u(k-n)$$

= Difference equation, where present output is dependent on present input and past inputs and outputs

Example:
Half bridge PWM converter



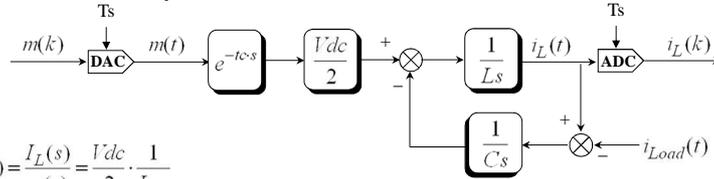
Controller output = modulation index ($-1 < m < 1$)



H(z) ?

Modelling of the PWM converter

The PWM can be modelled by a DAC



$$H(s) = \frac{I_L(s)}{m(s)} = \frac{V_{dc}}{2} \cdot \frac{1}{L \cdot s}$$

$$\Rightarrow H(z) = (1 - z^{-1}) \cdot Z\left[\frac{H(s)}{s}\right] = (1 - z^{-1}) \cdot \frac{V_{dc}}{2 \cdot L} \cdot Z\left[\frac{e^{-tc \cdot s}}{s^2}\right] = z^{-n} \cdot (1 - z^{-1}) \cdot \frac{V_{dc}}{2 \cdot L} \cdot Z\left[\frac{e^{\theta \cdot s}}{s^2}\right] \quad \text{with } tc = n \cdot Ts - \theta, 0 \leq \theta < Ts$$

Using Eq. (3), with a double pole:

$$Z\left[\frac{e^{\theta \cdot s}}{s^2}\right] = \frac{1}{(2-1)!} \cdot \lim_{s \rightarrow 0} \frac{d^{2-1}}{ds^{2-1}} \left[(s-0)^2 \cdot \frac{e^{\theta \cdot s}}{s^2} \cdot \frac{1}{1 - e^{sTs} \cdot z^{-1}} \right]$$

$$= \lim_{s \rightarrow 0} \frac{d}{ds} \left[e^{\theta \cdot s} \cdot \frac{1}{1 - e^{sTs} \cdot z^{-1}} \right] = \frac{\theta \cdot (1 - z^{-1}) + Ts \cdot z^{-1}}{(1 - z^{-1})^2}$$

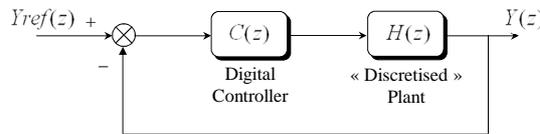
In this example $n=1$ and $\theta=0$

$$\Rightarrow H(z) = \frac{V_{dc} \cdot Ts}{2 \cdot L} \cdot \frac{z^{-2}}{1 - z^{-1}}$$

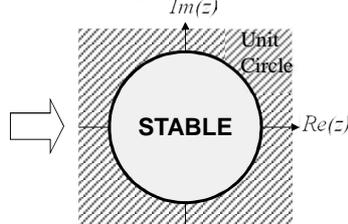
System behavior & stability

= determined by the roots of the closed-loop TF polynomials

- **Stability of digital closed-loops**



Mapping s-plane to z-plane



z-plane stable pole location

Closed-loop transfer function:

$$CL(z) = \frac{Y(z)}{Y_{ref}(z)} = \frac{C(z) \cdot H(z)}{1 + C(z) \cdot H(z)}$$

Closed-loop stability \Leftrightarrow

Modulus of the closed-loop TF poles (= Roots of the characteristic equation $1 + C(z) \cdot H(z) = 0$) < 1



Analysis of digital closed-loops

39

- **Robustness**

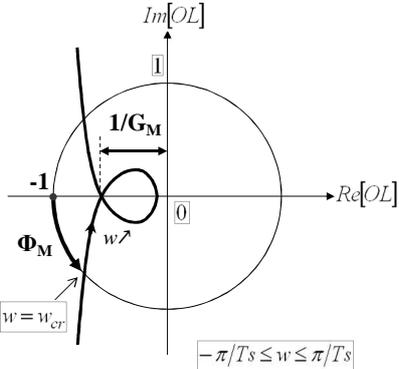
Open-loop transfer function: $OL(z) = C(z) \cdot H(z)$

Phase margin Φ_M : $\Phi_M = 180^\circ + \arg [OL(e^{j\omega_{cr}Ts})]$

where ω_{cr} is such that $|OL(e^{j\omega_{cr}Ts})| = 1$

Gain margin G_M : $G_M = \frac{1}{|OL(e^{j\omega_\pi Ts})|}$

where ω_π is such that $\arg [OL(e^{j\omega_\pi Ts})] = -180^\circ$



NB: Matlab plots & margins → Functions 'nyquist', 'bode', 'margin'

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Analysis of digital closed-loops

40

- **Influence of the poles on the transient behavior**

Illustration with the step response analysis of a system CL(z) having only simple poles:

$$y(k) = CL(1) + \sum_{\substack{i=1 \\ \text{real} \\ \text{poles}}}^n c_i \cdot z_i^k + \sum_{\substack{j=1 \\ \text{complex} \\ \text{conjugate} \\ \text{poles}}}^m |c_j| \cdot |z_j|^k \cdot \cos(k \cdot \theta_j + \varphi_j)$$

- **Steady-state:** $\lim_{k \rightarrow \infty} y(k) = CL(1)$
- **Contribution of real poles z_i**

Sum of exponential terms $\begin{cases} \xrightarrow[k \rightarrow \infty]{} 0 & \text{if } |z_i| < 1 \\ \xrightarrow[k \rightarrow \infty]{} \pm \infty & \text{if } |z_i| > 1 \end{cases}$
- **Contribution of complex conjugate poles z_j**

Oscillating regime $\begin{cases} \xrightarrow[k \rightarrow \infty]{} 0 & \text{if } |z_j| < 1 \Rightarrow \text{Damped oscillations} \\ \xrightarrow[k \rightarrow \infty]{} \pm \infty & \text{if } |z_j| > 1 \Rightarrow \text{Undamped oscillations} \end{cases}$

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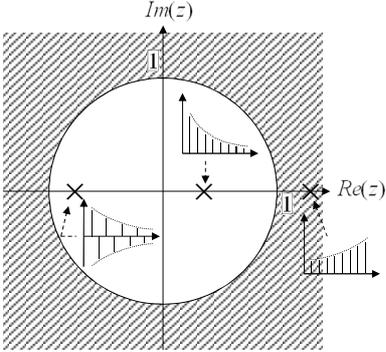


Analysis of digital closed-loops

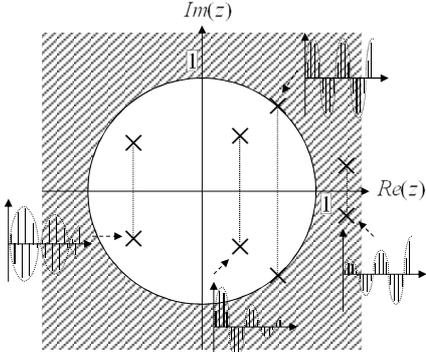
41

- **Influence of the poles on the transient behavior**

Contribution of real poles



Contribution of complex poles



NB: Poles closer to origin → Faster transient regime

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Analysis of digital closed-loops

42

- **Particular case: 2nd order systems**

As seen previously, a common controller design method consists to derive the controller parameters from a pole placement such that the dominant closed-loop dynamics is of second order

Reminder: Continuous-time theory

The design specifications imply constraints on the cut-off frequency w_n and the damping ratio ζ

Discrete closed-loops:
Pole mapping from s-plane to z-plane:

$$Den_{L_{des}}(s) = (s^2 + 2 \cdot \zeta \cdot w_n \cdot s + w_n^2) \cdot P_{aux}(s)$$

Dominant poles Additional poles, faster

Rise time (10% → 90%):
 $t_r \approx (2.6 \cdot \zeta^2 - 0.45 \cdot \zeta + 1.2) / w_n$

Peak overshoot: $M_p = e^{-\pi \cdot \zeta / \sqrt{1 - \zeta^2}}$

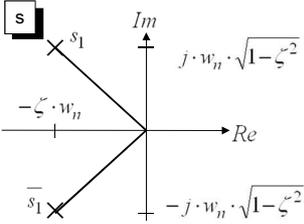
Settling time (to 1%): $t_s \approx 4.6 / \zeta \cdot w_n$

$s_1 = -w_n \cdot (\zeta - j \cdot \sqrt{1 - \zeta^2}) \Rightarrow z_1 = e^{-w_n T s (\zeta - j \cdot \sqrt{1 - \zeta^2})}$

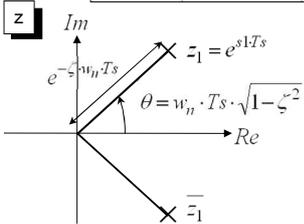
$t_r, M_p \Rightarrow \zeta, w_n \Rightarrow z_1, \bar{z}_1$

$t_s = 4.6 / \zeta \cdot w_n$
 $\Rightarrow |z_1| < e^{-4.6 T s / t_s} \approx 0.01 T s / t_s$
=> condition for Ts

s



z



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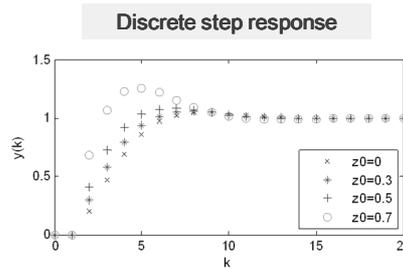
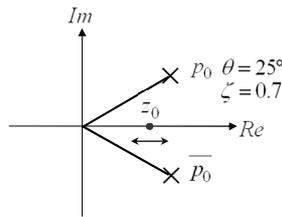
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- Particular case: 2nd order systems

Influence of a zero

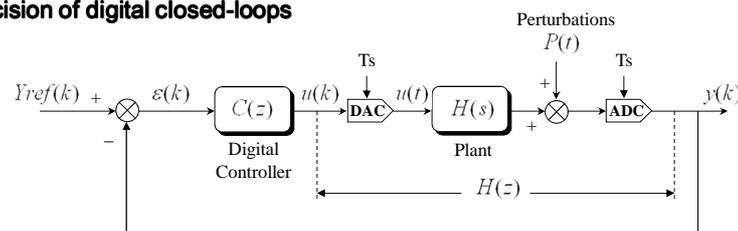
$$CL(z) = K \cdot \frac{z - z_0}{(z - p_0)(z - \overline{p_0})}$$

where $z_0 \in \mathbb{R}$ and $K = (1 - p_0)(1 - \overline{p_0}) / (1 - z_0)$
 (→ unit static gain)



- Increasing overshoot when the zero is moving towards +1 → Take care...
- The reference tracking performance can be improved by designing appropriate zeros in the closed-loop transfer function

- Precision of digital closed-loops



Same conclusions as for continuous-time closed-loops

- Precision versus the input
 To achieve zero steady-state error, we require
 → at least 1 integrator (pole @ $z = 1$) in the open-loop TF $C(z) \cdot H(z)$ for a step input
 → at least 2 integrators in the open-loop TF for a ramp input
 → ...
- Perturbation rejection
 To reject disturbances of class N → at least N integrators in $C(z) \cdot H(z)$

45

SOLEIL
SYNCHROTRON

Discrete-time controller synthesis

2 main ways to synthesize discrete-time controllers:

1. Emulation design

```

graph LR
    A((Continuous-time design)) --> B(Controller discretization)
    B --> C((Digital implementation))
  
```

2. Direct discrete-time design

```

graph LR
    A((Plant discretization)) --> B(Controller design)
    B --> C((Digital implementation))
  
```

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46

SOLEIL
SYNCHROTRON

Discrete-time controller synthesis

- Emulation design**
 - 1st step: Continuous-time controller design.** At this stage the sampling is ignored (But the impact on the phase margin of the control delay & anti-aliasing filter should preferably be taken into account → preserve stability margin)
 - 2nd step: Discretization of the continuous-time controller (Followed by simulations to check performance)**
 ≠ Methods:
 - Approximate s , i.e. $C(s) \rightarrow C(z)$
 - Pole-zero matching
 - 3rd step: Derivation of the controller algorithm (difference equation)**

Approximation methods:

- Euler $s \rightarrow \frac{1}{Ts} \cdot (1 - z^{-1})$
- Tustin's or bilinear approximation $s \rightarrow \frac{2}{Ts} \cdot \frac{1 - z^{-1}}{1 + z^{-1}}$

Example: Discretization of a PI controller using Tustin's approximation

$$C(s) = K_P \cdot \left(1 + \frac{1}{T_i \cdot s} \right) \quad \square \quad C(z) = C(s) \Big|_{s = \frac{2}{Ts} \frac{1 - z^{-1}}{1 + z^{-1}}} = K_P \cdot \left(\frac{1 + Ts/(2 \cdot T_i) + (-1 + Ts/(2 \cdot T_i)) \cdot z^{-1}}{1 - z^{-1}} \right)$$

Matlab: `sysd = c2d(sys,Ts,'tustin')`

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Discrete-time controller synthesis

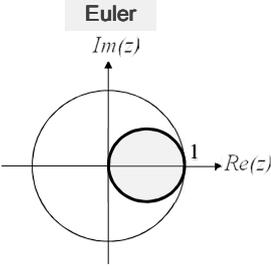
47

- **Comparison between Euler and Tustin's approx.**

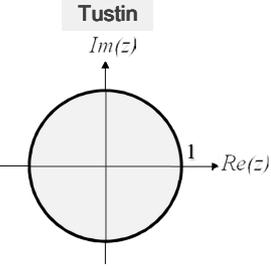
- Stability
= preserved

Half-plane
 $Re[s] < 0$

Euler



Tustin



An unstable continuous-time system can be mapped to a stable discrete system

Perfect correspondence

- Mapping of the poles

Euler: $s - s_0 \rightarrow (z \cdot (1 - s_0 \cdot Ts) - 1) / Ts \cdot z \Rightarrow z_0 = 1 / (1 - s_0 \cdot Ts) = 1 + (s_0 \cdot Ts) + (s_0 \cdot Ts)^2 + (s_0 \cdot Ts)^3 + \dots$

Tustin: $s - s_0 \rightarrow (z \cdot (2 - s_0 \cdot Ts) - (2 + s_0 \cdot Ts)) / Ts \cdot (z + 1)$

$\Rightarrow z_0 = (1 + s_0 \cdot Ts / 2) / (1 - s_0 \cdot Ts / 2) = 1 + (s_0 \cdot Ts) + \frac{1}{2} \cdot (s_0 \cdot Ts)^2 + \frac{1}{4} \cdot (s_0 \cdot Ts)^3 + \dots$

To be compared to $z_0 = e^{s_0 \cdot Ts} = 1 + (s_0 \cdot Ts) + \frac{1}{2} \cdot (s_0 \cdot Ts)^2 + \frac{1}{6} \cdot (s_0 \cdot Ts)^3 + \dots$ ↗ ≠ poles

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Discrete-time controller synthesis

48

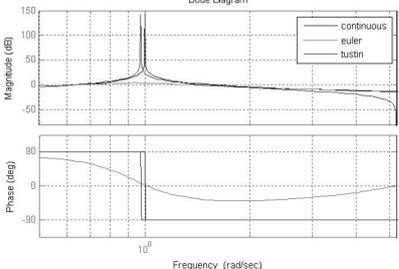
- **Comparison between Euler and Tustin's approx.**

- Pole and zero locations not preserved → Frequency response is changed
- Increasing the sampling frequency → Smaller approximation errors

Example 1: $C(s) = \frac{s}{1 + s^2}$ TF with resonance @ $w_0 = 1 \text{ rad/s}$

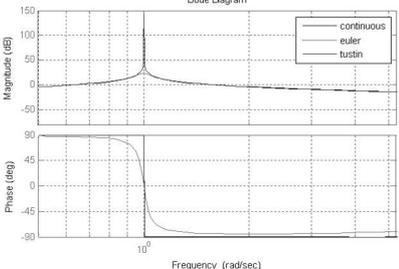
Ts = 0.6

Bode Diagram



Ts = 0.06

Bode Diagram

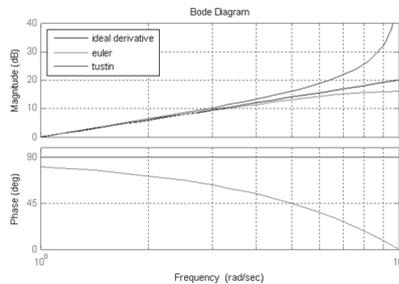


=> Better result with Tustin

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- Comparison between Euler and Tustin's approx.

Example 2: Ideal derivative



Euler: Filtering effect @ high frequencies

Tustin:

Magnitude $\rightarrow \infty$ when $\omega \rightarrow \pi/T_s$

Noise amplification @ high frequencies

=> Euler more appropriate for discretization of high-pass filters

=> Tustin more appropriate for discretization of low-pass filters

- Other discretization method = Matched transform

Matlab:

`sysd = c2d(sys,Ts,'matched')`

$$C(s) = K \cdot \frac{\prod_j (s - r_j)}{\prod_j (s - \sigma_j)} \xrightarrow{Z} C(z) = Kc \cdot \frac{\prod_j (z - e^{r_j T_s})}{\prod_j (z - e^{\sigma_j T_s})} \quad Kc \text{ is set so to obtain the same static gain}$$

No frequency distortion => Well-adapted for the discretization of transfer functions including resonances (ex: notch filter, ...)

- Direct discrete-time design

- A system controlled using an emulation controller always suffer performance degradation compared with its continuous-time counter-part

- To reduce the degree of degradation, very fast sampling can be needed, as

{ADC – Digital controller – DAC} should behave the same as the analogue controller (generally PID type = very simple control algorithm)



Bad use of the potentialities of the digital controller

In this case, direct discrete-time design offers an alternative solution, since in this design the sampling is considered from the beginning of the design process

1st step: Discretization of the continuous-time plant

2nd step: Choice of controller type and synthesis methodology

3rd step: Derivation of the controller algorithm (difference equation)

- Choice of the sampling period

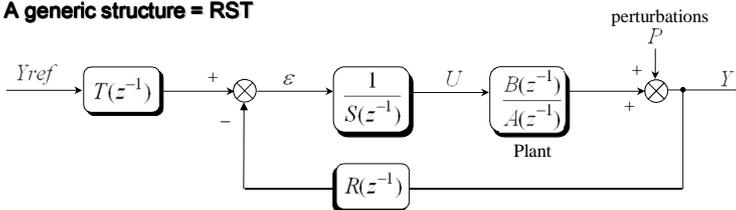
- T_s too small => Fast and expensive control hardware
 - => Numerical issues: Recall the relation between poles in s-domain and z-domain: $z_i = e^{s_i T_s}$ => For $T_s \rightarrow 0$ we have $z_i \rightarrow 1 \quad \forall s_i$
 - > Makes trouble when working in finite precision
 - => Systems with control delays that are not multiples of the sampling period: Plant discretization may bring about unstable zeros
 - > Limitation on the possible method to compute the regulator
 - => If the sampling frequency of the outer current loop is small enough in comparison with the inner voltage loop bandwidth: No need to include the digital model of the voltage source -> Lower order controller, less complexity
- T_s too large => Loss of information, aliasing (violation of the sampling theorem)
 - => Regulation may not react enough readily to disturbances affecting the Σ
 - => Plant discretization may give birth to poles having negative real part: Not desirable as the step response caused by such poles cycles back and forth between positive and negative deviations from the steady-state value
 - To prevent this: $\text{Re}[z_i] > 0 \Rightarrow 1/T_s > 2/\pi \cdot |\text{Im}[s_i]$

Rule of thumb:

Choice of T_s based on the closed-loop bandwidth F^{CL}_B $\frac{1}{T_s} = (6 \text{ to } 25) \cdot F^{CL}_B$

- RST controller structure

- Digital control -> Enables implementation of new controller structures
- A generic structure = RST

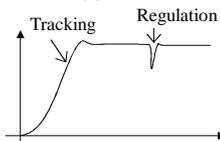


R, S and T are 3 polynomials to be determined (usually by pole placement)

- The control signal is calculated as
$$U(z) = \frac{T(z^{-1})}{S(z^{-1})} \cdot Y_{ref}(z) - \frac{R(z^{-1})}{S(z^{-1})} \cdot Y(z)$$

=> Combination of Feedforward and Feedback that can be tuned separately

- General approach with RST: Decouple the regulation pb from the tracking pb



Polynomials R and S

Polynomial T

- RST controller structure

- Closed-loop system equation:

$$Y = \frac{B \cdot T}{A \cdot S + B \cdot R} \cdot Y_{ref} + \frac{A \cdot S}{A \cdot S + B \cdot R} \cdot P$$

- Synthesis of the RST control law using pole-zero placement:

1st step: Choose arbitrarily a desired closed-loop transfer function or model

$$CL_{des}(z^{-1}) = B_m(z^{-1}) / A_m(z^{-1})$$

2nd step: Cancel poles and zeros of the plant TF (stable ones)

$$A(z^{-1}) = A^-(z^{-1}) \cdot A^+(z^{-1}) \quad \Rightarrow \quad R(z^{-1}) = A^+(z^{-1}) \cdot R_1(z^{-1})$$

$$B(z^{-1}) = B^-(z^{-1}) \cdot B^+(z^{-1}) \quad \Rightarrow \quad S(z^{-1}) = B^+(z^{-1}) \cdot S_1(z^{-1})$$

$A^-(z^{-1})$: "non-compensable poles" = unstable & poorly damped poles + poles with negative real part

$B^-(z^{-1})$: "non-compensable zeros" = unstable & poorly damped zeros + plant pure delay z^{-d} ($d \geq 1$) + zeros having negative real part

Cannot be a factor of $A \cdot S + B \cdot R$ (\Rightarrow CL unstable), thus: $B_m = B^- \cdot B_{m_1}$

3rd step: Perturbation rejection in steady-state

The open-loop TF must contain the classes of perturbation \Rightarrow Introduce an appropriate nb of integral terms in the controller by means of the polynomial S

$$S_1(z^{-1}) = (1 - z^{-1})^{n-l} \cdot S_2(z^{-1}) \quad n: \text{perturbation class} \quad l: \text{plant class}$$

- RST controller structure

- Synthesis of the RST control law (cont'd)

4th step: Compute R and S = Solve the following Diophantine equation such that the poles of the CL regulated system are in the required position

$$A^-(z^{-1}) \cdot (1 - z^{-1})^{n-l} \cdot S_2(z^{-1}) + B^-(z^{-1}) \cdot R_1(z^{-1}) = A_m(z^{-1})$$

5th step: Compute T

$$T(z^{-1}) = A^+(z^{-1}) \cdot B_{m_1}(z^{-1})$$

NB: To ensure unity gain to the CL TF, we must have $T(1) = R(1)$

Particular case: System with stable zeros (apart from the pure delay z^{-1} systematically present = consequence of the ZOH) $\Rightarrow B^- = z^{-1}$

Then the tracking TF is $\frac{Y}{Y_{ref}} = \frac{B \cdot T}{A \cdot S + B \cdot R} = \frac{z^{-1} \cdot B^+ \cdot A^+ \cdot B_{m_1}}{A^+ \cdot A^- \cdot B^+ \cdot S_1 + z^{-1} \cdot B^+ \cdot A^+ \cdot R_1} = \frac{z^{-1} \cdot B_{m_1}}{A_m} = \frac{B_m}{A_m}$

Choosing $B_{m_1}(z^{-1}) = A_m(z^{-1})$, thus $T(z^{-1}) = A^+(z^{-1}) \cdot A_m(z^{-1})$

$$\Rightarrow \frac{Y}{Y_{ref}} = z^{-1}$$

NB: Requires an accurate modelling of the system \rightarrow Identification may be necessary. Moreover, if the Σ parameters are likely to vary (ex: magnet saturation $\rightarrow L_{load} = f(I)$), adaptive control may be required

• RST controller structure

• Synthesis of the RST control law (cont'd)

Choice of A_m

Degree(A_m) \leq degree of the first member of the Diophantine eq.

If CL desired behavior = 2nd order Σ : $A_m(z^{-1}) = (1 - z_1 \cdot z^{-1}) \cdot (1 - \bar{z}_1 \cdot z^{-1}) \cdot P_{aux}(z^{-1})$

where $\begin{cases} z_1, \bar{z}_1 = e^{-\zeta \cdot \omega_n \cdot Ts} \cdot e^{\pm j \cdot \omega_n \cdot Ts \cdot \sqrt{1 - \zeta^2}} = \text{dominant poles} \\ P_{aux}(z^{-1}) \text{ contains the } \{\text{degree}(A_m) - 2\} \text{ remaining poles } z_i, \\ \text{chosen for example such that } |z_i| \leq 0.1 (\Rightarrow \text{fast transient response as compared to the one due to } z_1, \bar{z}_1) \end{cases}$

• PID = Special case of RST controller

$$\begin{aligned} Num_{PID}(z^{-1}) &= R(z^{-1}) = r_0 + r_1 \cdot z^{-1} + r_2 \cdot z^{-2} & T(z^{-1}) &= R(z^{-1}) \\ Den_{PID}(z^{-1}) &= S(z^{-1}) = (1 - z^{-1}) \cdot (1 + s_1 \cdot z^{-1}) \end{aligned}$$

• IP controller

$$\begin{aligned} Num_{PI}(z^{-1}) &= R(z^{-1}) = r_0 + r_1 \cdot z^{-1} & T(z^{-1}) &= R(1) = r_0 + r_1 \\ Den_{PI}(z^{-1}) &= S(z^{-1}) = 1 - z^{-1} \end{aligned}$$

Polynomial T = simple gain chosen to ensure unity gain to the closed-loop TF

• RST controller structure

Example: Design a RST for the current loop of the previously presented Buck converter

System model:

$$\frac{B(z^{-1})}{A(z^{-1})} = CL_V(z^{-1}) \cdot \frac{z^{-n}}{R_{load}} \cdot \frac{(e^{-a \cdot Ts} - e^{-a \cdot Ts}) \cdot z^{-1} + 1 - e^{-a \cdot Ts}}{1 - z^{-1} \cdot e^{-a \cdot Ts}}$$

$CL_V(z^{-1}) = \text{Voltage closed-loop TF}$

$$a = R_{load} / L_{load}$$

$tc = n \cdot Ts - \theta = \text{Control delay}$

Assume $1/Ts \ll \text{voltage CL bandwidth}$, $Ts \ll 1/a$, $tc \ll Ts$

$$\Rightarrow \frac{B(z^{-1})}{A(z^{-1})} \approx \frac{b_0 \cdot z^{-1}}{1 - z^{-1}} \quad b_0 = \frac{Ts}{L_{load}}$$

Analog signals often sampled @ a rate $> 1/Ts$:
If cut-off freq. of anti-aliasing filter $\gg F_{CLB}$,
no need to take it into account in the model

Diophantine equation:

$$\begin{aligned} (1 - z^{-1})^2 + b_0 \cdot z^{-1} \cdot (r_0 + r_1 \cdot z^{-1}) &= (1 - z_1 \cdot z^{-1}) \cdot (1 - \bar{z}_1 \cdot z^{-1}) \\ \Rightarrow r_0 &= \frac{2 - (z_1 + \bar{z}_1)}{b_0} \quad r_1 = \frac{z_1 \cdot \bar{z}_1 - 1}{b_0} \end{aligned}$$

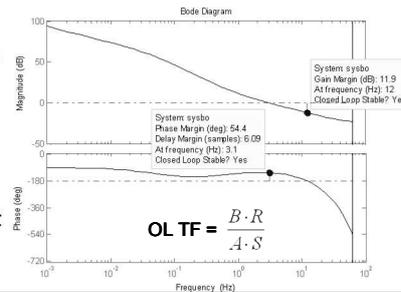
Choice for T:

- If no tracking requirement: $T(z^{-1}) = r_0 + r_1$

\Rightarrow No undesirable zero in the CL TF = no overshoot

- Fast tracking required:

$$T(z^{-1}) = (1 - z_1 \cdot z^{-1}) \cdot (1 - \bar{z}_1 \cdot z^{-1}) / b_0$$



A lot more... But time is over

Thank you for your attention

Questions?