The Geometry Of Supersymmetric Partition Functions

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Outline:

- Introduction and motivation
- Defining $\mathcal{N} = 1$ supersymmetric theories on $\mathcal{M}_4$
- What does $Z_{\mathcal{M}_4}$ depend on ?
- Example: $\mathcal{M}_4 = S^3 \times S^1$
- 3d $\mathcal{N} = 2$ supersymmetric theories on $\mathcal{M}_3$
- Example: $\mathcal{M}_3 = S^3$
- Outlook

Based on:

- C.C., T. Dumitrescu, G.Festuccia, Z. Komargodski, 1309.5876
- C.C., I. Shamir, to appear

Supersymmetry : recent developments

We all appreciate the beauty and importance of supersymmetry as a simplifying assumption in our study of quantum field theories.

Many recent developements stem from considering supersymmetric QFT's on curved space, in particular on compact Riemannian manifolds.

What does it buy us?

- The partitition function $Z_{\mathcal{M}}$ (the path integral) on $\mathcal{M}$ is somewhat better defined. And interesting by itself.
- $Z_{\mathcal{M}}$ can often be computed *exactly* by localization.
- We can compute $Z_{\mathcal{M}}(J)$ for some sources J: access to some extra observables.

Supersymmetry on spheres, lightning review

Most of the recent work on this subject has focussed on supersymmetry on S^d or $S^{d-1} \times S^1$.

- $Z(S^3 \times S^1)$ for 4d $\mathcal{N} = 1$ theories: "superconformal index" [Romelsberger, 2005, 2007; Kinney, Maldacena, Minwalla, Raju, 2005]
- $Z(S^4)$ for 4d $\mathcal{N} = 2$ (and for BPS Wilson loops) [Pestun, 2007]
- $Z(S^3)$ for $\mathcal{N} = 2$ theories in 3d [Kapustin, Willett, Yaakov, 2010; Jafferis, 2010; Hama, Hosomichi, Lee, 2010, 2011; Imamura, Yokoyama, 2011] Many useful applications. Discovery of a new useful "observable": $F = -\ln(Z)$ for any 3d CFT: conjectured c-function.
- Superconformal index $Z(S^2 \times S^1)$ for 3d $\mathcal{N} = 2$ [Kim, 2009; Imamura, Yokoyama, 2011]
- In 2d, $\mathcal{N} = (2, 2)$ on S^2 [Benini, Cremonesi, 2012; Doroud, Gomis, Le Floch, Lee, 2012] and also in 5d, for instance [Jafferis, Pufu, 2012]

Originally, case-by-case approach.

A general method was proposed based on background supergravity fields. [Festuccia, Seiberg, 2011]

Rigid supersymmetry on curved manifolds

Given a d-dimensional supersymmetric field theory $\mathcal{T}$ in flat space and a Riemannian manifold $(\mathcal{M}, g_{\mu\nu})$, can we define a corresponding supersymmetric theory on $(\mathcal{M}, g_{\mu\nu})$?

$$\begin{array}{ccc} (\mathcal{T}, \mathbb{R}^d, \delta_{\mu\nu}) & \rightarrow & (\mathcal{T}', \mathcal{M}, g_{\mu\nu}) \\ \delta\mathcal{T} & \rightarrow & \delta'\mathcal{T}' \end{array}$$

For definiteness, consider a supersymmetric quantum field theory described by some UV Lagrangian $\mathcal{L}_0$.

$$\delta\mathcal{L}_0 = \partial_\mu(\cdots)\,, \qquad \tilde{\delta}\mathcal{L}_0 = \partial_\mu(\cdots)\,.$$

$$\{\delta, \tilde{\delta}\} \sim P\,.$$

We will only consider deformations of $\mathcal{L}_0$ which do not modify the UV behavior. Compact geometry gives “IR cut-off”.

Given any flat space Lagrangian, we can define the theory on any Riemannian manifold by "minimal coupling" to the metric. At first order,

$$\mathcal{L} = \mathcal{L}_0 + h_{\mu\nu} T^{\mu\nu}\,, \qquad\qquad g_{\mu\nu} = \delta_{\mu\nu} + 2h_{\mu\nu}\,.$$

Possible because we consider theories with a conserved energy-momentum operator $T_{\mu\nu}$.

Of course we also know how to covariantize this (general covariance). The resulting Lagrangian is not supersymmetric in general.

Example: $\mathcal{N} = 1$ chiral multiplet:

$$\mathcal{L}_0 = g^{\mu\nu}\partial_\mu\phi\partial_\nu\tilde{\phi} + i\tilde{\psi}\tilde{\sigma}^\mu\nabla_\mu\psi - F\tilde{F} + \alpha R\tilde{\phi}\phi + \cdots\,.$$

This is not supersymmetric. We need additional corrections.

Remark: We do not require nor use conformal invariance.

Background supergravity fields

In any supersymmetric theory, we have a conserved supercurrent $S_{\mu\alpha}$, which sits in the same supersymmetry multiplet as $T_{\mu\nu}$.

$$\mathcal{S}_\mu \sim \cdots + \theta S_\mu + \theta\gamma^\nu\tilde{\theta}\, T_{\mu\nu} + \cdots$$

The detailed structure of the supercurrent multiplet can vary. The general supermultiplet $\mathcal{S}$ can often be improved to a simpler supercurrent. [Komargodski, Seiberg, 2010; Dumitrescu, Seiberg, 2011]

Thus it is natural to consider the supersymmetric partners of the metric. Rigid supersymmetry in curved space is a “background gauging” of the supercurrent multiplet.
$\Rightarrow$ Consider background supergravity [Festuccia, Seiberg, 2011]

Metric and its superpartners form a “background superfield”.

- We can think of rigid supersymmetry as some $M_p \to \infty$ of a full-fledged supergravity theory.
- For any given supercurrent there exists a corresponding supergravity multiplet $(g_{\mu\nu}, \Psi_\mu, X)$. E.g. "old-minimal" or "new-minimal" in 4d. [Stelle, West, 1978; Ferrara, van Nieuwenhuizen, 1978; Sohnius, West, 1981]
- We should not impose any gravitational equation of motion. Need to consider off-shell formalism for the supergravity of interest.
- In the rigid limit, $\Psi_\mu = 0$, $\delta\Psi_\mu = 0$. (*Much* simpler than SUGRA.)
- Given a set of background fields $(g_{\mu\nu}, X)$, we have one rigid supersymmetry for each spinor ζ solving the generalised Killing spinor equation

$$\delta_\zeta \Psi_\mu = D(g, X)\zeta = 0\,.$$

Curved space supersymmetry for $\mathcal{N} = 1$ theories

Consider $\mathcal{N} = 1$ supersymmetric theories in four dimensions, with an R-symmetry.

We have the $\mathcal{R}$-multiplet

$$j_\mu^{(R)}\,, \qquad S_{\mu\alpha}, \qquad \tilde{S}_\mu^{\dot\alpha}\,, \qquad T_{\mu\nu}\,, \qquad \mathcal{F}_{\mu\nu}$$

which couples to the "new minimal" supergravity multiplet of [Sohnius, West, 1981] :

$$A_\mu^{(R)}\,, \qquad \Psi_{\mu\alpha}, \qquad \tilde{\Psi}_\mu^{\dot\alpha}\,, \qquad g_{\mu\nu}\,, \qquad B_{\mu\nu}$$

We often use the dual field strength of $B_{\mu\nu}$, denoted V_μ.

There is a complete classification of supersymmetric backgrounds in this case.
[Dumitrescu, Festuccia, Seiberg, 2012; Klare, Tomassielo, Zaffaroni, 2012]

Supersymmetry and complex geometry

4d Killing spinor equations :

$$\begin{aligned}(\nabla_\mu - iA_\mu^{(R)})\zeta &= \tfrac{i}{2}V_\mu\zeta - iV^\nu\sigma_{\mu\nu}\zeta \\ (\nabla_\mu + iA_\mu^{(R)})\tilde\zeta &= -\tfrac{i}{2}V_\mu\tilde\zeta + iV^\nu\tilde\sigma_{\mu\nu}\tilde\zeta\end{aligned}$$

The spinors ζ_α, $\tilde\zeta^{\dot\alpha}$ are non-vanishing sections of $S\otimes L$, $S\otimes L^{-1}$.

The most important result is [Dumitrescu, Festuccia, Seiberg, 2012; Klare, Tomassielo, Zaffaroni, 2012]

One supercharge on $\mathcal{M}_4$ $\quad\Leftrightarrow\quad$ $\mathcal{M}_4$ a complex manifold

The background metric $g_{\mu\nu}$ must be Hermitian.

The complex structure is expressed in term of the Killing spinor :

$$J^\mu{}_\nu = -\frac{2i}{|\zeta|^2}\zeta^\dagger\sigma^\mu{}_\nu\zeta$$

The background supergravity fields are determined in term of the metric and complex structure:

$$\begin{aligned} V_\mu &= \frac{1}{2}\nabla_\nu J^\nu{}_\mu + U_\mu\,, \\ A_\mu^{(R)} &= -\frac{1}{4}J_\mu{}^\nu\partial_\nu \log\sqrt{g} - \frac{1}{4}(2\delta_\mu{}^\nu - iJ_\mu{}^\nu)\nabla_\rho J^\rho{}_\nu\,, \end{aligned}$$

Note a certain "ambiguity" U_μ, which is anti-holomorphic and otherwise arbitrary.

The Killing spinor ζ transforms by a phase under adapted (holomorphic) coordinate transformations. This can be compensated by a R-symmetry transformation. In that sense, ζ determines a scalar supercharge. Analog of topological twist.

Global symmetry and holomorphic G-bundle

When our theory has a global symmetry group G , we can couple it to a background vector multiplet.

We ask that it preserve the Killing spinor ζ. This requires

$$F^{(0,2)} = 0\,, \qquad D = -\frac{1}{2} J^{\mu\nu} F_{\mu\nu}$$

The first equation implies that we have an holomorphic G-bundle.

We will focus on G Abelian : Holomorphic line bundles.

Enters the partition function

One can take any R-symmetric $\mathcal{N} = 1$ supersymmetric quantum field theory and couple it to a given complex four-manifold $\mathcal{M}_4$. Consider $\mathcal{M}_4$ compact.

When the theory has a global symmetry, we also couple it to an holomorphic G-bundle over $\mathcal{M}_4$.

We can (in principle) compute the partition function $Z_{\mathcal{M}_4}$:

$$Z_{\mathcal{M}_4} = Z_{\mathcal{M}_4}(J^{\mu}{}_{\nu}, g_{\mu\nu}, U_{\mu}, A_{\mu}, \lambda)$$

What does it depend on ?

$$Z_{\mathcal{M}_4} = Z_{\mathcal{M}_4}(J^\mu{}_\nu, g_{\mu\nu}, U_\mu, A_\mu, \lambda)$$

$Z_{\mathcal{M}_4}$ could *a priori* depend on :

- The choice of complex structure $J^\mu{}_\nu$
- The choice of Hermitian metric $g_{\mu\nu}$
- The $(0,1)$-form U_μ which parametrizes the ambiguity in V_μ
- The connection A_μ of the holomorphic G-bundle
- The couplings λ of the original flat-space theory

All this data can be varied continuously. There might also be discrete choices, such as for the topology of G-bundles.

We will focus on the geometric data. The dependence on the couplings λ is best studied with other tools. [To appear.]

Strategy to study the parameter dependence of $Z_{\mathcal{M}_4}$

Linearized analysis around flat space, $g_{\mu\nu} = \delta_{\mu\nu} + \Delta g_{\mu\nu}$.

Choose a single supercharge Q (corresponding to ζ).

- Consider small variations of all the continuous geometric data above, $\Delta g_{\mu\nu}$, $\Delta J^{\mu}{}_{\nu}$, ΔU_{μ}, ΔA_{μ}
- These variations couple to various operators in the $\mathcal{R}$-multiplet
- Study which of these operators are Q-exact. The partition function does not depend on these operators
- Because Q is a scalar under adapted coordinate transformations, we can argue that this result is true also non-linearly, for any $\mathcal{M}_4$

A fully non-linear analysis is also possible. [To appear.] Same conclusions.

- Caveat: We did not consider the effect of anomalies.

Basics of deformation theory

Consider a complex structure $J^\mu{}_\nu$ on $\mathcal{M}_4$. Introduce adapted coordinates z^i, $\bar{z}^{\bar{i}}$. We have $J^i{}_j = i\,\delta^i{}_j\,, \qquad J^{\bar{i}}{}_{\bar{j}} = -i\,\delta^{\bar{i}}{}_{\bar{j}}$.

Non-zero elements of the general variation $\Delta J^\mu{}_\nu$ are

$$\Delta J^i{}_{\bar{j}}\,, \qquad \partial_{\bar{j}}\left(\Delta J^i{}_{\bar{k}}\right) - \partial_{\bar{k}}\left(\Delta J^i{}_{\bar{j}}\right) = 0$$

and its complex conjugates $\Delta J^{\bar{i}}{}_j$. We quotient by diffeomorphisms,

$$\Delta J^i{}_{\bar{j}} = 2i\,\partial_{\bar{j}}\,\varepsilon^i$$

Thus, first order deformations of the complex structure correspond to

$$\Theta^i = \Delta J^i{}_{\bar{j}}\, d\bar{z}^{\bar{j}}\,, \qquad \left[\Theta^i\right] \in H^{0,1}\left(\mathcal{M}_4, T^{1,0}\mathcal{M}_4\right)$$

This cohomology contains the complex structure moduli.

Basics of deformation theory

We also vary the metric $\Delta g_{\mu\nu}$. To preserve the compatibility with the complex structure, we have

- $\Delta g_{i\bar{j}}$ unconstrained
- $\Delta g_{ij} = \frac{i}{2}\left(g_{i\bar{k}}\Delta J^{\bar{k}}{}_{j} + g_{j\bar{k}}\Delta J^{\bar{k}}{}_{i}\right)$ and its complex conjugate

Similarly, for Abelian background gauge fields, we have

$$\partial_{\bar{i}}\Delta A_{\bar{j}} - \partial_{\bar{j}}\Delta A_{\bar{i}} = 0$$

modulo (complexified) gauge transformations. The holomorphic line bundle moduli sit in $H^{0,1}\left(\mathcal{M}_4\right)$.

The deformation Lagrangian

At first order around flat space, the supergravity background fields couple to the $\mathcal{R}$-multiplet of the flat space theory. We have the deformation Lagrangian

$$\Delta\mathcal{L} = -\frac{1}{2}\Delta g^{\mu\nu}T_{\mu\nu} + A^{(R)\mu}j^{(R)}_{\mu} + \frac{i}{4}\varepsilon^{\mu\nu\rho\lambda}B_{\mu\nu}\mathcal{F}_{\rho\lambda}$$

(And similar for the background gauge fields.)

We study the Q-cohomology of the $\mathcal{R}$-multiplet. In particular,

$$\left\{Q,\, \tilde{S}^{\dot\alpha}{}_{\mu}\right\} = 2i\,(\tilde\sigma^{\nu}\zeta)^{\dot\alpha}\,\mathcal{T}_{\mu\nu}$$

with

$$\mathcal{T}_{\mu\nu} = T_{\mu\nu} + \frac{i}{4}\varepsilon_{\mu\nu\rho\lambda}\mathcal{F}^{\rho\lambda} - \frac{i}{4}\varepsilon_{\mu\nu\rho\lambda}\partial^{\rho}j^{(R)\lambda} - \frac{i}{2}\partial_{\nu}j^{(R)}_{\mu}$$

We can see that the eight operators $\mathcal{T}_{\mu\bar{i}}$ are Q-exact. They are also the only Q-closed operators in the $\mathcal{R}$-multiplet.

The deformation Lagrangian

We plug the supersymmetric values of the (linearized) background supergravity fields in $\Delta\mathcal{L}$, and find :

$$
\begin{aligned}
\Delta\mathcal{L} \;=\; & -\Delta g^{\bar{i}\bar{j}}\mathcal{T}_{\bar{i}\bar{j}} - i\sum_{j=\bar{j}} \Delta J^{\bar{i}}{}_{j}\mathcal{T}_{\bar{j}\bar{i}} \\
& + i\Delta J^{w}{}_{\bar{w}}\left(T_{ww} + \frac{i}{2}\partial_w j_w^{(R)}\right) + i\Delta J^{z}{}_{\bar{z}}\left(T_{zz} + \frac{i}{2}\partial_z j_z^{(R)}\right) \\
& + i\Delta J^{w}{}_{\bar{z}}\left(T_{wz} + \frac{i}{2}\mathcal{F}_{wz} - \frac{i}{4}\partial_w j_z^{(R)} + \frac{3i}{4}\partial_z j_w^{(R)}\right) \\
& + i\Delta J^{z}{}_{\bar{w}}\left(T_{wz} - \frac{i}{2}\mathcal{F}_{wz} - \frac{i}{4}\partial_z j_w^{(R)} + \frac{3i}{4}\partial_w j_z^{(R)}\right)
\end{aligned}
$$

Note that $\Delta g^{\bar{i}\bar{j}}$ and $\Delta J^{\bar{i}}{}_{j}$ couple to operators which are Q-exact. Moreover, the whole Lagrangian is Q-exact for a trivial complex structure deformation, as follows from diff invariance.

We conclude :

- $Z_{\mathcal{M}_4}$ is independent of $\Delta g_{\bar{i}\bar{j}}$.
- $Z_{\mathcal{M}_4}$ is independent on the anti-holomorphic complex structure deformations $\Delta J^{\bar{i}}{}_{j}$, and only depends on the holomorphic ones through its cohomology class $H^{0,1}\left(\mathcal{M}_4, T^{1,0}\mathcal{M}_4\right)$. Thus it only depends on holomorphic complex structure moduli.

$Z_{\mathcal{M}_4}$ is independent of the Hermitian metric.
It is a locally holomorphic function of the complex structure moduli

Similar analysis for the ambiguity U and the G-bundle moduli.
We conclude :

$Z_{\mathcal{M}_4}$ is locally holomorphic in the G-bundle moduli

Moreover $Z_{\mathcal{M}_4}$ is only depends on the cohomology of U, and is independent of U if there exists a FZ multiplet.

Example I : The $S^3 \times S^1$ partition function

Consider the following quotient of $\mathbb{C}^2 - \{(0,0)\}$:

$$(z_1, z_2) \sim (p\, z_1, q\, z_2)\,, \qquad 0 < |p| \leq |q| < 1$$

It is a complex manifold known as a primary Hopf surface, that we denote $\mathcal{M}_4^{p,q}$.

$$\mathcal{M}_4^{p,q} \cong S^3 \times S^1$$

We can write it in term of real angles:

$$z_1 = p^x \cos\frac{\theta}{2}\, e^{i\varphi}\;, \qquad z_2 = q^x \sin\frac{\theta}{2}\, e^{i\chi}$$

The quotient corresponds to the identification $x \sim x + 1$.

Example I : The $S^3 \times S^1$ partition function

For generic p, q, $\mathcal{M}_4^{p,q}$ is compatible with two supercharges.
For the particular choice $p = \bar{q}$, we have four supercharges.

The $S^3 \times S^1$ partition function on this space is known as the supersymmetric index

$$\mathcal{I}(p, q, u) = \mathrm{Tr}_{S^3}\left((-1)^F p^{J_3 + J_3' - \frac{R}{2}} q^{J_3 - J_3' - \frac{R}{2}} u^{Q_f}\right)$$

Explicit formula known for general gauge theories. [Romelsberger, 2007]
Our new insight is that

- The fugacities p, q above are naturally complex parameters, corresponding to complex structure moduli of $\mathcal{M}_4^{p,q}$.
- The fugacity u corresponds to holomorphic line bundle moduli.
- $\mathcal{I}$ is a meromorphic function on the $\mathcal{M}_4^{p,q}$ moduli space.
- We can compute the index for any Hermitian metric. Same answer. See also [C.C., I. Shamir, to appear]

Example II : The $T^2 \times S^2$ partition function

Another example is a quotient of $\mathbb{C} \times \mathbb{C}P^1$

$$(w, z) \sim (w + 1, e^{i\alpha} z) \sim (w + \tau, e^{i\beta} z)$$

This is $T^2 \times S^2$. The partition function $Z_{T^2 \times S^2}$ computes an index

$$\mathcal{I}(q, x, t) = \mathrm{Tr}_{S^2 \times S^1} \left((-1)^F q^P x^{J_3} t^{J_0} \right)$$

This is related to the elliptic genus of certain $\mathcal{N} = (0, 2)$ theories on T^2.
It can be computed exactly. [C.C., I. Shamir, to appear]

$\mathcal{N}=2$ Supersymmetric field theories on $\mathcal{M}_3$

We can define 3d $\mathcal{N}=2$ supersymmetric field theories on three-manifolds. [C.C., T. Dumitrescu, G.Festuccia, Z. Komargodski, 2012]

Consider $\mathcal{N}=2$ theories with an R-symmetry. The 3d $\mathcal{R}$-multiplet couples to a supergravity multiplet :

$$A_\mu^{(R)}\,, \qquad \Psi_{\mu\alpha}, \qquad \tilde{\Psi}_\mu^{\dot{\alpha}}\,, \qquad g_{\mu\nu}\,, \qquad V_\mu\,, \qquad H$$

Rigid supersymmetry is governed by

$$\begin{aligned}
(\nabla_\mu - iA_\mu)\zeta_\alpha &= -\frac{1}{2}H(\gamma_\mu\zeta)_\alpha - \frac{1}{2}\epsilon_{\mu\nu\rho}V^\nu(\gamma^\rho\zeta)_\alpha - iV_\mu\zeta_\alpha\,,\\
(\nabla_\mu + iA_\mu)\tilde{\zeta}_\alpha &= -\frac{1}{2}H(\gamma_\mu\tilde{\zeta})_\alpha + \frac{1}{2}\epsilon_{\mu\nu\rho}V^\nu(\gamma^\rho\tilde{\zeta})_\alpha + iV_\mu\tilde{\zeta}_\alpha\,.
\end{aligned}$$

Supersymmetric three-manifolds

Given one Killing spinor ζ on $\mathcal{M}_3$, we construct a vector

$$\xi^\mu = \frac{\zeta^\dagger \gamma^\mu \zeta}{\zeta^\dagger \zeta}$$

It satisfies a certain differential equation which implies that we can describe $\mathcal{M}_3$ in term of coordinates $\tau, z, \bar{z}$ such that

$$\xi = \partial_\tau$$

with z a complex coordinates, and that the transition functions between patches are "holomorphic":

$$\tau' = \tau + t(z, \bar{z}) \,, \qquad z' = f(z) \,.$$

Conversely, on any manifold with such a structure we can define one supercharge.

Supersymmetric three-manifolds

One supercharge on $\mathcal{M}_3$

$\Leftrightarrow$

$\mathcal{M}_3$ admits a transversally holomorphic foliation
(or is "integrable almost contact")

The background metric $g_{\mu\nu}$ must be adapted to that structure :

$$ds^2 = \eta^2 + c(\tau, z, \bar{z})^2 dz d\bar{z}\,, \qquad \eta = d\tau + h(\tau, z, \bar{z})dz + \bar{h}(\tau, z, \bar{z})\ ,$$

with $\eta_\mu = \xi_\mu$. Transversally Hermitian metric.

We studied first order deformations of such structures on $\mathcal{M}_3$.
Non-trivial deformations sit in some cohomology group $H^{0,1}\left(\mathcal{M}_3, T^{1,0}\mathcal{M}_3\right)$, in analogy with the 4d case.

Constraints on 3d partition functions

We can again compute the partition function on $\mathcal{M}_3$.
(There is also an analog of holomorphic G-bundles for the global symmetries.)
We find in particular that

- The partition function is independent of the adapted metric
- It depends holomorphically on the moduli of the 3d transversally holomorphic foliation.

Transversally holomorphic foliations are classified. [Brunella, 1996]
Includes Seifert manifolds, which admit two supercharges.

Application: S^3 and its squashings

The partition function on S^3 has been much studied. Various "squashings" of the round sphere were considered in the literature.
It was observed that

- The dependence on the metric is only through a single "squashing parameter", generally complex.
- Some metric squashings do not affect the partition function.

We studied transversally holomorphic foliations on S^3. There is a one-dimensional family of such structures connected to the round sphere of [Kapustin, Willett, Yaakov, 2010]. It can be obtained from Hopf surfaces.

This explains the observations above : The "squashing parameter" corresponds to a choice of transversally holomorphic foliation.

Conclusions

Summary

- We studied curved space supersymmetric partition functions, $Z_{\mathcal{M}}$, of R-symmetric $\mathcal{N} = 1$ theories in 4d and $\mathcal{N} = 2$ theories in 3d.
- $Z_{\mathcal{M}}$ is essentially independent of the metric, while it depends on the complex structure in 4d (and on its 3d analog).
- $Z_{\mathcal{M}}$ is locally holomorphic in the geometric moduli.

Outlook

- The holomorphy of $Z_{\mathcal{M}_4}$ could be used to constrain the partition function further, simplifying explicit computations. In examples, it seems to be meromorphic. What is the meaning of the poles ?
- In the case of two supercharges, an explicit localization computation of $Z_{\mathcal{M}}$ is tractable.
- Generalizations to 2d, 5d, $\mathcal{N} = 2$ in 4d...