HPFBU

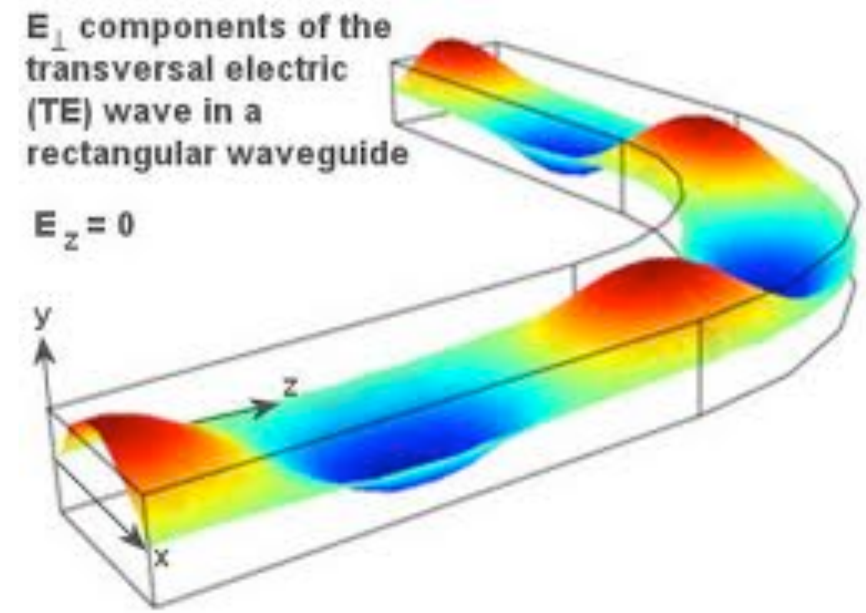
Parçacık Hızlandırıcıları için **RF Sistemlere Giriş**

Dr. Öznur METE
University of Manchester
The Cockcroft Institute of Accelerator Science and Technology
İletişim Bilgileri
oznur.mete@cockcroft.ac.uk
oznur.mete@manchester.ac.uk
www.cern.ch/omete

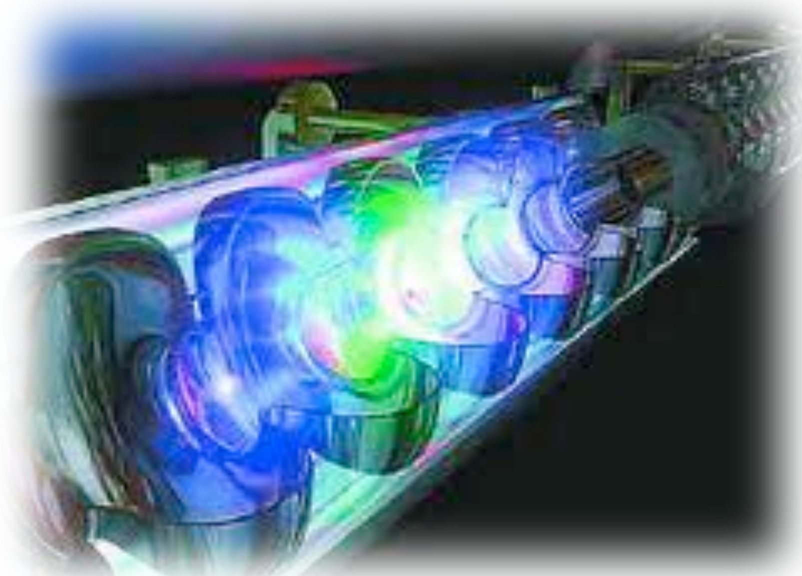
www.cern.ch/ACCTR

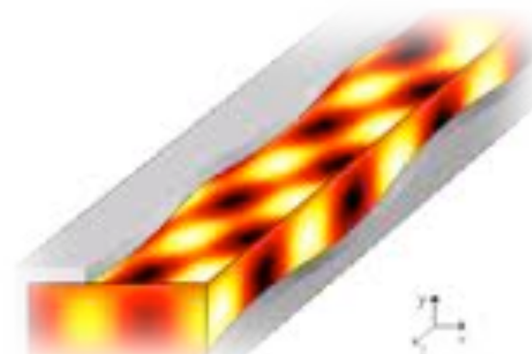
www.cern.ch/HFiz

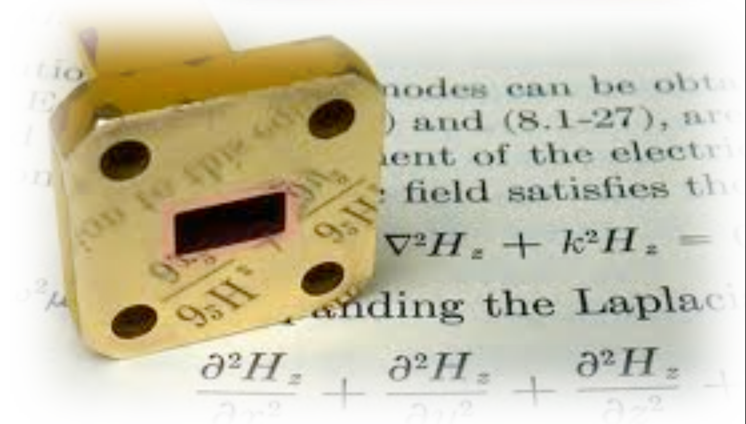
Parçacık hızlandırıcıları için RF sistemler



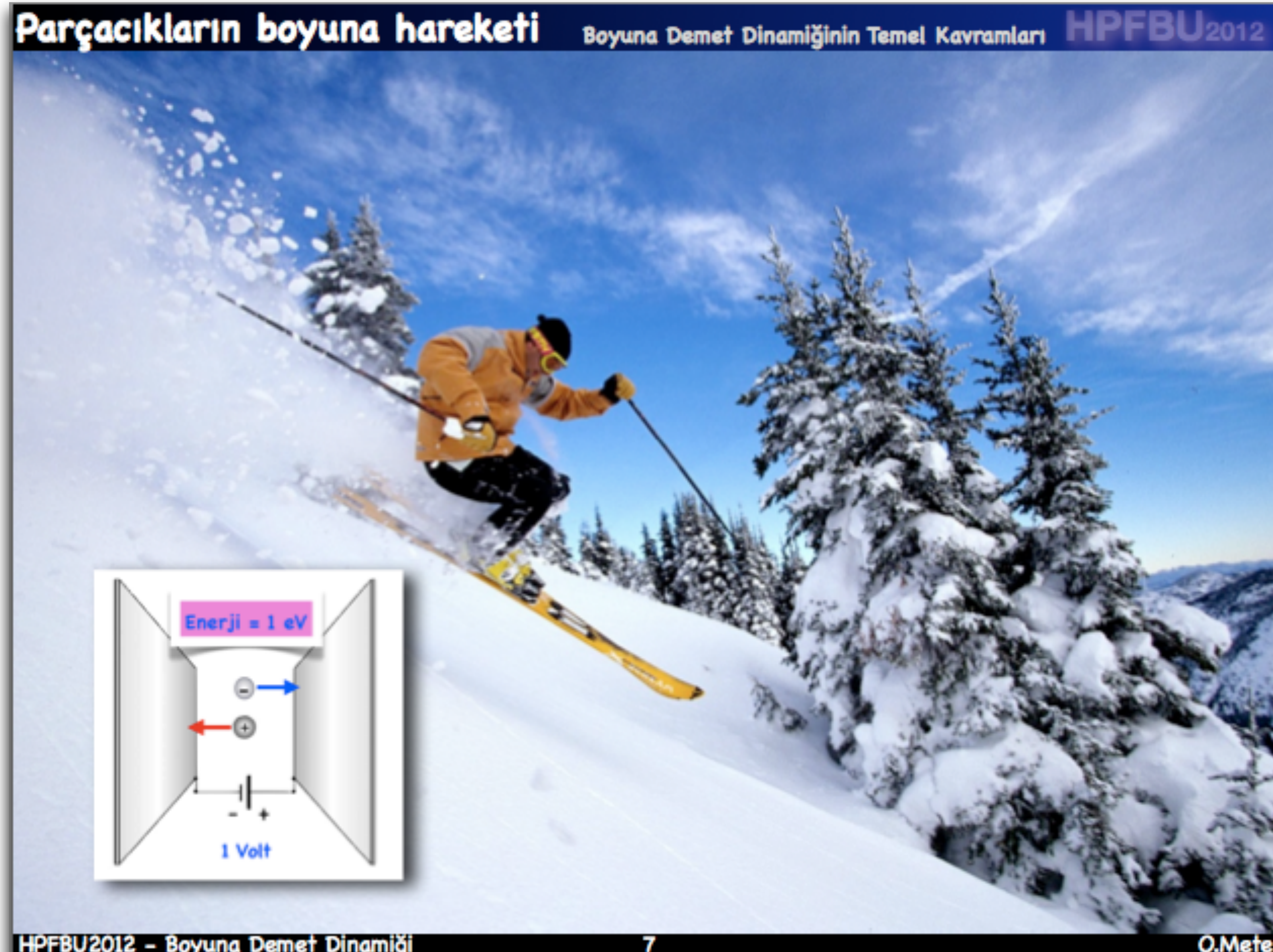
- Güç kaynakları - klystronlar
- Güç iletim yolları - dalga kılavuzları
- Hızlandırma kovukları - Rezonans kovukları
- Bağlaşım parçalar
- ...







- Static acceleration: fundamental limitation to the final energy.
- High frequency to achieve high voltages.
- Modern accelerators use powerful radio-frequency systems (from MHz to GHz).



Parçacıkların boyuna hareketi
Boyuna Demet Dinamiğinin Temel Kavramları
HPFBU2012
Enerji = 1 eV
1 Volt
HPFBU2012 - Boyuna Demet Dinamiği
7
O.Mete

Parçacıkların boyuna hareketi

- Parçacıkların hızlandırıcı alanın şiddeti, dolayısıyla kazanacakları enerji, **"kırılma (breakdown)"** denen bir olgu ile sınırlıdır.
- Kırılma sınırı düşünüldüğünde, daha yüksek gerilimlere durgun elektrik alanı çok kısa atmalar halinde uygulayarak ulaşılabilir.
    - Bu şekilde, kırılma sınırının altında, elde edilen en yüksek gerilim 10 MV mertebesindedir.
- Daha da yüksek gerilimler elde etmek istersek, o zaman yüksek frekanslı elektromagnetik alanlar kullanmalıyız[1,2,3].
    - İçlerinde bu şekilde alanlar uyarılabilecek özel RF kovukları tasarlamalıyız.
    - Parçacıklar RF kovuklarının içinden birkez ya da kazanmaları istenilen enerjiye göre birçok kez geçerler.
    - Yüksek frekanslı RF normal iletken kovuklar kullanılarak, kırılma noktasının altında elde edilebilen en yüksek elektrik alan **100 MV/m** 'dir. Bu değere ulaşabilen kovuklar **CERN'de CLIC projesi** dahilinde üretimekte, SLAC ve KEK işbirliğinde test eilmektedir.

[1] G. Ising
[2] R. Wideroe
[3] M.S. Livingstone

- **TEM, Transverse Electromagnetic wave mode:** Electric and magnetic field components are perpendicular to each other, and both are transverse to the direction of propagation. No longitudinal electric field component.
- **TM, Transverse Magnetic wave mode:** Longitudinal electric field component.
- **TE, Transverse Electric wave mode:** Longitudinal magnetic field component.
- Both TE and T.M. modes have a characteristic cut-off frequencies. Waves of frequencies below the cut-off frequency can not propagate; power and signal transmission is only possible for frequencies above cut-off frequencies.
- Therefore, waveguides operating in TE of T.M. modes can be seen as high-pass filters.

Waveguides and Cavity Resonators

- Parallel-plate waveguides for TE and T.M. modes,
  - All transverse field components can be expressed in terms of Ez for T.M. modes,
  - and in terms of Hz for TE mode.
- Attenuation constant due to imperfect conducting walls for T.M. and TE modes,
  - It depends on the mode and the frequency; for some modes attenuation can increase with frequency, for some other modes attenuation can reach a minimum as the frequency exceed the cut-off freq. by a some amount.
- Hollow metal pipes of arbitrary cross section,
  - Field, current, charge distribution, propagation and attenuation characteristics of rectangular and cylindrical waveguides (for T.M. and TE modes, no support for TEM).
- Propagation is possible by an open dielectric-slab waveguide. Fields are confined within the dielectric region and rapidly decays away from the slab surface. Therefore the the waves supported by the dielectric-slab waveguides are called the surface waves.
- Hollow conducting box with proper dimensions can be a resonance device. Box provides large areas for current flow. Such a box, which is a segment of a waveguide is called a cavity resonator.
  - Different mode patterns of the fields in the rectangular and cylindrical cavity resonators.

Assumptions:

> Waves propagate in the +z-direction,

> $\gamma = \alpha + i\beta$ propagation constant (to be determined),

> For an harmonic dependence with an angular frequency ω,

the dependence on z and t for all field components can be described by the exponential factor,

$$e^{-\gamma z} e^{j\omega t} = e^{(j\omega t - \gamma z)} = e^{-\alpha z} e^{j(\omega t - \beta z)}$$

> For cosine reference instantaneous $\vec{E}$ field in Cartesian coordinates,

$$\vec{E}(x,y,z;t) = \mathcal{R}e\left[\vec{E}^{\circ}(x,y)\,e^{(j\omega t-\gamma z)}\right]$$

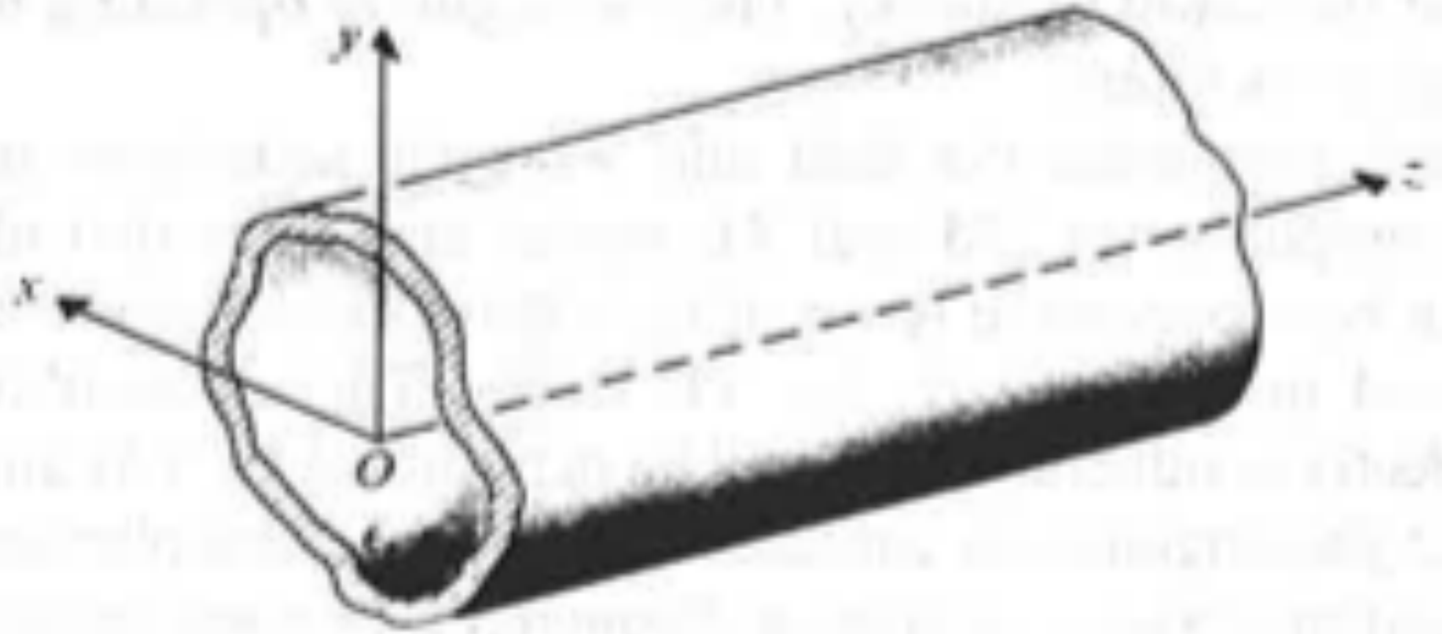
> Similarly for $\vec{H}$,

$$\vec{H}(x,y,z;t) = \mathcal{R}e\left[\vec{H}^{\circ}(x,y)\,e^{(j\omega t-\gamma z)}\right]$$

> Here $\vec{E}^{\circ}$ and $\vec{H}^{\circ}$ are phasors only depending on the ~~trans~~ cross sectional coordinates.

> Using a phasor representation in equations relating field quantities, replace partial derivatives with respect to t and z with $(j\omega)$ and $(-\gamma)$.

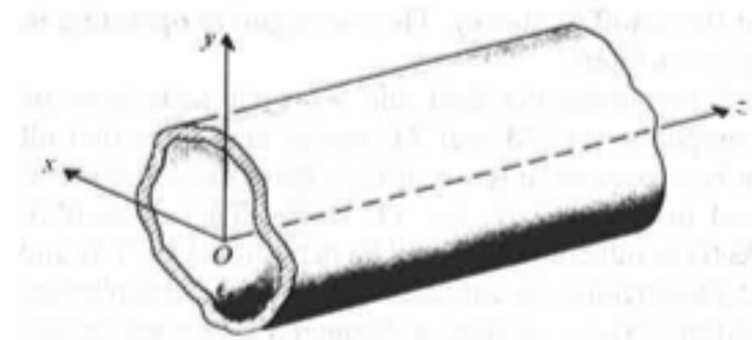
> Consider a straight in the form of a dielectric-filled metal tube with an arbitrary cross section, lying along z-axis.



> Electric and magnetic field intensities in the charge-free dielectric region satisfy the following homogeneous vector Helmholtz's equations:

$$\nabla^2 E + k^2 E = 0$$

$$\nabla^2 H + k^2 H = 0$$



> E, H : three dimentional vector phasors, k: wavenumber.

$$k = \omega\sqrt{\mu\epsilon}$$

> Three dimentional Laplacian operator, ∇^2, may be broken into two parts:

$$\nabla^2 \longrightarrow \nabla^2_{u_1 u_2} \; , \; \nabla^2_z$$

> For waveguides with rectangular cross section we use Cartesian coordinates.

$$\nabla^2 E = \left(\nabla_{xy}^2 + \nabla_z^2\right) E = \left(\nabla_{xy}^2 + \frac{\partial^2}{\partial z^2}\right) E$$

$$= \nabla_{xy}^2 E + \gamma^2 E$$

$$= -k^2 E \qquad \text{(from Helmholtz's equation)}$$

$$\boxed{\begin{aligned} \nabla_{xy}^2 E + (\gamma^2 + k^2) E &= 0 \\ \nabla_{xy}^2 H + (\gamma^2 + k^2) H &= 0 \end{aligned}}$$

Governing equations for waveguides with rectangular crosssection.

> These equations are three second-order partial differential equations (one for each component of E and H),

> The exact solutions of ~~thesecomp~~ these component equations depend on

- The cross sectional geometry,
- Boundary conditions that a particular field component must satisfy at conductor-dielectric interfaces.

> Note: By writing $\nabla^2_{r\phi}$ for the transversal operator ∇^2_{xy}, the equations become the governing equations for waveguides with a circular cross section.

> Various components of $\vec{E}$ and $\vec{H}$ are not all independent.

> So it is not necessary to solve all six second-order partial differential equations for six components of $\vec{E}$ and $\vec{H}$.

> We can find interrelationships among six components in Cartesian coordinates by expanding the two source-free curl equations,

$$\nabla \times E = -j\omega\mu H \qquad \nabla \times H = j\omega\epsilon E$$

Note: Maxwell equations governing electromagnetic phenomena in source-free media:

$$\nabla \times E + \frac{\partial B}{\partial t} = 0$$

$$\nabla \times H - \frac{\partial D}{\partial t} = 0$$

For external sources in vacuum,

$$D = \epsilon_0 E$$

$$B = \mu_0 H$$

In dielectric media: $\epsilon_0 \rightarrow \epsilon_r \epsilon_0 = \epsilon$

$\mu_0 \rightarrow \mu_r \mu_0 = \mu$

In phasor representation $\frac{\partial}{\partial t} \rightarrow (j\omega)$

From $\nabla \times E = -j\omega\mu H$

$$\frac{\partial E_z^0}{\partial y} + \gamma E_y^0 = -j\omega\mu H_x^0 \quad (1a)$$

$$-\gamma E_x^0 - \frac{\partial E_z^0}{\partial x} = -j\omega\mu H_y^0 \quad (1b)$$

$$\frac{\partial E_y^0}{\partial x} - \frac{\partial E_x^0}{\partial y} = -j\omega\mu H_z^0 \quad (1c)$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

from $\nabla \times H = j\omega\epsilon E$

$$\frac{\partial H_z^\circ}{\partial y} + \gamma H_y^\circ = j\omega\epsilon E_x^\circ \qquad (2a)$$

$$-\gamma H_x^\circ - \frac{\partial H_z^\circ}{\partial x} = j\omega\epsilon E_y^\circ \qquad (2b)$$

$$\frac{\partial H_y^\circ}{\partial x} - \frac{\partial H_x^\circ}{\partial y} = j\omega\epsilon E_z^\circ \qquad (2c)$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

Note:

> partial derivatives with respect to $z \rightarrow (-\gamma)$,

> all the component field quantities in the equations are phasors that depend on x and y,

> the common $e^{-\gamma z}$ factor for z dependence having been omitted.

> In the equations $h^2 = \gamma^2 + k^2$.

Example : Take 1a and 2b, eliminate E_y^0 and obtain H_x^0

$$\frac{\partial E_z^0}{\partial y} + \gamma E_y^0 = -j\omega\mu H_x^0$$

$$-\gamma H_x^0 - \frac{\partial H_z^0}{\partial x} = j\omega\epsilon E_y^0$$

$$-\gamma H_x = j\omega\epsilon E_y^0 + \frac{\partial H_z^0}{\partial x}$$

$$E_y = \left(-j\omega\mu H_x - \frac{\partial E_z^0}{\partial y}\right)\frac{1}{\gamma}$$

$$-\gamma H_x = \frac{j\omega\epsilon}{\gamma}\left(-j\omega\mu H_x - \frac{\partial E_z}{\partial y}\right) + \frac{\partial H_z}{\partial x}$$

$$-\gamma H_x = \frac{\omega^2\epsilon\mu}{\gamma} H_x - \frac{j\omega\epsilon}{\gamma}\frac{\partial E_z}{\partial y} + \frac{\partial H_z}{\partial x}$$

$$H_x\left(-\gamma - \frac{\omega^2\epsilon\mu}{\gamma}\right) = -\frac{j\omega\epsilon}{\gamma}\frac{\partial E_z}{\partial y} + \frac{\partial H_z}{\partial x}$$

$$H_x\Big(\underbrace{-\gamma^2 - \omega^2\epsilon\mu}_{-k^2}\Big) = -j\omega\epsilon\frac{\partial E_z}{\partial y} + \gamma\frac{\partial H_z}{\partial x}$$

$$-\gamma^2 - \omega^2\epsilon\mu$$

$$k = \omega\sqrt{\mu\epsilon}$$

$$k^2 = \omega^2\mu\epsilon$$

$$H_x^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial H_z^0}{\partial x} - j\omega\epsilon \frac{\partial E_z^0}{\partial y}\right),$$

$$H_y^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial H_z^0}{\partial y} + j\omega\epsilon \frac{\partial E_z^0}{\partial x}\right),$$

$$E_x^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial E_z^0}{\partial x} + j\omega\mu \frac{\partial H_z^0}{\partial y}\right),$$

$$E_y^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial E_z^0}{\partial y} - j\omega\mu \frac{\partial H_z^0}{\partial x}\right).$$

Conclusions

- The wave behaviour in a waveguide can be analysed by using the equations for the longitudinal components, E^0_z and H^0_z

$$\Delta^2_{xy}E + (\gamma^2 + k^2)E = 0$$

$$\Delta^2_{xy}H + (\gamma^2 + k^2)H = 0$$

$$k = \omega\sqrt{\mu\epsilon}$$

- Then, equations 1(a,b,c) and 2(a,b,c) can be solved to determine the other components.

Q: Which mode would you use to accelerate particles in a cavity?

- It is convenient to classify the propagating waves in a uniform waveguide into three types according to whether E_z or H_z exists.
  - Transverse electromagnetic (TEM) waves:
    Neither E_z not H_z
  - Transverse magnetic (TM) waves:
    Nonzero E_z but H_z=0
  - Transverse electric (TE) waves:
    Nonzero H_z but E_z=0

- **TM** field components for **a rectangular waveguide**:

$$E_x^0(x,y) = -\frac{\gamma}{h^2}\left(\frac{m\pi}{a}\right)E_0\,cos\left(\frac{m\pi}{a}x\right)sin\left(\frac{n\pi}{b}y\right)$$

$$E_y^0(x,y) = -\frac{\gamma}{h^2}\left(\frac{n\pi}{b}\right)E_0\,sin\left(\frac{m\pi}{a}x\right)cos\left(\frac{n\pi}{b}y\right)$$

$$H_x^0(x,y) = \frac{j\omega\epsilon}{h^2}\left(\frac{n\pi}{b}\right)E_0\,sin\left(\frac{m\pi}{a}x\right)cos\left(\frac{n\pi}{b}y\right)$$

$$H_y^0(x,y) = -\frac{j\omega\epsilon}{h^2}\left(\frac{m\pi}{a}\right)E_0\,cos\left(\frac{m\pi}{a}x\right)sin\left(\frac{n\pi}{b}y\right)$$

Propagation constant

$$\gamma = j\beta = j\sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

Cut-off frequency

$$(f_c)_{mn} = \frac{1}{2\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m}{a}\right)^2 - \left(\frac{n}{b}\right)^2}$$

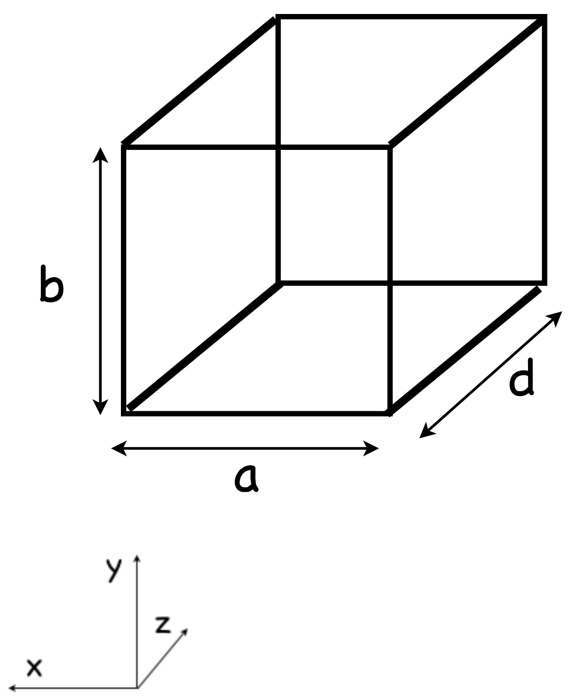
HW: Can all modes propagate in a rectangular waveguide?

HW: What happens when γ>0 and γ<0 ?

HW: Can you derive f_c starting from the definition of γ ?

TODAY

- **TMmnp** field components for **a rectangular resonance cavity**:
  - Consider a rectangular waveguide with both ends closed by a conducting wall.
  - And the interior dimensions of the cavity are a, b and d.



$$E_z(x,y,z) = E_0 sin\left(\frac{m\pi}{a}x\right) sin\left(\frac{n\pi}{b}y\right) cos\left(\frac{p\pi}{d}z\right)$$

$$E_x(x,y,z) = -\frac{1}{h^2}\left(\frac{m\pi}{a}\right)\left(\frac{p\pi}{d}\right) E_0\, cos\left(\frac{m\pi}{a}x\right) sin\left(\frac{n\pi}{b}y\right) sin\left(\frac{p\pi}{d}z\right)$$

$$E_y(x,y,z) = -\frac{1}{h^2}\left(\frac{n\pi}{b}\right)\left(\frac{p\pi}{d}\right) E_0\, sin\left(\frac{m\pi}{a}x\right) cos\left(\frac{n\pi}{b}y\right) sin\left(\frac{p\pi}{d}z\right)$$

$$H_x(x,y,z) = \frac{j\omega\epsilon}{h^2} E_0\, sin\left(\frac{m\pi}{a}x\right) cos\left(\frac{n\pi}{b}y\right) cos\left(\frac{p\pi}{d}z\right)$$

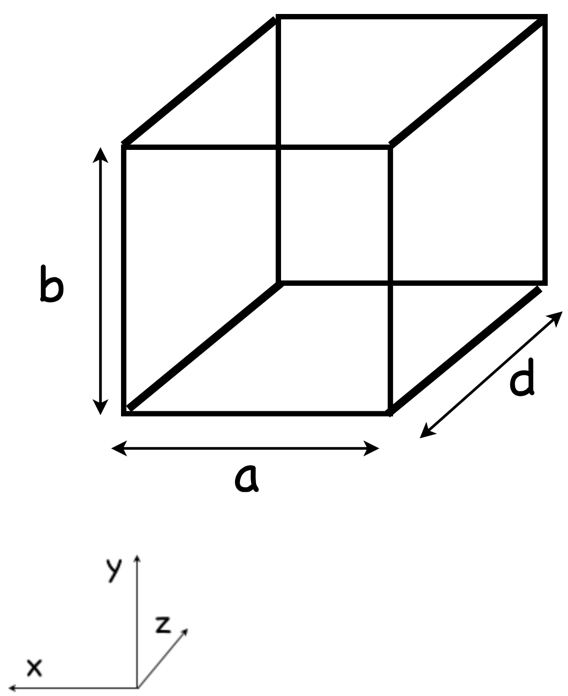
$$H_y(x,y,z) = -\frac{j\omega\epsilon}{h^2} E_0\, cos\left(\frac{m\pi}{a}x\right) sin\left(\frac{n\pi}{b}y\right) cos\left(\frac{p\pi}{d}z\right)$$

$$h^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

Resonant frequency for the mode mnp

$$f_{mnp} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2}$$

- **TMmnp** field components for **a rectangular resonance cavity**:
  - Consider a rectangular waveguide with both ends closed by a conducting wall.
  - And the interior dimensions of the cavity are a, b and d.



$$E_z(x,y,z) = E_0 sin\left(\frac{m\pi}{a}x\right) sin\left(\frac{n\pi}{b}y\right) cos\left(\frac{p\pi}{d}z\right)$$

$$E_x(x,y,z) = -\frac{1}{h^2}\left(\frac{m\pi}{a}\right)\left(\frac{p\pi}{d}\right) E_0\, cos\left(\frac{m\pi}{a}x\right) sin\left(\frac{n\pi}{b}y\right) sin\left(\frac{p\pi}{d}z\right)$$

$$E_y(x,y,z) = -\frac{1}{h^2}\left(\frac{n\pi}{b}\right)\left(\frac{p\pi}{d}\right) E_0\, sin\left(\frac{m\pi}{a}x\right) cos\left(\frac{n\pi}{b}y\right) sin\left(\frac{p\pi}{d}z\right)$$

$$H_x(x,y,z) = \frac{j\omega\epsilon}{h^2} E_0 \ldots$$

$$H_y(x,y,z) = -\frac{j\omega\epsilon}{h^2} \ldots$$

For a given mode, the resonance frequency depends on the material and the geometry.

Resonant frequency for the mode mnp

$$h^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

$$f_{mnp} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2}$$

SUNDAY

RF Project:

a) Write the instantaneous field expressions for the TM_{11} mode in a rectangular waveguide of sides a and b.

b) Use Matlab/Octave and draw electric and magnetic field lines in a typical xy plane and in a typical yz plane.

c) Repeat the same exercise for a rectangular resonant cavity for TM010 mode.

Figures of merit

- **Quality factor,**

$$Q = \frac{\omega U}{P}$$

- **Shunt impedance**
  - "Ohm's law" resistance

$$r_s = \frac{V_0^2}{P}$$

- **Effective shunt impedance**
  - impedance including transient time factor

$$r = \frac{(V_0 T)^2}{P} = r_s T^2$$

- **Shunt impedance per unit length**

$$Z = \frac{r_s}{L} = \frac{E_0^2}{P/L}$$

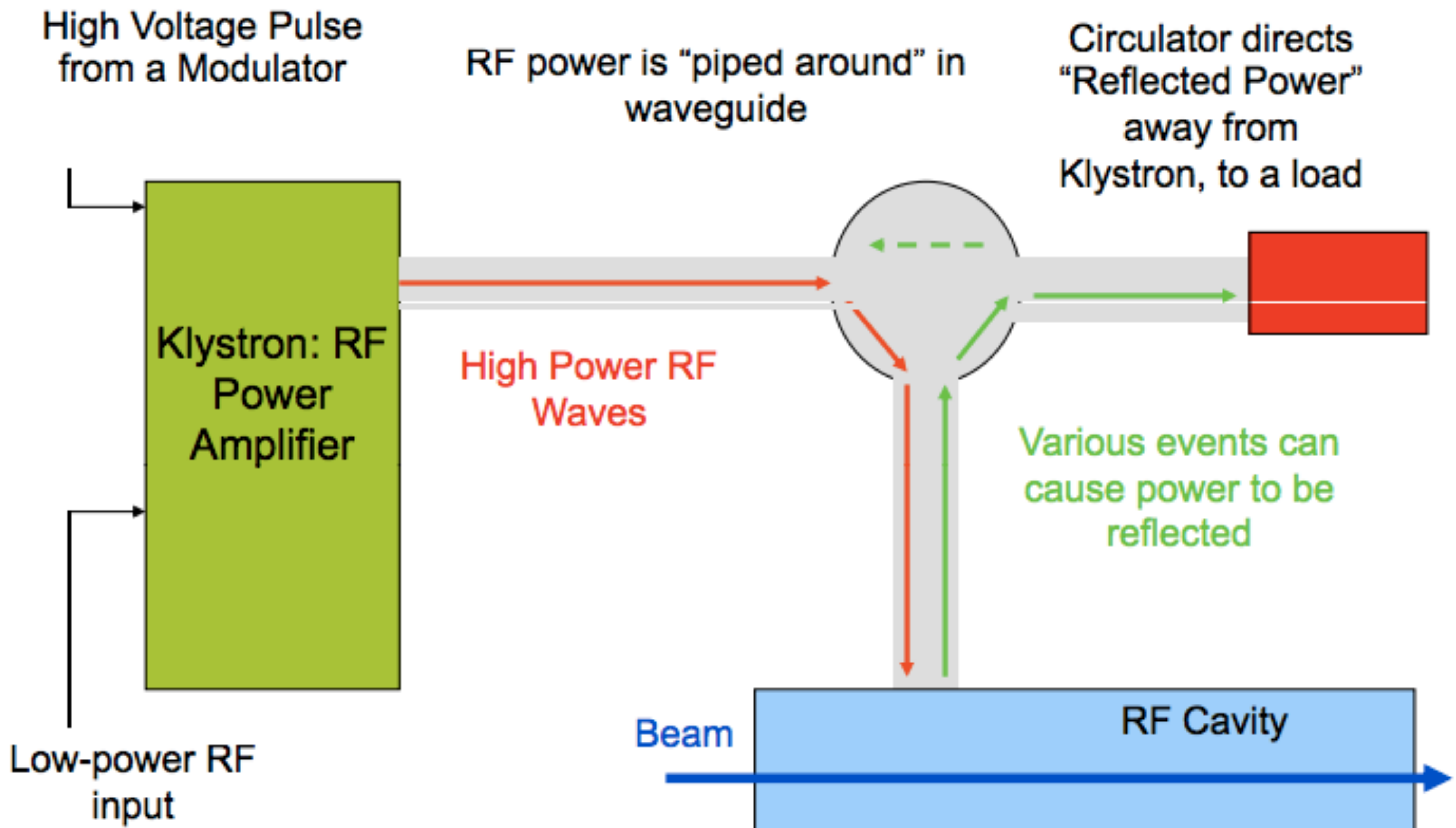
- **Effective shunt impedance per unit length**

$$ZT^2 = \frac{r}{L} = \frac{(E_0 T)^2}{P/L}$$

- **"R over Q"**
  - Efficiency of acceleration per unit of stored energy.

$$\frac{r}{Q} = \frac{(V_0 T)^2}{\omega U}$$

Components of a High Power RF System

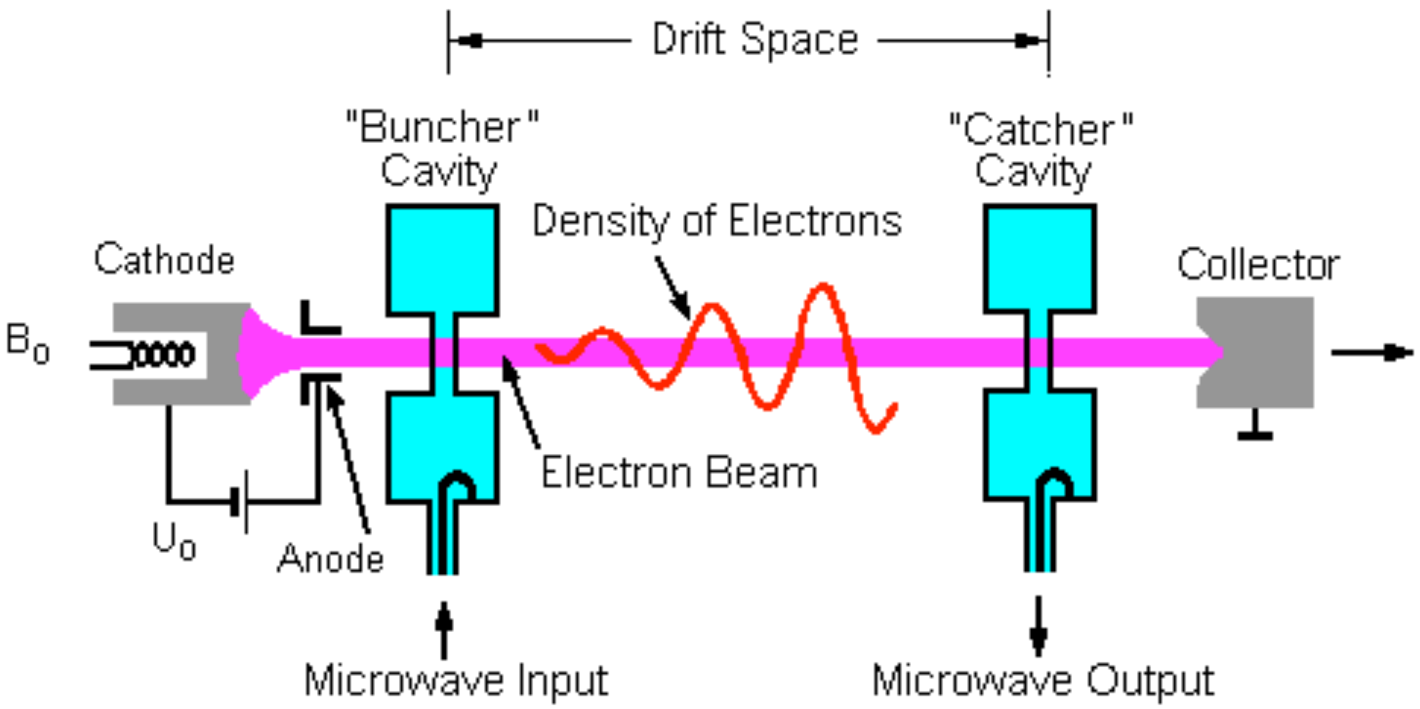


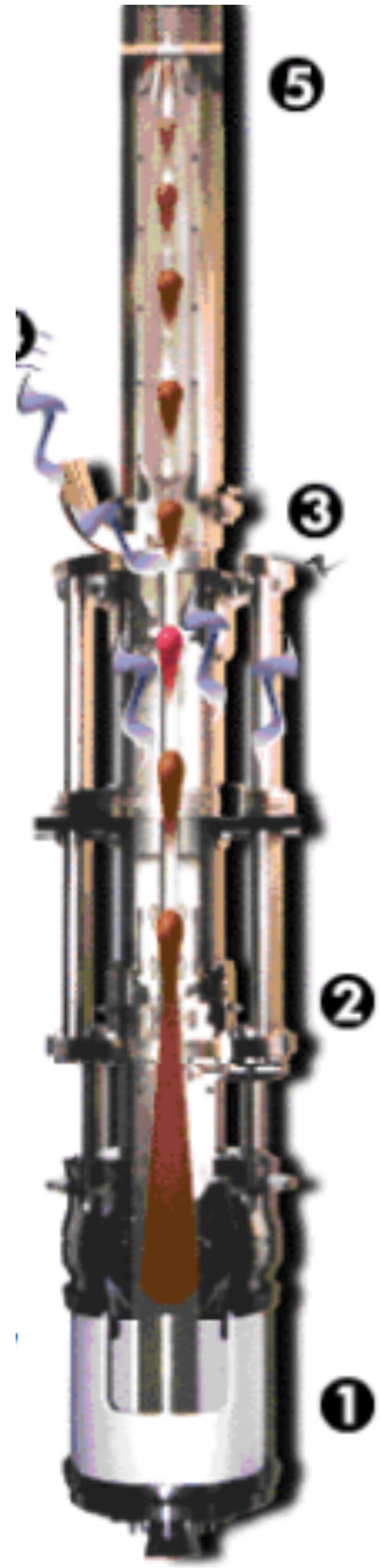
- Power delivered to the beam: $P_B = EI$
- Total power provided by the RF power source: $P_T = P + P_B$

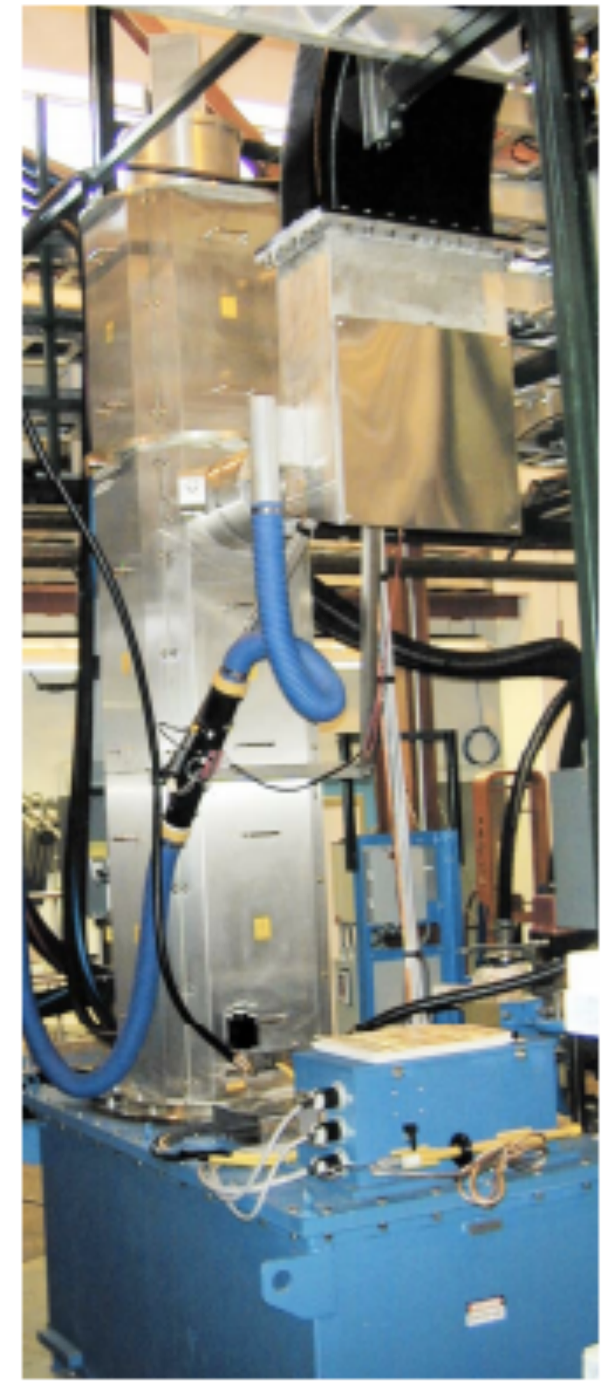
Circulation losses

http://uspas.fnal.gov/materials/09VU/Lecture4.pdf

Klystrons







- **A klystron is an amplifier for RF waves**
- **It is a small scale accelerator/RF cavity system**
- **Transfers beam power into RF power**
    1. A high voltage pulse accelerates an electron beam
    2. Low power RF excites the first cavity and bunches the electron beam
    3. This electrons “ring the bell” in the next cavity
    4. Electron bunches excite the cavity and generates RF power
    5. Power is transferred by wave guides to the equipment to be driven by RF power.

http://uspas.fnal.gov/materials/09VU/Lecture4.pdf

Example Problem

- Consider a 10-cm-long copper ($1/\sigma = 1.7\times10^{-8}\ \Omega$ m) TM_{010} pillbox cavity with resonant frequency of 500 MHz and axial field E=1.5MV/m.
  a) For a proton with kinetic energy of 100 MeV, calculate the transit-time factor ignoring the effects of the aperture, and assuming that the velocity remains constant in the gap
  b) If the proton arrives at the center of the gap 45 degrees before the crest, what is the energy gain?
  c) Calculate the RF power dissipated in the cavity walls
  d) Suppose this cavity is used to accelerate a 100 mA beam. What is the total RF power that must be provided by the klystron?
  e) Calculate the shunt impedance, the effective shunt impedance, the shunt impedance per unit length, and the effective shunt impedance per unit length
  f) Assume the drift tube bore radius is 2 cm. Calculate the transit-time factor, including the aperture effects, for the proton on-axis, and off-axis by 1 cm. Assume that $I_0(x) = 1 + x^2/4 \quad J_0(x) = 1 - x^2/4$

http://uspas.fnal.gov/materials/09VU/Lecture4.pdf

Vector Formulas

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$
$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$
$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$
$$\nabla\times\nabla\psi = 0$$
$$\nabla\cdot(\nabla\times\mathbf{a}) = 0$$
$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$
$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$$
$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$$
$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$
$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$
$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

If $\mathbf{x}$ is the coordinate of a point with respect to some origin, with magnitude $r = |\mathbf{x}|$, $\mathbf{n} = \mathbf{x}/r$ is a unit radial vector, and $f(r)$ is a well-behaved function of r, then

$$\nabla\cdot\mathbf{x} = 3 \qquad \nabla\times\mathbf{x} = 0$$
$$\nabla\cdot[\mathbf{n}f(r)] = \frac{2}{r}f + \frac{\partial f}{\partial r} \qquad \nabla\times[\mathbf{n}f(r)] = 0$$
$$(\mathbf{a}\cdot\nabla)\mathbf{n}f(r) = \frac{f(r)}{r}[\mathbf{a} - \mathbf{n}(\mathbf{a}\cdot\mathbf{n})] + \mathbf{n}(\mathbf{a}\cdot\mathbf{n})\frac{\partial f}{\partial r}$$
$$\nabla(\mathbf{x}\cdot\mathbf{a}) = \mathbf{a} + \mathbf{x}(\nabla\cdot\mathbf{a}) + i(\mathbf{L}\times\mathbf{a})$$

where $\mathbf{L} = \frac{1}{i}(\mathbf{x}\times\nabla)$ is the angular-momentum operator.

Theorems from Vector Calculus

In the following ϕ, ψ, and $\mathbf{A}$ are well-behaved scalar or vector functions, V is a three-dimensional volume with volume element d^3x, S is a closed two-dimensional surface bounding V, with area element da and unit outward normal $\mathbf{n}$ at da.

$$\int_V \nabla \cdot \mathbf{A}\, d^3x = \int_S \mathbf{A} \cdot \mathbf{n}\, da \qquad \text{(Divergence theorem)}$$

$$\int_V \nabla\psi\, d^3x = \int_S \psi\mathbf{n}\, da$$

$$\int_V \nabla \times \mathbf{A}\, d^3x = \int_S \mathbf{n} \times \mathbf{A}\, da$$

$$\int_V (\phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi)\, d^3x = \int_S \phi\mathbf{n} \cdot \nabla\psi\, da \qquad \text{(Green's first identity)}$$

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, d^3x = \int_S (\phi\nabla\psi - \psi\nabla\phi) \cdot \mathbf{n}\, da \qquad \text{(Green's theorem)}$$

In the following S is an open surface and C is the contour bounding it, with line element $d\mathbf{l}$. The normal $\mathbf{n}$ to S is defined by the right-hand-screw rule in relation to the sense of the line integral around C.

$$\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, da = \oint_C \mathbf{A} \cdot d\mathbf{l} \qquad \text{(Stokes's theorem)}$$

$$\int_S \mathbf{n} \times \nabla\psi\, da = \oint_C \psi\, d\mathbf{l}$$

Explicit Forms of Vector Operations

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian $(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical (ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \mathbf{e}_3\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical (r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right] + \mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi).\right]$$

Further Reading

- The physics of particle accelerators, Klaus Wille, Chapter 5,
- Field and Wave Electromagnetics, David K. Cheng, Chapter 10,
- Classical Electrodynamics, J. D. Jackson, Chapter 8.