Comparison between charged-particle (CP) and light/laser (L) optics

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Hamiltonians in Different Prescriptions

The following are the Hamiltonians, in the different prescriptions of light beam optics and charged-particle beam optics for magnetic systems. $\hat{H}_{0,p}$ are the paraxial Hamiltonians, with lowest order wavelength-dependent contributions.

<table>
<thead>
<tr>
<th>Light Beam Optics</th>
<th>Charged-Particle Beam Optics</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{H}_{0,p} =$</td>
<td>$\hat{H}_{0,p} =$</td>
</tr>
<tr>
<td>$- n(r) + \frac{1}{2\aleph_0} \hat{p}_\perp^2$</td>
<td>$- p_0 - q A_z + \frac{1}{2p_0} \hat{\Gamma}_\perp^2$</td>
</tr>
<tr>
<td>$- i\chi \beta \Sigma \cdot u$</td>
<td>$- \frac{\hbar}{2p_0} \left{ \mu \gamma \Sigma_\perp \cdot B_\perp + (q + \mu) \Sigma Z B_z \right}$</td>
</tr>
<tr>
<td>$+ \frac{1}{2\aleph_0} \lambda^2 w^2 \beta$</td>
<td>$+ i \frac{\hbar}{\aleph_0 c} \epsilon B_z$</td>
</tr>
</tbody>
</table>

Maxwell, Matrix

$\hat{H}_{0,p} =$

Dirac Formalism

$\hat{H}_{0,p} =$

Notation

Refractive Index, $n(r) = c \sqrt{\varepsilon(r) \mu(r)}$

Resistance, $h(r) = \sqrt{\mu(r) / \varepsilon(r)}$

$u(r) = - \frac{1}{2n(r)} \nabla n(r)$

$w(r) = \frac{1}{2h(r)} \nabla h(r)$

$\Sigma$ and $\beta$ are the Dirac matrices.

$\hat{\Gamma}_\perp = \hat{p}_\perp - q A_\perp$

$\mu_a$ anomalous magnetic moment.

$\epsilon_a$ anomalous electric moment.

$\mu = 2m_0 \mu_a / \hbar$,

$\epsilon = 2m_0 \epsilon_a / \hbar$

$\gamma = E / m_0 c^2$
Plan of attack:

• brief summary of matrix approach to light ray optics (Marta Divall’s talk at GANIL School also refers)

• extension of ideas to CP optics

• differences between light (L) and charged-particle (CP) optics

• deviations from ideal beams in L and CP

• some examples of similarities and differences
Each optical system will have an axis, and all light rays will be assumed to propagate at small angles to it (Paraxial Approximation).

We define all rays relative to the relevant optical axis.
Ray Vector

A light ray can be defined by two co-ordinates:

- position, $x$
- slope, $\theta$

These parameters define a **ray vector**, which will change with distance as the ray propagates through the optics.

$$
\begin{bmatrix}
    x \\
    \theta
\end{bmatrix}
$$
Ray Matrices

For many optical components, we can define 2 x 2 ray matrices. An element’s effect on a ray is found by multiplying its ray vector.

Optical system ↔ 2 x 2 ray matrix

These matrices are often called ABCD Matrices.
Ray matrices as derivatives

Since the displacements and angles are assumed to be small, we can think in terms of partial derivatives.

\[
x_{\text{out}} = \frac{\partial x_{\text{out}}}{\partial x_{\text{in}}} x_{\text{in}} + \frac{\partial x_{\text{out}}}{\partial \theta_{\text{in}}} \theta_{\text{in}}
\]

\[
\theta_{\text{out}} = \frac{\partial \theta_{\text{out}}}{\partial x_{\text{in}}} x_{\text{in}} + \frac{\partial \theta_{\text{out}}}{\partial \theta_{\text{in}}} \theta_{\text{in}}
\]

We can write these equations in matrix form.
For cascaded elements, we simply multiply ray matrices.

\[
\begin{bmatrix}
  x_{\text{in}} \\
  \theta_{\text{in}}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  0_1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  0_2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  0_3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  x_{\text{out}} \\
  \theta_{\text{out}}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  x_{\text{out}} \\
  \theta_{\text{out}}
\end{bmatrix} = 0_3 \left\{ 0_2 \left( 0_1 \begin{bmatrix}
  x_{\text{in}} \\
  \theta_{\text{in}}
\end{bmatrix} \right) \right\} = 0_3 \cdot 0_2 \cdot 0_1
\begin{bmatrix}
  x_{\text{in}} \\
  \theta_{\text{in}}
\end{bmatrix}
\]

Notice that the order looks opposite to what it should be, but it makes sense when you think about it.
Some typical ray matrices

Ray matrix for free space or a medium

If $x_{in}$ and $\theta_{in}$ are the position and slope upon entering, let $x_{out}$ and $\theta_{out}$ be the position and slope after propagating from $z = 0$ to $z$.

$$x_{out} = x_{in} + z \theta_{in}$$

$$\theta_{out} = \theta_{in}$$

Rewriting these expressions in matrix notation:

$$\begin{bmatrix} x_{out} \\ \theta_{out} \end{bmatrix} = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{in} \\ \theta_{in} \end{bmatrix}$$

$$O_{space} = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}$$
Ray Matrix for an Interface

At the interface, clearly:

\[ x_{\text{out}} = x_{\text{in}}. \]

Now calculate \( \theta_{\text{out}} \).

Snell's Law says:

\[ n_1 \sin(\theta_{\text{in}}) = n_2 \sin(\theta_{\text{out}}) \]

which becomes for small angles:

\[ n_1 \theta_{\text{in}} = n_2 \theta_{\text{out}} \]

\[ \Rightarrow \theta_{\text{out}} = \left[ n_1 / n_2 \right] \theta_{\text{in}} \]

\[ O_{\text{interface}} = \begin{bmatrix} 1 & 0 \\ 0 & n_1 / n_2 \end{bmatrix} \]
Ray matrix for a curved interface

If the interface is curved:

\[
\theta_1 = \theta_{\text{in}} + \frac{x_{\text{in}}}{R} \quad \text{and} \quad \theta_2 = \theta_{\text{out}} + \frac{x_{\text{in}}}{R}
\]

Snell's Law: \( n_1 \theta_1 = n_2 \theta_2 \)

\[
\Rightarrow n_1 (\theta_{\text{in}} + \frac{x_{\text{in}}}{R}) \approx n_2 (\theta_{\text{out}} + \frac{x_{\text{in}}}{R})
\]

\[
\Rightarrow \theta_{\text{out}} \approx (n_1 / n_2)(\theta_{\text{in}} + \frac{x_{\text{in}}}{R}) - \frac{x_{\text{in}}}{R}
\]

\[
\Rightarrow \theta_{\text{out}} \approx (n_1 / n_2)\theta_{\text{in}} + (n_1 / n_2 - 1)\frac{x_{\text{in}}}{R}
\]

\[
O_{\text{curved interface}} = \begin{bmatrix}
1 & 0 \\
\frac{n_1}{n_2} - 1 / R & \frac{n_1}{n_2}
\end{bmatrix}
\]

Now the output angle also depends on the input position
A thin lens is just two curved interfaces

We’ll neglect the glass in between (it’s a really thin lens!), and take \( n_1 = 1 \).

\[
O_{\text{curved interface}} = \begin{bmatrix}
1 & 0 \\
(n_1 / n_2 - 1) / R & n_1 / n_2
\end{bmatrix}
\]

\[
O_{\text{thin lens}} = O_{\text{curved interface}}_2 O_{\text{curved interface}}_1 = \begin{bmatrix}
1 & 0 \\
(n-1) / R_2 & n
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
(1/n-1) / R_1 & 1 / n
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 \\
(n-1) / R_2 + n(1/n-1) / R_1 & n(1/n)
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
(n-1) / R_2 + (1-n) / R_1 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 \\
(n-1)(1/R_2 - 1/R_1) & 1
\end{bmatrix}
\]

This can be written: \[
\begin{bmatrix}
1 & 0 \\
-1/f & 1
\end{bmatrix}
\]

where: \[ 1/f = (n-1)(1/R_1 - 1/R_2) \]

The Lens-Maker’s Formula
Ray matrix for a lens

\[ \frac{1}{f} = (n-1)(1/R_1 - 1/R_2) \]

The quantity \( f \) is the **focal length** of the lens. It’s the single most important parameter of a lens. It can be positive or negative.

If \( f > 0 \), the lens deflects rays toward the axis.

If \( f < 0 \), the lens deflects rays away from the axis.
A system images an object when $B = (x / \theta) = 0$.

When $B = 0$, all rays from a point $x_{in}$ arrive at a point $x_{out}$, independent of angle.

$$\begin{align*}
\begin{bmatrix} x_{out} \\ \theta_{out} \end{bmatrix} &= \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \begin{bmatrix} x_{in} \\ \theta_{in} \end{bmatrix} = \begin{bmatrix} A x_{in} \\ C x_{in} + D \theta_{in} \end{bmatrix}
\end{align*}$$

$x_{out} = A x_{in}$

When $B = 0$, $A$ is the magnification.
Significance of zero values of matrix elements

\[ A = 0 \Rightarrow x_{out} \text{ independent of } x_{in} \Rightarrow \text{parallel-to-point focusing (focus)} \]

\[ B = 0 \Rightarrow x_{out} \text{ independent of } \theta_{in} \Rightarrow \text{point-to-point focusing (image of object)} \]

\[ C = 0 \Rightarrow \theta_{out} \text{ independent of } x_{in} \Rightarrow \text{parallel-to-parallel imaging (telescopic)} \]

\[ D = 0 \Rightarrow \theta_{out} \text{ independent of } \theta_{in} \Rightarrow \text{point-to-parallel imaging (e.g. headlamp)} \]

\[
\begin{bmatrix}
  x_{out} \\
  \theta_{out}
\end{bmatrix} =
\begin{bmatrix}
  \text{par} \rightarrow \text{pt} & \text{pt} \rightarrow \text{pt} \\
  \text{par} \rightarrow \text{par} & \text{pt} \rightarrow \text{par}
\end{bmatrix}
\begin{bmatrix}
  x_{in} \\
  \theta_{in}
\end{bmatrix}
\]
Thin Lens Equation
(using matrices)

\[ O = \begin{bmatrix} 1 & d_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & d_o \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & d_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d_o \\ -1/f & 1-d_o/f \end{bmatrix} = \begin{bmatrix} 1-d_i/f & d_o+d_i-d_o d_i/f \\ -1/f & 1-d_o/f \end{bmatrix} \]

\[ B = d_o + d_i - d_o d_i / f \]
\[ = d_o d_i \left[ 1/d_o + 1/d_i - 1/f \right] \]
\[ = 0 \quad \text{if} \quad \frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f} \]
Imaging Magnification

If the imaging condition,
\[
\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}
\]
is satisfied, then:

\[
O = \begin{bmatrix}
1 - d_i / f & 0 \\
-1 / f & 1 - d_o / f
\end{bmatrix}
\]

So:

\[
O = \begin{bmatrix}
M & 0 \\
-1 / f & 1 / M
\end{bmatrix}
\]

\[
A = 1 - d_i / f = 1 - d_i \left( \frac{1}{d_o} + \frac{1}{d_i} \right)
\]

\[
\Rightarrow M = -\frac{d_i}{d_o}
\]

\[
D = 1 - d_o / f = 1 - d_o \left( \frac{1}{d_o} + \frac{1}{d_i} \right)
\]

\[
= -\frac{d_o}{d_i} = 1 / M
\]
Consecutive lenses

Suppose we have two lenses right next to each other (with no space in between).

\[
O_{tot} = \begin{bmatrix}
1 & 0 \\
-1/f_2 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
-1/f_1 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
-1/f_1 -1/f_2 & 1
\end{bmatrix}
\]

\[1/f_{tot} = 1/f_1 + 1/f_2\]

So two consecutive lenses act as one whose focal length is the sum of the individual focal lengths.

[As a result, we define a measure of inverse lens focal length, the dioptre, where \(1 \text{ dioptre} = 1 \text{ m}^{-1}\).]
Lenses can also map angle ↔ position

From the object to the image, we have:

1) A distance $f$
2) A lens of focal length $f$
3) A distance $f$

\[
\begin{bmatrix}
    x_{\text{out}} \\
    \theta_{\text{out}}
\end{bmatrix} =
\begin{bmatrix}
    1 & f \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    1 & 0 \\
    -1/f & 1
\end{bmatrix}
\begin{bmatrix}
    1 & f \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    x_{\text{in}} \\
    \theta_{\text{in}}
\end{bmatrix}
\]

\[
= 
\begin{bmatrix}
    1 & f \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    1 & f \\
    -1/f & 1
\end{bmatrix}
\begin{bmatrix}
    x_{\text{in}} \\
    \theta_{\text{in}}
\end{bmatrix}
\]

\[
= 
\begin{bmatrix}
    0 & f \\
    -1/f & 0
\end{bmatrix}
\begin{bmatrix}
    x_{\text{in}} \\
    \theta_{\text{in}}
\end{bmatrix}
= 
\begin{bmatrix}
    f \theta_{\text{in}} \\
    -x_{\text{in}} / f
\end{bmatrix}
\]

So $x_{\text{out}} \propto \theta_{\text{in}}$

and this arrangement maps position to angle:

$\theta_{\text{out}} \propto x_{\text{in}}$
If an optical system lacks axial symmetry, we must analyse its x- and y-directions separately: e.g. cylindrical lenses

A "spherical lens" focuses in both transverse directions.
A "cylindrical lens" focuses in only one transverse direction.

When using cylindrical lenses, we must perform two separate ray-matrix analyses, one for each transverse direction. This compares with a Quadrupole lens in charged-particle optics.
Complex Beam Parameter for Gaussian Beams

Gaussian distribution is a solution of the paraxial Helmholtz equation TEM\textsubscript{00} mode

- The ABCD method can be extended to Gaussian beams using the complex beam radius, \( q(z) \):

\[
\frac{1}{q(z)} = \frac{1}{R(z)} - i \frac{\lambda}{\pi w(z)^2}
\]

- After finding the optical ABCD matrix as before, the complex radius of curvature is transformed in the following way

\[
q'(z) = \frac{Aq(z) + B}{Cq(z) + D}
\]

- Characterisation of the beam allows us to predict its form elsewhere

- No time to discuss details here ….
Ray Matrix for a Curved Mirror

Consider a mirror with radius of curvature $R$, with its optical axis perpendicular to the mirror:

$$\theta_1 = \theta_{in} - \theta_s \quad \theta_s \approx x_{in} / R$$

$$\theta_{out} = \theta_1 - \theta_s = (\theta_{in} - \theta_s) - \theta_s$$

$$\approx \theta_{in} - 2x_{in} / R$$

$$O_{\text{mirror}} = \begin{bmatrix} 1 & 0 \\ -2/R & 1 \end{bmatrix}$$

Like a lens, a curved mirror will focus a beam. Its focal length is $R/2$. Note that a flat mirror has $R = \infty$ and hence an identity ray matrix.
Laser Cavities / Resonators

Two flat mirrors, the **flat-flat** laser cavity, is difficult to align and maintain aligned.

Two concave curved mirrors, the usually **stable** laser cavity, is generally easy to align and maintain aligned.

e.g. confocal or concentric

Two convex mirrors, the **unstable** laser cavity, is impossible to align!

(coULD ALSO BE flat–plane, etc.)
Unstable Resonators

An unstable cavity (or unstable resonator) can work if you do it properly!
In fact, it produces a large diameter beam, useful for high-power lasers, which must have large beams.

The mirror curvatures determine the beam size, which, for a stable resonator, is small (100 μm to 1 mm).

An unstable resonator can have a very large beam, but the gain must be high… and the beam has a hole in it.
Stability of a resonator - ray matrix analysis

Consider paraxial ray within resonator – if ray escapes after finite number of bounces, resonator is unstable.

ABCD matrix analysis for a single round trip:

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
-2/R_1 & 1
\end{bmatrix} \begin{bmatrix}
1 & L \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
-2/R_2 & 1
\end{bmatrix} \begin{bmatrix}
1 & L \\
0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 - 2L/R_2 & 2L - 2L^2/R_2 \\
4L/R_1R_2 - 2/R_1 - 2/R_2 & 1 - 2L/R_2 - 4L/R_1 + 4L^2/R_1R_2
\end{bmatrix}
\]

After N round trips, the output ray is related to the initial ray by:

\[
\begin{bmatrix}
x_{\text{out}} \\
\theta_{\text{out}}
\end{bmatrix} = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}^N \begin{bmatrix}
x_{\text{in}} \\
\theta_{\text{in}}
\end{bmatrix}
\]

Define new variable, \( \chi \), such that:

\[
\cos \chi = 1 - \frac{2L}{R_1} - \frac{2L}{R_2} + \frac{2L^2}{R_1R_2}
\]

Can show that:

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}^N = \frac{1}{\sin \chi} \begin{bmatrix}
A \sin(N \chi) - \sin[(N-1)\chi] & B \sin(N \chi) \\
C \sin(N \chi) & D \sin(N \chi) - \sin[(N-1)\chi]
\end{bmatrix}
\]
\[ x_{out} = \frac{1}{\sin \chi} \left[ (A \sin N \chi - \sin(N-1)\chi) x_{in} + (B \sin N \chi) \theta_{in} \right] \]

Note that the output ray position \( x_{out} \) remains finite when \( N \) goes to infinity, as long as \( \chi \) is a real number.

If \( \chi \) becomes complex, then

\[
\sin N \chi = \frac{1}{2i} \left( e^{iN\chi} - e^{-iN\chi} \right)
\]

one of these exponential terms grows to infinity as \( N \) gets large

Thus the condition for resonator stability is that \( \chi \) is real, or \( |\cos \chi| \leq 1 \)

\[
0 \leq \left( 1 - \frac{L}{R_1} \right) \left( 1 - \frac{L}{R_2} \right) \leq 1
\]

Defining the “\( g \) parameters” of the resonator by:

\[
g_1 = 1 - \frac{L}{R_1} \quad g_2 = 1 - \frac{L}{R_2}
\]

Then, the stability requirement is:

\[
0 \leq g_1 g_2 \leq 1
\]

Shaded regions show the stable solutions
Analysis of Gaussian laser beams gives the same result for stable cavity modes!

- Planar – rarely used, large mode size, very sensitive to misalignment
- Concentric – also very sensitive to misalignment, so rarely used
- Confocal – smallest average spot size, relatively insensitive to misalignment, commonly used
- Hemispherical – very insensitive to misalignment, very common design

... etc.
Aberrations

Aberrations are distortions that occur in images, usually due to imperfections in lenses - some unavoidable, some avoidable.

They include:

- Chromatic aberration
- Spherical aberration
- Astigmatism
- Coma
- Curvature of field
- Pincushion and Barrel distortion

Most aberrations can’t be modelled with ray matrices. Designers beat them with lenses of multiple elements, that is, several lenses in a row. Some zoom lenses can have as many as a dozen or more elements.
Chromatic Aberration

Because the lens material has a different refractive index for each wavelength, the lens will have a different focal length for each wavelength. Recall the lens-maker’s formula:

\[ \frac{1}{f(\lambda)} = (n(\lambda) - 1)(1/R_1 - 1/R_2) \]

Here, the refractive index is larger for blue than red, so the focal length is less for blue than red.

Can model spherical aberration using ray matrices, but only one colour at a time.

c.f. strong chromatic aberration in magnetic quadrupole lenses.
Chromatic aberration can be minimised using additional lenses

In an achromat, the second lens cancels the dispersion of the first.

Achromats use two different materials, and one has a negative focal length.
Spherical Aberration in Mirrors

For all rays to converge to a point a distance $f$ away from a curved mirror, we require a paraboloidal surface.

But this only works for rays with $\theta_{in} = 0$. 
Spherical Aberration in Lenses

So we use spherical surfaces, which work better for a wider range of input angles.

Nevertheless, off-axis rays see a different focal length, so lenses have spherical aberration, too.
Minimising spherical aberration in a focus

\[ q \equiv \frac{R_2 + R_1}{R_2 - R_1} \]

- \( R_1 \) = Front surface radius of curvature
- \( R_2 \) = Back surface radius of curvature

*plano-convex lenses* (with their flat surface facing the focus) are best for minimising spherical aberration when *focusing*.

*One-to-one imaging* works best with a *symmetrical lens* \((q = 0)\).
Spherical aberration can also be minimised using additional lenses

(general point here is that aberrations may be minimised by utilising symmetry.

Also true in CP optics)
Astigmatism

When the optical system lacks perfect cylindrical symmetry, we say it has astigmatism. A simple cylindrical lens or off-axis curved-mirror reflection will cause this problem.

Cure astigmatism with another cylindrical lens or off-axis curved mirror.

- We’ll see that CP systems are intrinsically astigmatic
Geometrical Optics terms for lens system
Charged-Particle (CP) Optics

Here we use predominantly magnetic field configurations to bend and focus CP beams (positive ions or electrons).

• Bending performed by (predominantly) uniform-field dipole magnets

• Focusing performed (predominantly) by quadrupole magnets

• Higher-order corrections applied via sextupole, octupole, etc., magnets

Lorentz Force

\[ \mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \]

- \( q \) = electric charge
- \( \mathbf{B} \) = magnetic induction
- \( \mathbf{E} \) = electric field
- \( \mathbf{v} \) = velocity
dipole bending magnet

quadrupole focusing magnet

quadrupole triplet in beam line

sextupole magnet
Deviations from ideal beams in L and CP

\( M^2 \) versus Emittance

\( M^2 \) - deviation of real laser beam envelope from ideal TEM_{00} Gaussian envelope

- fundamental beam divergence a consequence of diffraction of finite aperture beam

Emittance – measure of finite phase space occupied by CP beam envelope (or an agreed fraction of it)

- normally a conserved quantity for beam transport systems (within well-defined conditions)
M-squared is a bit like emittance.

M-squared measurement is like a solenoid or quadrupole scan.

\[
M^2 = \left( \frac{D_m \cdot \Theta_m}{d_0 \cdot \Theta_0} \right) = \left( \frac{D_m}{d_0} \right)^2.
\]

\[
\theta = \frac{M^2 \lambda}{\pi w_0^2}
\]

\[
w(z) = w_0 \sqrt{1 + \left( M^2 \frac{\lambda z}{\pi w_0^2} \right)^2}
\]

M² in real Gaussian beams

M² remains invariant through ABCD optical systems.
CP Beam Emittance, $\varepsilon$

$\varepsilon$ defined as 2D phase space area divided by $\pi$

Liouville’s Theorem:

- The volume occupied in phase space by a system of particles is constant.
- This is a general physics theorem in canonical co-ordinates, not limited to accelerators.
- Application of external forces or emission of radiation needs to be treated carefully [LT holds for fixed E & conservative forces]
- Can define normalised emittance, $\varepsilon_n = \varepsilon / \beta \gamma$
As beam propagates, shape of the phase ellipse will change.

How the conserved phase space appears at different points in a FODO cell. The development of a constant-emittance beam in phase space at (a) a narrow waist, (b) and (c) places where the beam is diverging, and (d) at a broad maximum at the centre of an F lens.
measured H phase space using solenoid scan

N.B. Even when LT holds, non-linear forces can give rise to an increase in the ‘effective’ phase space area occupied by the beam, termed ‘filamentation’.
Transport of charged particle beams

- Basic calculations (to 3rd order) can be carried out by matrix techniques quite similar to ABCD

- **BUT** - inherent astigmatism of magnetic-optical components demands that we use separate matrix elements for x (H) and y (V) motion

- **AND** chromaticity of systems means we need to add 2 more components to our matrix, giving a 6 x 6 matrix.

- Matrix components are \((x, \theta, y, \phi, \ell, \delta)\)

Extra co-ordinates are:

\(\ell = \) bunch length of particle beam (strictly the difference in path length between the arbitrary trajectory and the central trajectory)

\(\delta = \delta p/p = \) fractional momentum width of particles in beam (which is same as fractional energy width \(\delta E/E\) for ultra-relativistic particles like electrons – but not in general for ions/protons)
Program **TRANSPORT** (K L Brown et al) is archetypal beam transport program, and almost all others use similar notation. Beam rep by 6 X 6 ‘R-matrix’:

\[
\begin{bmatrix}
x \\
\theta \\
y \\
\phi \\
\ell \\
\delta \\
\end{bmatrix} =
\begin{bmatrix}
R_{11} & R_{12} & R_{13} & R_{14} & R_{15} & R_{16} \\
R_{21} & R_{22} & R_{23} & R_{24} & R_{25} & R_{26} \\
R_{31} & R_{32} & R_{33} & R_{34} & R_{35} & R_{36} \\
R_{44} & R_{45} & R_{46} & etc \\
R_{55} & R_{56} & R_{66} \\
\end{bmatrix}
\begin{bmatrix}
x_0 \\
\theta_0 \\
y_0 \\
\phi_0 \\
\ell_0 \\
\delta \\
\end{bmatrix}
\]

Due to symmetries in system, many matrix elements are always **zero**

\[
\begin{bmatrix}
x \\
\theta \\
y \\
\phi \\
\ell \\
\delta \\
\end{bmatrix} =
\begin{bmatrix}
R_{11} & R_{12} & 0 & 0 & 0 & R_{16} \\
R_{21} & R_{22} & 0 & 0 & 0 & R_{26} \\
0 & 0 & R_{33} & R_{34} & 0 & 0 \\
0 & 0 & R_{43} & R_{44} & 0 & 0 \\
R_{51} & R_{52} & 0 & 0 & 1 & R_{56} \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_0 \\
\theta_0 \\
y_0 \\
\phi_0 \\
\ell_0 \\
\delta \\
\end{bmatrix}
\]

…assuming here (as is normal) that all **bends** are in **horizontal** plane.
Transport of a ray though a system of beam line elements is as before for L optics:

\[ x_n = (R_n R_{n-1} \ldots R_0) x_0 \]

Complete system is represented by one matrix:

\[ R_{\text{system}} = R_n R_{n-1} \ldots R_0 \]

…and, as for L optics, optical imaging conditions can be specified by setting certain matrix elements to zero. e.g.:

\[ R_{11} = 0 \Rightarrow \text{H plane parallel-to-point focusing} \]
\[ R_{12} = 0 \Rightarrow \text{H plane point-to-point imaging} \]
\[ R_{21} = 0 \Rightarrow \text{H plane parallel-to-parallel transformation} \]
\[ R_{22} = 0 \Rightarrow \text{H plane point-to-parallel transformation} \]

… with corresponding conditions for V plane (\( R_{33} \) to \( R_{44} \))
Curvilinear TRANSPORT Coordinate System

(a) Schematic representation of an optical system with vanishing $(x|a)$. Such a system is also referred to as point-to-point focusing. (b) Schematic representation of an optical system with $(a|x) = 0$. Such a system is also referred to as parallel-to-parallel focusing (it is also called a telescope). (c) Schematic representation of an optical system with $(x|x) = 0$. Such a system is also referred to as parallel-to-point focusing. (d) Schematic representation of an optical system with $(a|a) = 0$. Such a system is also referred to as point-to-parallel focusing.
In bend (H) plane also have **chromaticity conditions**:

\[ R_{16} = 0 \implies \text{achromatic} \quad \text{and} \quad R_{26} = 0 \implies \text{angle-achromatic} \]

Now we can also impose conditions on **bunch length**: 
e.g.

\[ R_{56} = 0 \implies \text{isochronous condition} \quad (\text{bunch length does NOT depend on particle momentum}) \ldots \]

plus if \( R_{51} = 0 \) and \( R_{52} = 0 \) also, this isochronous condition will be maintained down the transport system

This should ideally be maintained to 2\(^{nd}\) order \((T_{566} = 0, \text{etc})\).

**Important** in many modern accelerator and free-electron laser applications – but often **difficult** to achieve (**see later**).
e.g. TRANSPORT matrices of a Drift and a Quadrupole

Combine drifts and quadrupoles as before to make doublets and triplets:

N.B. Essential in CPO because quadrupole is highly astigmatic – need combinations to mimic rotationally-symmetric LO lenses.

Definitions:
- \( L \) = the effective length of the quadrupole
- \( a \) = the radius of the aperture
- \( B_0 \) = the field at radius \( a \)
- \( k^2_q = \frac{(B_0/a)(1/B_0)}{q} \) = the magnetic rigidity (momentum) of the central trajectory.
optics of a quadrupole singlet & doublet

Imagine designing laser optics with lenses like these!
FD and DF planes of a stigmatic quadrupole doublet

point-to-point quadrupole sextet

A point-to-parallel-to-point focusing quadrupole sextet showing particle trajectories that all start at the center of the object. Note that this sextet is a DFDDFD system.
TRANSPORT design calculation of a quadrupole triplet focusing on a target (tgt). horiz.(x) and vert.(y) envelopes of the phase ellipse (σ-matrix).

This symmetrical triplet $\frac{1}{2}f - d - \frac{1}{2}f$ corresponds to an optical lens.

This indicates the complexity of beam transport with magnetic elements!
TRANSPORT bending magnet (dipole) matrix (first-order only):

**First-order wedge bending magnet matrix**

<table>
<thead>
<tr>
<th></th>
<th>$\cos k_x L$</th>
<th>$\frac{1}{k_x} \sin k_x L$</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>$\frac{h}{k_x^2} (1 - \cos k_x L)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-k_x \sin k_x L$</td>
<td>$- \frac{h}{k_x} (1 - \cos k_x L)$</td>
<td>$\cos k_x L$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{h}{k_x^2} \sin k_x L$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>$\cos k_y L$</td>
<td>$\frac{1}{k_y} \sin k_y L$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>$-k_y \sin k_y L$</td>
<td>$\cos k_y L$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$- \frac{h}{k_x} \sin k_x L$</td>
<td>$- \frac{h}{k_x^2} (1 - \cos k_x L)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$1 - \frac{h^2}{k_x^3} (k_x L - \sin k_x L)$</td>
</tr>
</tbody>
</table>

**Definitions:**

- $h = 1/\rho_0$, $k_x^2 = (1 - n)h^2$, $k_y^2 = nh^2$
- $\alpha = hL$ = the angle of bend
- $L$ = path length of the central trajectory.

No time to go into details of dipoles here ....
Examples of systems, and comparison with L optics

What follows are some arbitrarily chosen examples to illustrate the similarities and differences between CP and L optics.

The references at the end contain many more examples.
Schematic layouts of achromatic magnet systems

Achromatic systems have broad momentum acceptance, or allow selection of a narrow momentum/energy spread.

**Fig. 9.6.** Two double sector field arrangements that both cause achromatic beam deflections. Note that there are opposite signs for the radii of deflection in (a) and opposite signs for the area $A_\alpha$ in (b).

An achromatic beam deflector consisting of two sector fields preceded and followed by quadrupole doublets. Note that the areas $A_{\alpha 1}$ and $-A_{\alpha 1}$ are of equal size. The quadrupole in the middle between the sector fields can be adjusted so that the overall system is not only dispersion free ($x|\delta_K| = (x|\delta_m) = 0$) but achromatic ($a|\delta_K| = (a|\delta_m) = 0$); i.e., particles of different rigidities ($\delta_K \neq 0$) are parallel in the middle of the center quadrupole and coincide at the end of the systems if they coincided at the beginning.

$R_{16} = R_{26} = 0$
Isochronous Systems

There is no point in generating an ultrashort electron (or CP) bunch if we cannot deliver this bunch unaltered to a target! Systems required need to be isochronous to at least 2\textsuperscript{nd} order in the particle co-ordinates.

Adjustability of the isochronism may be required, for example, in a recirculating accelerator such as an energy-recovery linac (ERL) e.g. Jefferson Lab ERL in Virginia.

adjustably isochronous 180-deg transport system for free-electron lasers (Gillespie, 1996). Dipoles D2 and D3 can be separated if required.

(sextupoles are required to cancel out the chromatic aberrations produced by the quadrupoles)
An isochronous time-of-flight mass spectrometer is shown that is nothing more than an accelerator storage ring with the quadrupole excitations chosen such that after a 180° bend the matrix elements \( \langle t|\delta_K \rangle \), \( \langle x|a \rangle \), and \( \langle y|b \rangle \) vanish exactly and \( \langle a|x \rangle \) as well as \( \langle b|y \rangle \) vanish approximately. Particles with start simultaneously at the target thus arrive at the same time after a deflection of \( n\pi \), with \( n = 1, 2, 3, \ldots \), independent of \( x, y, a, b \) and \( \delta_K \). If introduced by the indicated first “kicker,” they can be ejected from the ring by the second “kicker” after several turns. Introducing ac driven electrostatic deflectors at the points A and/or B, we can use such a ring also as a high-resolving mass separator, since charged particles can then pass only if the electrostatic fields go through zero. If this ac frequency is high, the achievable mass resolving powers can be very high.
**Non-isochronous systems**

*Comparison: electron bunch compressors with laser pulse compressors/expanders*

**Problem:** Coherent X-ray sources (single-pass FELs), linear colliders and laser plasma wakefield accelerators all require high peak currents/very short electron bunches, so we need bunch compression techniques.

**Magnetic electron bunch compression – Principle:**

- Energy modulator: RF structure, laser, wakefield
- Non-isochronous section of beam line
- In practice, may need multi-stage compression
Types of magnetic bunch compressor

- chicane
  \[ R_{56} < 0 \quad T_{566} > 0 \]
- S-chicane
  \[ R_{56} < 0 \quad T_{566} > 0 \]
- FODO arc
  \[ R_{56} > 0 \quad T_{566} > 0 \]
- Dog-leg
  \[ R_{56} > 0 \quad T_{566} > 0 \]
Compare laser beam expanders/compressors

Equivalents of magnetic dipoles are optical prisms or diffraction gratings.

Prism compressor. The red lines represent rays of longer wavelengths and the blue lines those of shorter wavelengths. The spacing of the red, green, and blue wavelength components after the compressor is drawn to scale. This setup has a negative dispersion.
chirped pulse amplification using pairs of gratings.

CPA is a powerful method of producing high-power ultrashort laser pulses.
Single-prism pulse compressor (Trebino)
4-f spectral filter used to carve out narrow-band pulse (Dundee-Daresbury group)

4-dipole magnetic chicane can be used to insert mask at symmetry plane SS’ (extensively used in FEL accelerator injectors)
Examples of dispersive systems (charged particle spectrometers)

BIG KARL
Spectrometer
(Jülich, KFZ)

Bending radius \( r_0 = 1.98 \) m  
\( B_{\text{max}} = 1.7 \) T  
gap = 6cm  
weight = ~ 50 tons (D1)  
~ 70 tons (D2)

Resolv. power: \( p/\Delta p = 20600 \)  
Dispersion = -2.0 to 26 cm/%  
Magnification \( M_x = 0.63 - 1.26 \)  
Magnification \( M_y = 25.4 - 1.94 \)  
Large range: \( E_{\text{min}} / E_{\text{max}} = 1.14 \)  
Solid angle: < 12.5 msr

Arrangement of the magnetic elements of the QQDDQ spectrometer BIG KARL. The central ray (optical axis) is shown as dashed curve. The outermost rays with the extreme radial distances are drawn as full lines. Four channels in the inner yokes allow NMR probes to be moved into the gaps of the dipoles for radial field measurements. The multipole element between Q1 and Q2 allows the correlation of vertical aberration.
A Q3D Spectrometer used for nuclear physics. Second-order corrections are made through curved entrance and exit faces of the bending magnets.
Grand Raiden High-Resolution Spectrometer
(Osaka, Japan)

Max. Magn. Rigidity: 5.1 Tm
Bending Radius: 3.0 m
Solid Angle: 3 msr
Useful References

Accelerators:
M S Livingston, J P Blewett, Particle Accelerators, McGraw-Hill, NY, 1962 (the original!)

Charged-particle optics:
High Energy Beam Optics, Klaus G Steffen, Wiley Interscience, 1964
Optics & Lasers:

A. Lipson, S. G. Lipson and H. Lipson, Optical Physics, Cambridge UP, 2011
Rick Trebino, Geometrical Optics Lecture Notes, Georgia Inst. of Technology

END
The transport of rays and phase ellipses in a drift and a focusing quadrupole lens

Matching of emittance and acceptance
Forces on ions in magnetic multipoles

Quadrupole

Hexapole

Octopole

Pole arrangements of magnetic quadrupoles, hexapoles, and octopoles are indicated. Also shown is a circle of radius $r_0$ along which the magnetic flux density is constant, and its direction varies as indicated. Finally, strings of zeros indicate lines along which $B_y$ the $y$ component of the magnetic flux density vanishes. These lines separate regions in which $B_y$ is parallel or antiparallel to the $y$ axis.