Comparison between charged-particle (CP) and light/laser (L) optics

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Hamiltonians in Different Prescriptions

The following are the Hamiltonians, in the different prescriptions of light beam optics and charged-particle beam optics for magnetic systems. $\widehat{H}_{0,p}$ are the paraxial Hamiltonians, with lowest order wavelength-dependent contributions.

Light Beam Optics	**Charged-Particle Beam Optics**
Maxwell, Matrix $\widehat{H}_{0,p} =$ $-n(\boldsymbol{r}) + \frac{1}{2n_0}\widehat{\boldsymbol{p}}_\perp^2$ $- \mathrm{i}\lambda\!\!\!^{-}\beta\boldsymbol{\Sigma}\cdot\boldsymbol{u}$ $+ \frac{1}{2n_0}\lambda\!\!\!^{-2}w^2\beta$	**Dirac Formalism** $\widehat{H}_{0,p} =$ $-p_0 - qA_z + \frac{1}{2p_0}\widehat{\boldsymbol{\pi}}_\perp^2$ $- \frac{\hbar}{2p_0}\left\{\mu\gamma\boldsymbol{\Sigma}_\perp\cdot\boldsymbol{B}_\perp + (q+\mu)\,\Sigma_z B_z\right\}$ $+ \mathrm{i}\frac{\hbar}{m_0 c}\epsilon B_z$

This is NOT the approach we are going to take !

Notation

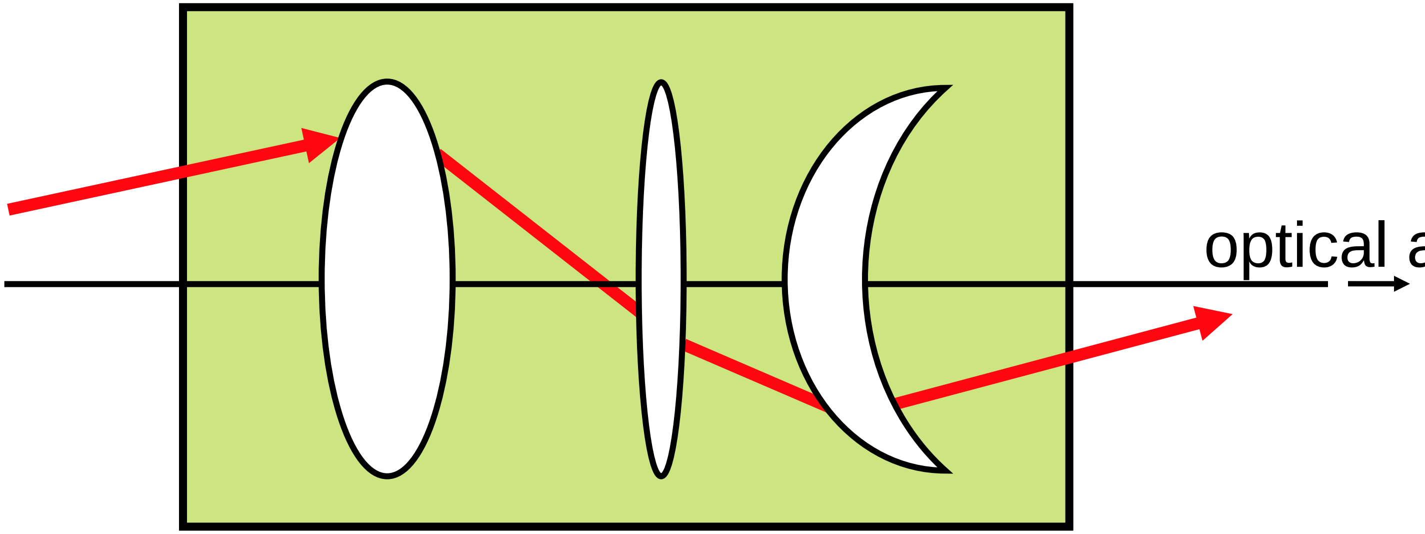
Refractive Index, $n(\boldsymbol{r}) = c\sqrt{\epsilon(\boldsymbol{r})\mu(\boldsymbol{r})}$
Resistance, $h(\boldsymbol{r}) = \sqrt{\mu(\boldsymbol{r})/\epsilon(\boldsymbol{r})}$
$\boldsymbol{u}(\boldsymbol{r}) = -\frac{1}{2n(\boldsymbol{r})}\boldsymbol{\nabla}n(\boldsymbol{r})$
$\boldsymbol{w}(\boldsymbol{r}) = \frac{1}{2h(\boldsymbol{r})}\boldsymbol{\nabla}h(\boldsymbol{r})$
$\boldsymbol{\Sigma}$ and β are the Dirac matrices.

$\widehat{\boldsymbol{\pi}}_\perp = \widehat{\boldsymbol{p}}_\perp - q\boldsymbol{A}_\perp$
μ_a anomalous magnetic moment.
ϵ_a anomalous electric moment.
$\mu = 2m_0\mu_a/\hbar\,, \quad \epsilon = 2m_0\epsilon_a/\hbar$
$\gamma = E/m_0c^2$

Plan of attack:

- brief summary of matrix approach to light ray optics (Marta Divall's talk at GANIL School also refers)
- extension of ideas to CP optics
- differences between light (L) and charged-particle (CP) optics
- deviations from ideal beams in L and CP
- some examples of similarities and differences

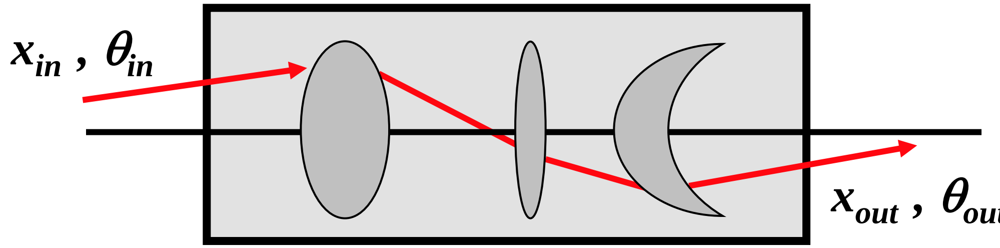
Light Ray Optics



Each optical system will have an axis, and all light rays will be assumed to propagate at small angles to it (**Paraxial Approximation**)

We define all rays relative to the relevant optical axis.

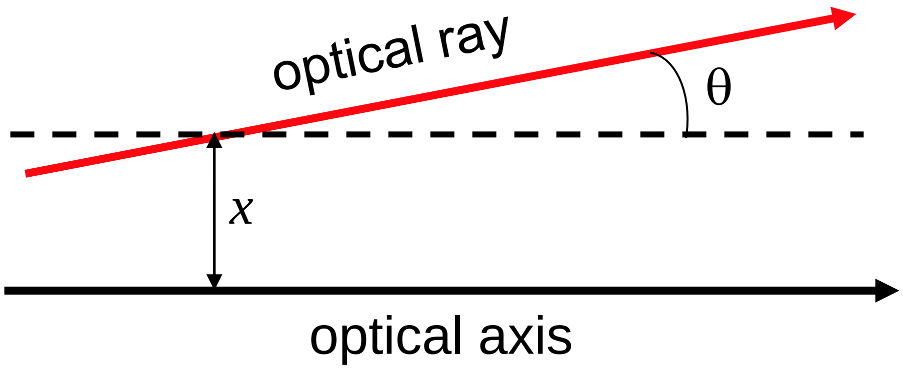
Ray Vector



A light ray can be defined by two co-ordinates:

position, x

slope, θ



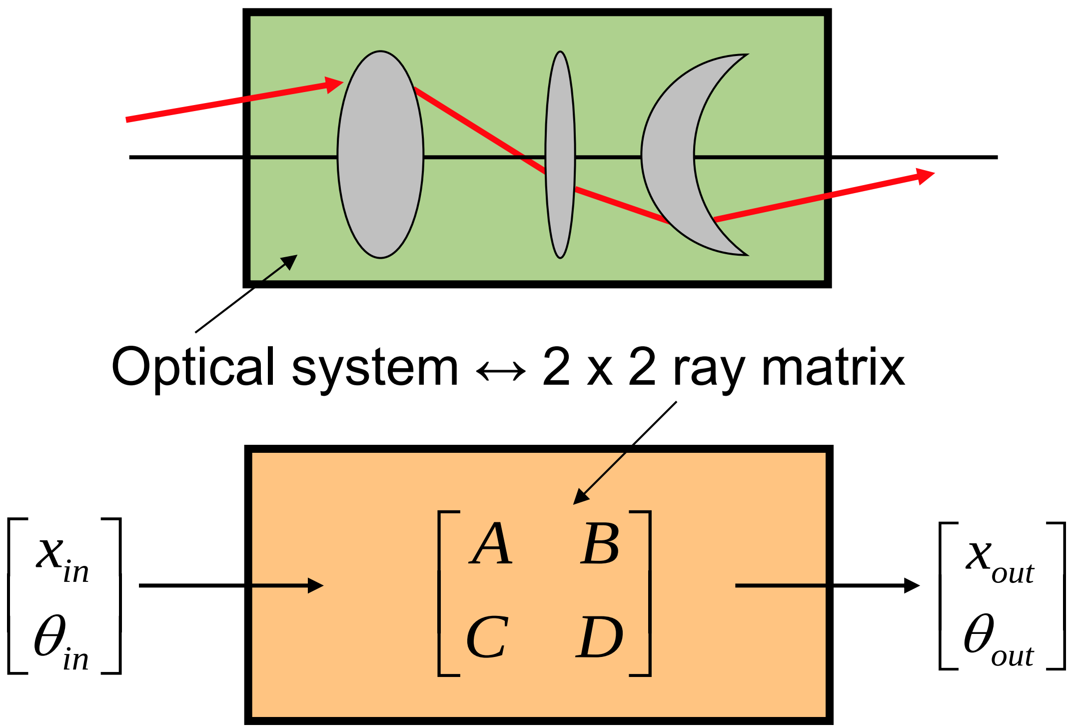
These parameters define a **ray vector**, which will change with distance as the ray propagates through the optics.

$$\begin{bmatrix} x \\ \theta \end{bmatrix}$$

Ray Matrices

For many optical components, we can define 2 x 2 **ray matrices**.

An element's effect on a ray is found by multiplying its ray vector.



$$\begin{bmatrix} x_{in} \\ \theta_{in} \end{bmatrix} \rightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix} \rightarrow \begin{bmatrix} x_{out} \\ \theta_{out} \end{bmatrix}$$

Ray matrices can describe simple and complex systems.

These matrices are often called ABCD Matrices.

Ray matrices as derivatives

$$x_{out} = \frac{\partial x_{out}}{\partial x_{in}} x_{in} + \frac{\partial x_{out}}{\partial \theta_{in}} \theta_{in}$$

Since the displacements and angles are assumed to be small, we can think in terms of partial derivatives.

$$\theta_{out} = \frac{\partial \theta_{out}}{\partial x_{in}} x_{in} + \frac{\partial \theta_{out}}{\partial \theta_{in}} \theta_{in}$$

spatial magnification $\frac{\partial x_{out}}{\partial x_{in}}$ → A

$\frac{\partial x_{out}}{\partial \theta_{in}}$ → B

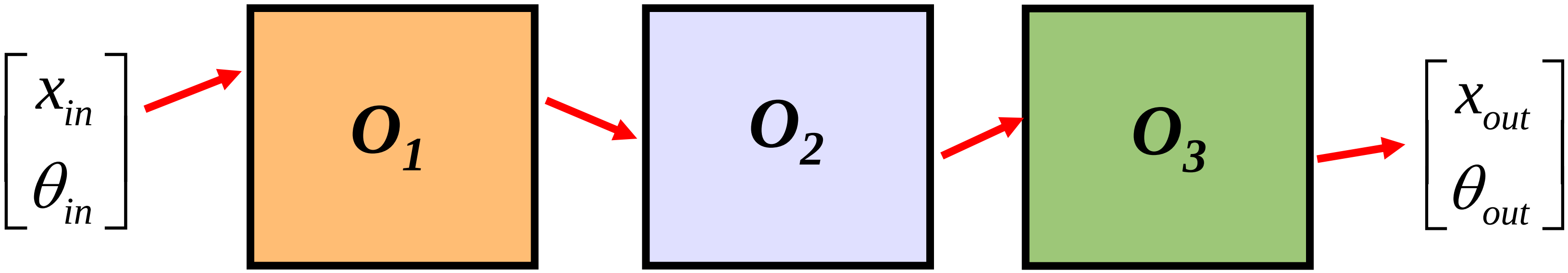
$$\begin{bmatrix} x_{out} \\ \theta_{out} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_{in} \\ \theta_{in} \end{bmatrix}$$

$\frac{\partial \theta_{out}}{\partial x_{in}}$ → C

$\frac{\partial \theta_{out}}{\partial \theta_{in}}$ → D **angular magnification**

We can write these equations in matrix form.

For cascaded elements, we simply multiply ray matrices.



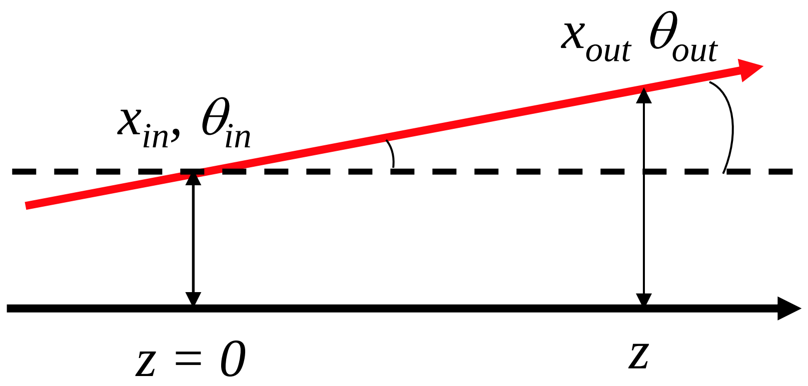
$$\begin{bmatrix} x_{out} \\ \theta_{out} \end{bmatrix} = O_3 \left\{ O_2 \left(O_1 \begin{bmatrix} x_{in} \\ \theta_{in} \end{bmatrix} \right) \right\} = O_3 \; O_2 \; O_1 \begin{bmatrix} x_{in} \\ \theta_{in} \end{bmatrix}$$

Notice that the order looks opposite to what it should be, but it makes sense when you think about it.

Some typical ray matrices

Ray matrix for free space or a medium

If x_{in} and θ_{in} are the position and slope upon entering, let x_{out} and θ_{out} be the position and slope after propagating from $z = 0$ to z.



$$x_{out} = x_{in} + z\,\theta_{in}$$

$$\theta_{out} = \theta_{in}$$

Rewriting these expressions in matrix notation:

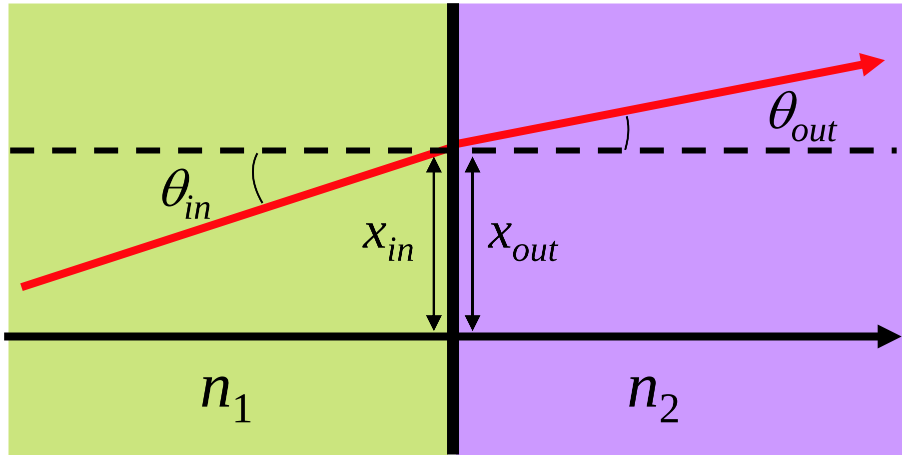
$$\begin{bmatrix} x_{out} \\ \theta_{out} \end{bmatrix} = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{in} \\ \theta_{in} \end{bmatrix}$$

$$O_{space} = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}$$

Ray Matrix for an Interface

At the interface, clearly:

$$x_{out} = x_{in}.$$



Now calculate θ_{out}.

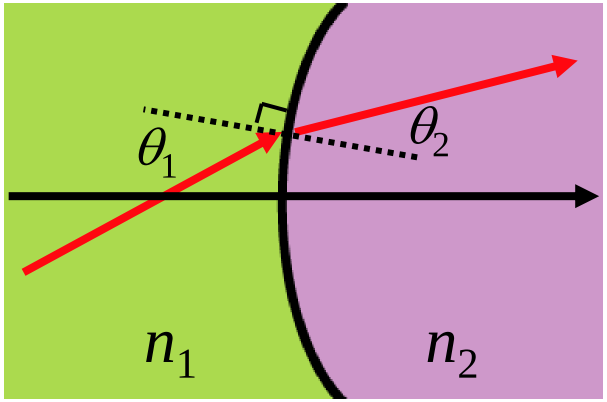
Snell's Law says: $n_1 \sin(\theta_{in}) = n_2 \sin(\theta_{out})$

which becomes for small angles: $n_1 \theta_{in} = n_2 \theta_{out}$

$$\Rightarrow \theta_{out} = [n_1 / n_2]\, \theta_{in}$$

$$O_{interface} = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{n_1}{n_2} \end{bmatrix}$$

Ray matrix for a curved interface



If the interface is curved:

$$\theta_1 = \theta_{in} + x_{in} / R \text{ and } \theta_2 = \theta_{out} + x_{in} / R$$

Snell's Law: $n_1 \theta_1 = n_2 \theta_2$

$$\Rightarrow n_1 (\theta_{in} + x_{in} / R) \approx n_2 (\theta_{out} + x_{in} / R)$$

$$\Rightarrow \theta_{out} \approx (n_1 / n_2)(\theta_{in} + x_{in} / R) - x_{in} / R$$

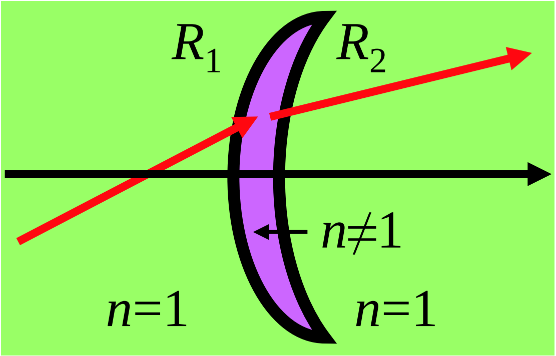
$$\Rightarrow \theta_{out} \approx (n_1 / n_2)\theta_{in} + (n_1 / n_2 - 1)\, x_{in} / R$$

$$O_{\substack{curved \\ interface}} = \begin{bmatrix} 1 & 0 \\ (\frac{n_1}{n_2} - 1) / R & \frac{n_1}{n_2} \end{bmatrix}$$

Now the output angle also depends on the input *position*

A thin lens is just two curved interfaces

We'll neglect the glass in between (it's a really thin lens!), and take $n_1 = 1$.



$$O_{\substack{curved \\ interface}} = \begin{bmatrix} 1 & 0 \\ (n_1/n_2 - 1)/R & n_1/n_2 \end{bmatrix}$$

$$O_{thin\ lens} = O_{\substack{curved \\ interface\,2}} \; O_{\substack{curved \\ interface\,1}} = \begin{bmatrix} 1 & 0 \\ (n-1)/R_2 & n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (1/n-1)/R_1 & 1/n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ (n-1)/R_2 + n(1/n-1)/R_1 & n(1/n) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ (n-1)/R_2 + (1-n)/R_1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ (n-1)(1/R_2 - 1/R_1) & 1 \end{bmatrix}$$

This can be written: $\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$

where: $1/f \; = \; (n-1)(1/R_1 - 1/R_2)$

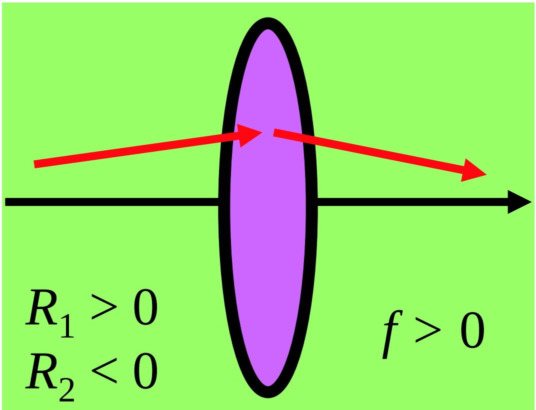
The Lens-Maker's Formula

Ray matrix for a lens

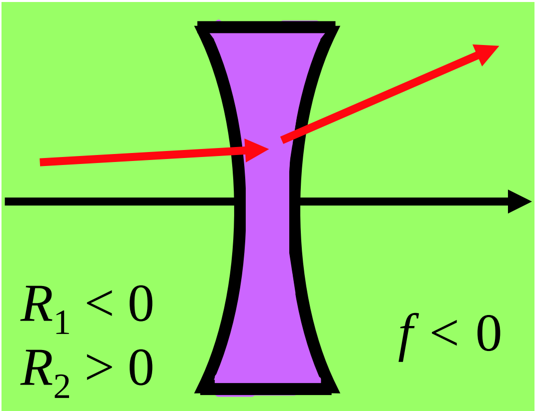
$$1/f = (n-1)(1/R_1 - 1/R_2)$$

$$O_{lens} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$$

The quantity f is the **focal length** of the lens. It's the single most important parameter of a lens. It can be positive or negative.



If $f > 0$, the lens deflects rays toward the axis.



If $f < 0$, the lens deflects rays away from the axis.

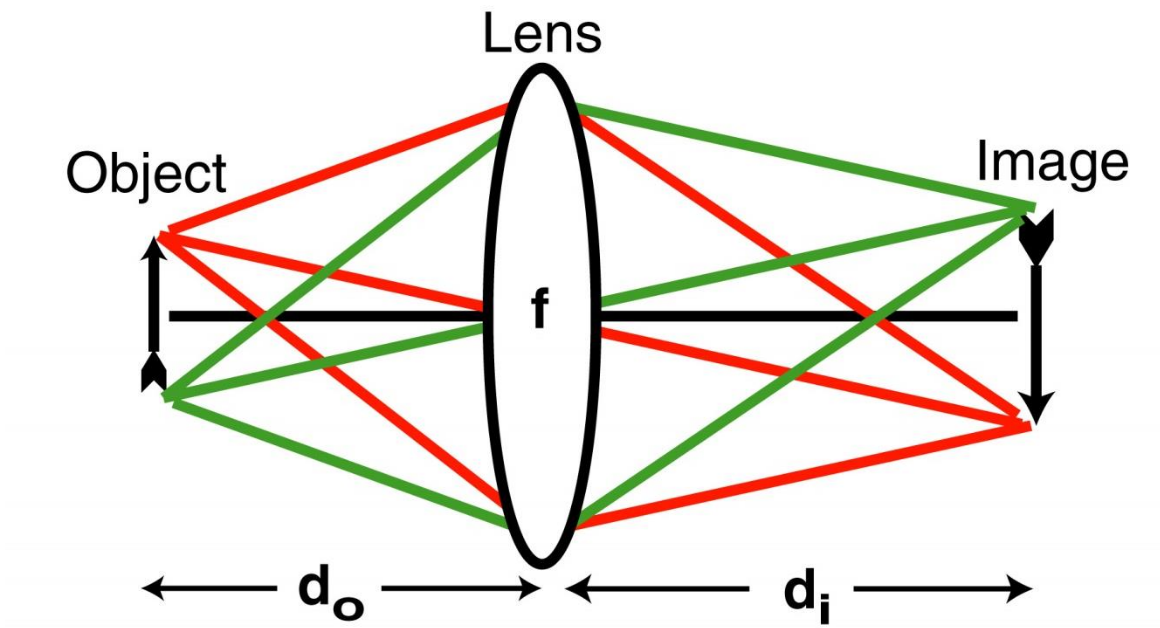
A system images an object when $B = (x/\theta) = 0$.

When $B = 0$, all rays from a point x_{in} arrive at a point x_{out}, independent of angle.

$$\begin{bmatrix} x_{out} \\ \theta_{out} \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \begin{bmatrix} x_{in} \\ \theta_{in} \end{bmatrix} = \begin{bmatrix} A\,x_{in} \\ C\,x_{in} + D\,\theta_{in} \end{bmatrix}$$

$$x_{out} = A\,x_{in}$$

When $B = 0$, A is the **magnification**.



Significance of zero values of matrix elements

$A = 0 \Rightarrow x_{out}$ independent of x_{in} $\Rightarrow$ parallel-to-point focusing (focus)

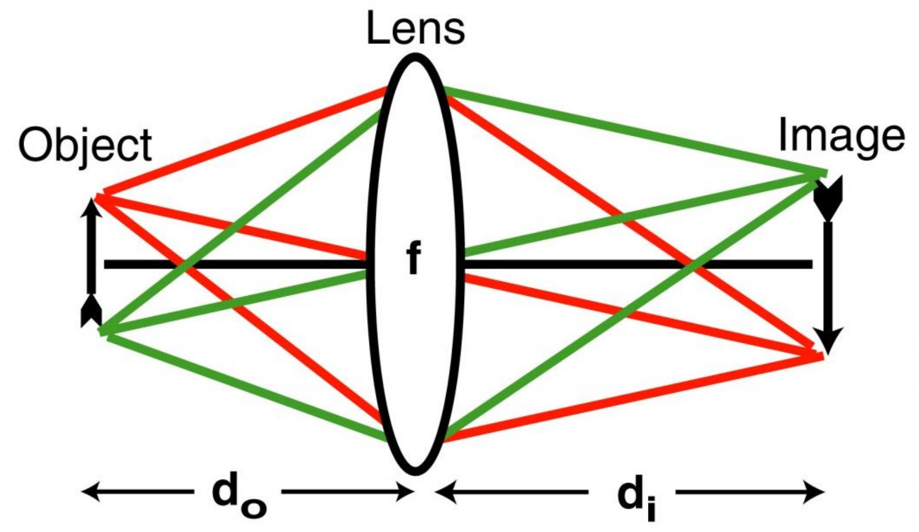
$B = 0 \Rightarrow x_{out}$ independent of θ_{in} $\Rightarrow$ point-to-point focusing (image of object)

$C = 0 \Rightarrow \theta_{out}$ independent of x_{in} $\Rightarrow$ parallel-to-parallel imaging (telescopic)

$D = 0 \Rightarrow \theta_{out}$ independent of θ_{in} $\Rightarrow$ point-to-parallel imaging (e.g. headlamp)

$$\begin{bmatrix} x_{out} \\ \theta_{out} \end{bmatrix} = \begin{bmatrix} par \to pt & pt \to pt \\ par \to par & pt \to par \end{bmatrix} \begin{bmatrix} x_{in} \\ \theta_{in} \end{bmatrix}$$

Thin Lens Equation
(using matrices)



$$O = \begin{bmatrix} 1 & d_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & d_o \\ 0 & 1 \end{bmatrix}$$

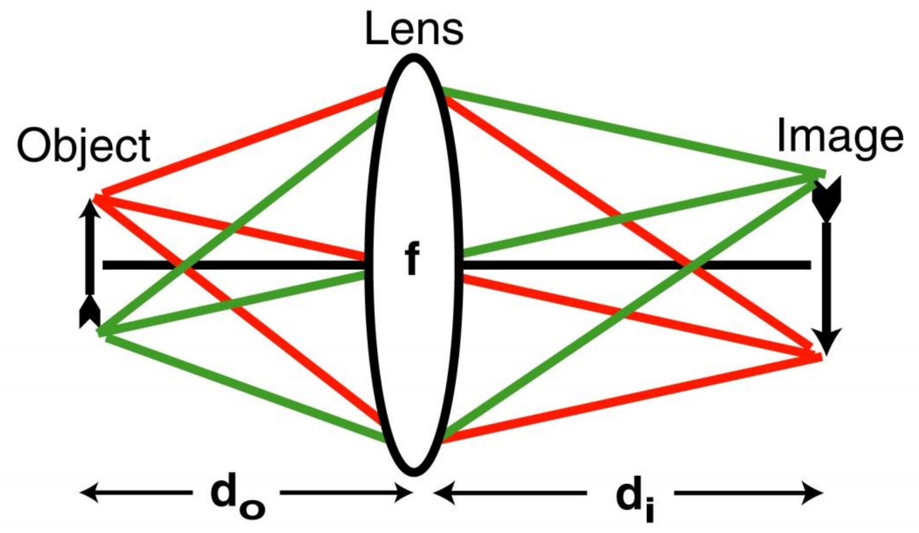
$$= \begin{bmatrix} 1 & d_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d_o \\ -1/f & 1-d_o/f \end{bmatrix}$$

$$= \begin{bmatrix} 1-d_i/f & \boxed{d_o + d_i - d_o d_i / f} \\ -1/f & 1-d_o/f \end{bmatrix}$$

$$B = d_o + d_i - d_o d_i / f$$
$$= d_o d_i \left[1/d_o + 1/d_i - 1/f\right]$$
$$= 0 \quad \text{if}$$

$$\boxed{\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}}$$

Imaging Magnification



If the imaging condition,

$$\frac{1}{d_o}+\frac{1}{d_i}=\frac{1}{f}$$

is satisfied, then:

$$O=\begin{bmatrix} 1-d_i/f & 0 \\ -1/f & 1-d_o/f \end{bmatrix}$$

$$A=1-d_i/f=1-d_i\left[\frac{1}{d_o}+\frac{1}{d_i}\right]$$

$$\Rightarrow \quad M=-\frac{d_i}{d_o}$$

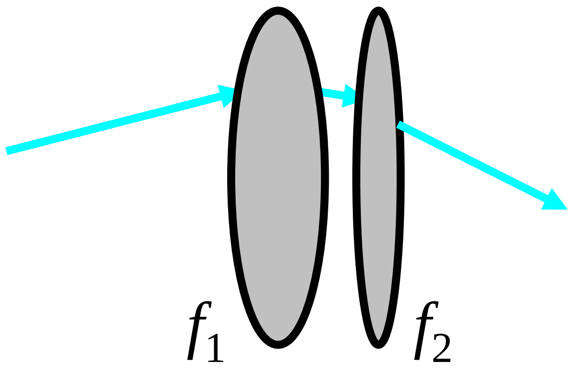
$$D=1-d_o/f=1-d_o\left[\frac{1}{d_o}+\frac{1}{d_i}\right]$$

$$=-\frac{d_o}{d_i}=1/M$$

So:

$$O=\begin{bmatrix} M & 0 \\ -1/f & 1/M \end{bmatrix}$$

Consecutive lenses



Suppose we have two lenses right next to each other (with no space in between).

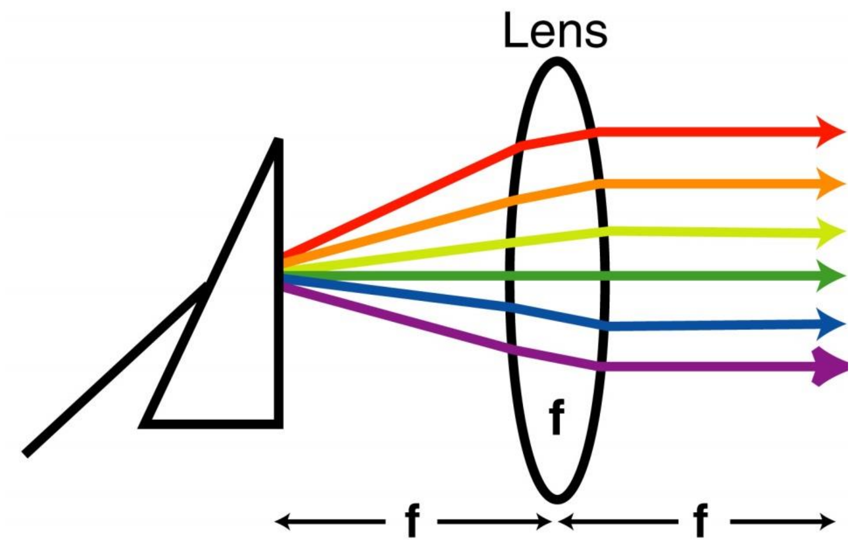
$$O_{tot} = \begin{bmatrix} 1 & 0 \\ -1/f_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f_1 - 1/f_2 & 1 \end{bmatrix}$$

$$1/f_{\text{tot}} = 1/f_1 + 1/f_2$$

So two consecutive lenses act as one whose focal length is the sum of the individual focal lengths.

[As a result, we define a measure of inverse lens focal length, the dioptre, where 1 dioptre = 1 m^{-1}]

Lenses can also map angle ↔ position



From the object to the image, we have:

1) A distance f
2) A lens of focal length f
3) A distance f

$$\begin{bmatrix} x_{out} \\ \theta_{out} \end{bmatrix} = \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{in} \\ \theta_{in} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & f \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} x_{in} \\ \theta_{in} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & f \\ -1/f & 0 \end{bmatrix} \begin{bmatrix} x_{in} \\ \theta_{in} \end{bmatrix} = \begin{bmatrix} f\theta_{in} \\ -x_{in}/f \end{bmatrix}$$

So $x_{out} \propto \theta_{in}$

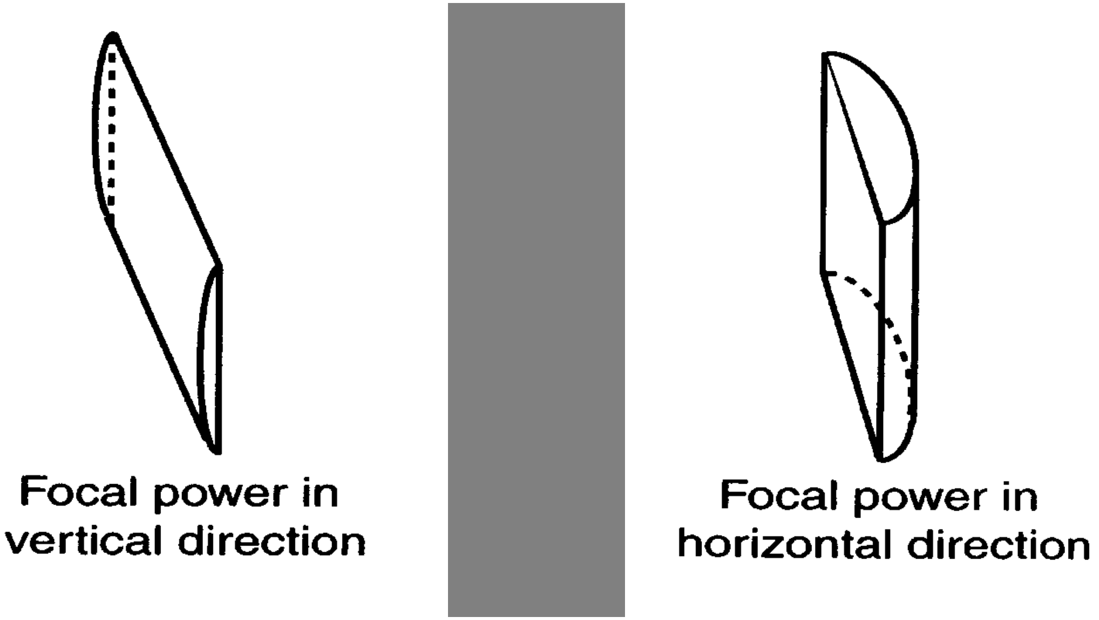
and this arrangement maps position to angle:

$\theta_{out} \propto x_{in}$

If an optical system lacks axial symmetry, we must analyse its x- and y-directions separately: e.g. cylindrical lenses

A "spherical lens" focuses in both transverse directions.

A "cylindrical lens" focuses in only one transverse direction.



When using cylindrical lenses, we must perform two separate ray-matrix analyses, one for each transverse direction.

This compares with a Quadrupole lens in charged-particle optics.

Complex Beam Parameter for Gaussian Beams

Gaussian distribution is a solution of the paraxial Helmholtz equation
TEM_{00} mode

- The ABCD method can be extended to Gaussian beams using the complex beam radius, *q(z):*

$$\frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{\pi w(z)^2}$$

- After finding the optical ABCD matrix as before, the complex radius of curvature is transformed in the following way

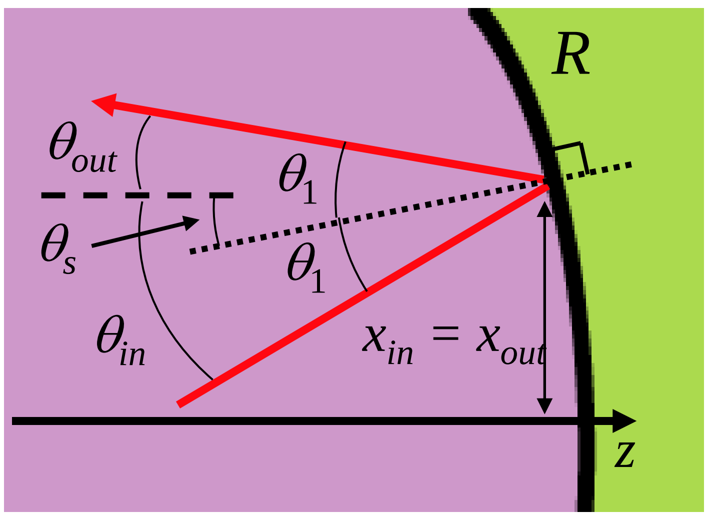
$$q'(z) = \frac{Aq(z)+B}{Cq(z)+D}$$

- Characterisation of the beam allows us to predict its form elsewhere

- *No time to discuss details here …..*

Ray Matrix for a Curved Mirror

Consider a mirror with radius of curvature R, with its optical axis perpendicular to the mirror:



$$\theta_1 = \theta_{in} - \theta_s \qquad \theta_s \approx x_{in} / R$$

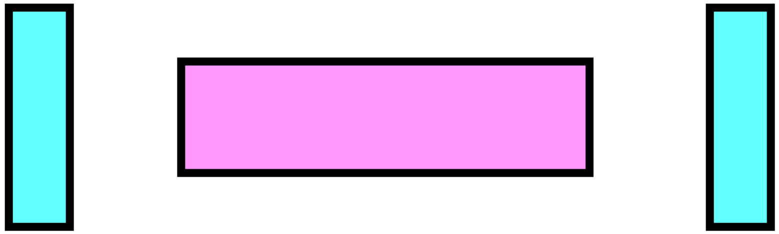
$$\theta_{out} = \theta_1 - \theta_s = (\theta_{in} - \theta_s) - \theta_s$$
$$\approx \theta_{in} - 2x_{in} / R$$

$$\Rightarrow \boxed{O_{mirror} = \begin{bmatrix} 1 & 0 \\ -2/R & 1 \end{bmatrix}}$$

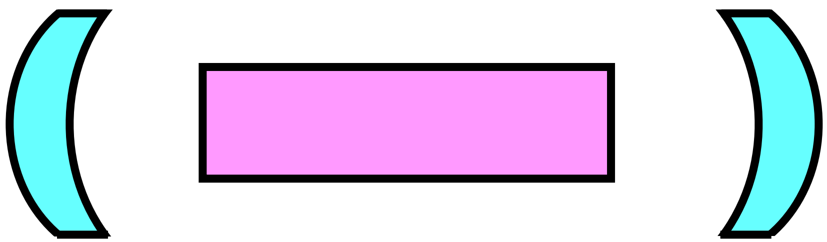
Like a lens, a curved mirror will focus a beam. Its focal length is $R/2$.

Note that a *flat* mirror has $R = \infty$ and hence an *identity* ray matrix.

Laser Cavities / Resonators

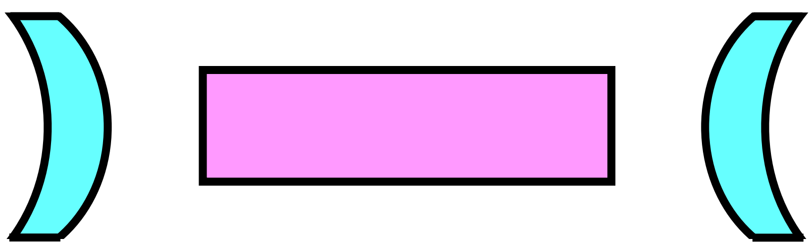


Two flat mirrors, the **flat-flat** laser cavity, is difficult to align and maintain aligned.



Two concave curved mirrors, the usually **stable** laser cavity, is generally easy to align and maintain aligned.

e.g. confocal or concentric



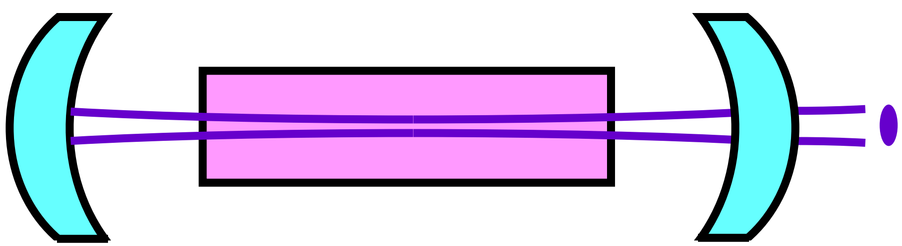
Two convex mirrors, the **unstable** laser cavity, is impossible to align!

(could also be flat–plane, etc.)

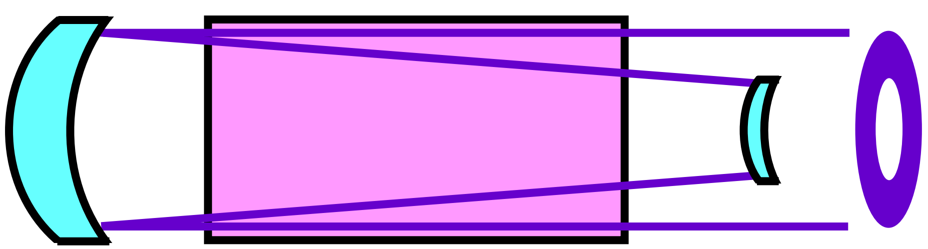
Unstable Resonators

An **unstable cavity** (or **unstable resonator**) can work if you do it properly!

In fact, it produces a large diameter beam, useful for high-power lasers, which must have large beams.



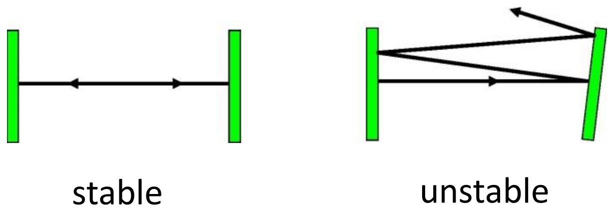
The mirror curvatures determine the beam size, which, for a stable resonator, is small (100 μm to 1 mm).



An unstable resonator can have a very large beam, but the gain must be high… and the beam has a hole in it.

Stability of a resonator - ray matrix analysis

Consider paraxial ray within resonator – if ray escapes after finite number of bounces, resonator is <u>unstable</u>.

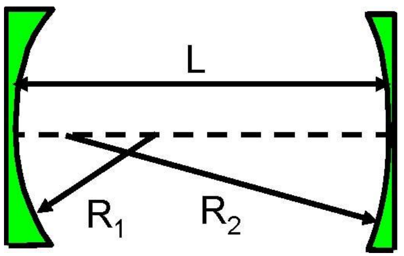


ABCD matrix analysis for a single round trip:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2/R_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2/R_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2L/R_2 & 2L-2L^2/R_2 \\ 4L/R_1R_2-2/R_1-2/R_2 & 1-2L/R_2-4L/R_1+4L^2/R_1R_2 \end{bmatrix}$$

2 concave mirrors. By convention, R > 0 for concave, and R < 0 for convex.



After N round trips, the output ray is related to the initial ray by:

$$\begin{bmatrix} x_{out} \\ \theta_{out} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^N \begin{bmatrix} x_{in} \\ \theta_{in} \end{bmatrix}$$

Define new variable, χ, such that: $\cos\chi = 1 - \frac{2L}{R_1} - \frac{2L}{R_2} + \frac{2L^2}{R_1R_2}$

Can show that:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^N = \frac{1}{\sin\chi} \begin{bmatrix} A\sin(N\chi)-\sin[(N-1)\chi] & B\sin(N\chi) \\ C\sin(N\chi) & D\sin(N\chi)-\sin[(N-1)\chi] \end{bmatrix}$$

$$x_{out} = \frac{1}{\sin\chi}[(A\sin N\chi - \sin(N-1)\chi)x_{in} + (B\sin N\chi)\theta_{in}]$$

Note that the output ray position x_{out} remains finite when N goes to infinity, as long as χ is a real number.

If χ becomes complex, then $\sin N\chi = \frac{1}{2i}(e^{iN\chi} - e^{-iN\chi})$

one of these exponential terms grows to infinity as N gets large

Thus the condition for resonator stability is that χ is real, or $|\cos\chi| \leq 1$

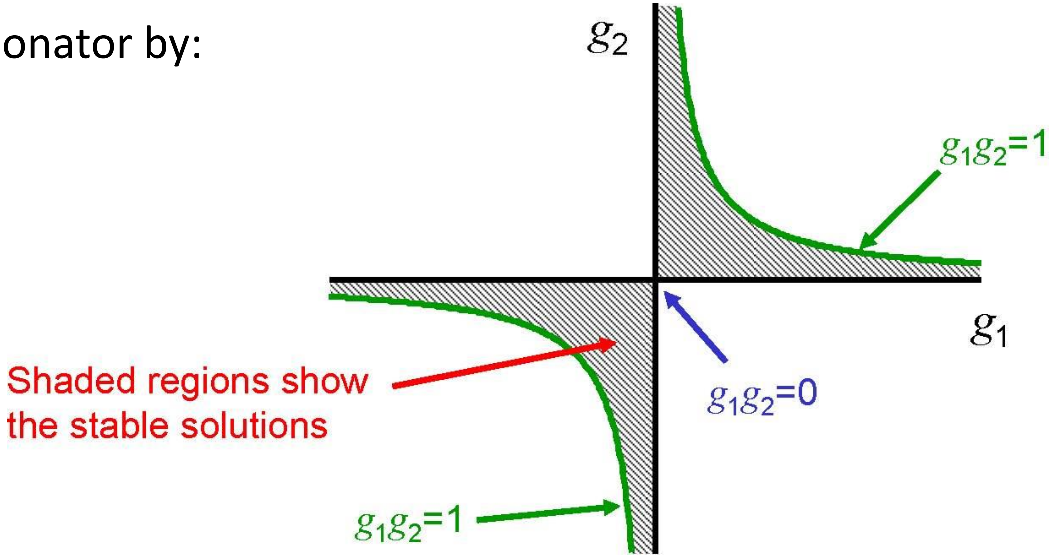
$$\rightarrow \quad 0 \leq \left(1 - \frac{L}{R_1}\right)\left(1 - \frac{L}{R_2}\right) \leq 1$$

Defining the "*g* parameters" of the resonator by:

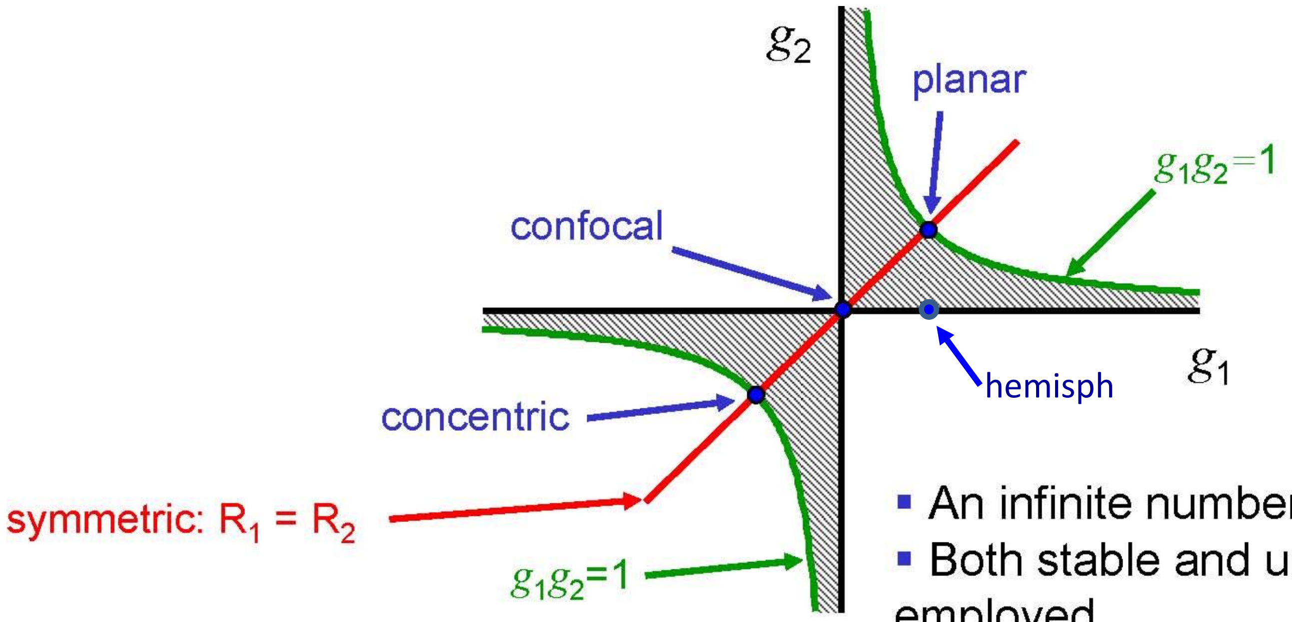
$$g_1 = (1 - \frac{L}{R_1}) \qquad g_2 = (1 - \frac{L}{R_2})$$

Then, the stability requirement is:

$$0 \leq g_1 g_2 \leq 1$$



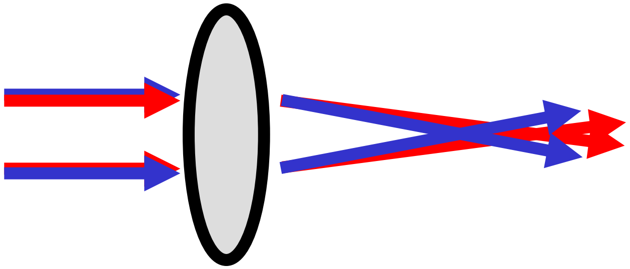
Analysis of Gaussian laser beams gives the same result for stable cavity modes!



- An infinite number of possible solutions
- Both stable and unstable solutions are employed
- Certain ones are particularly interesting

Planar – rarely used, large mode size, very sensitive to misalignment
Concentric – also very sensitive to misalignment, so rarely used
Confocal – smallest average spot size, relatively insensitive to misalignment, commonly used
Hemispherical – very insensitive to misalignment, very common design
… etc.

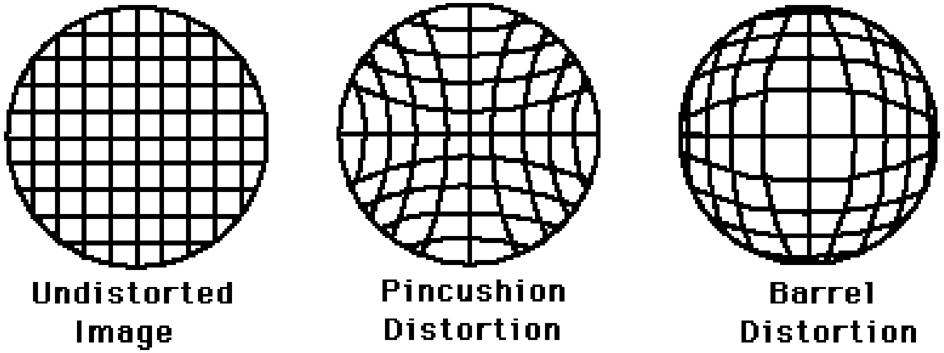
Aberrations



Aberrations are distortions that occur in images, usually due to imperfections in lenses - some unavoidable, some avoidable.

They include:

- Chromatic aberration
- Spherical aberration
- Astigmatism
- Coma
- Curvature of field
- Pincushion and Barrel distortion



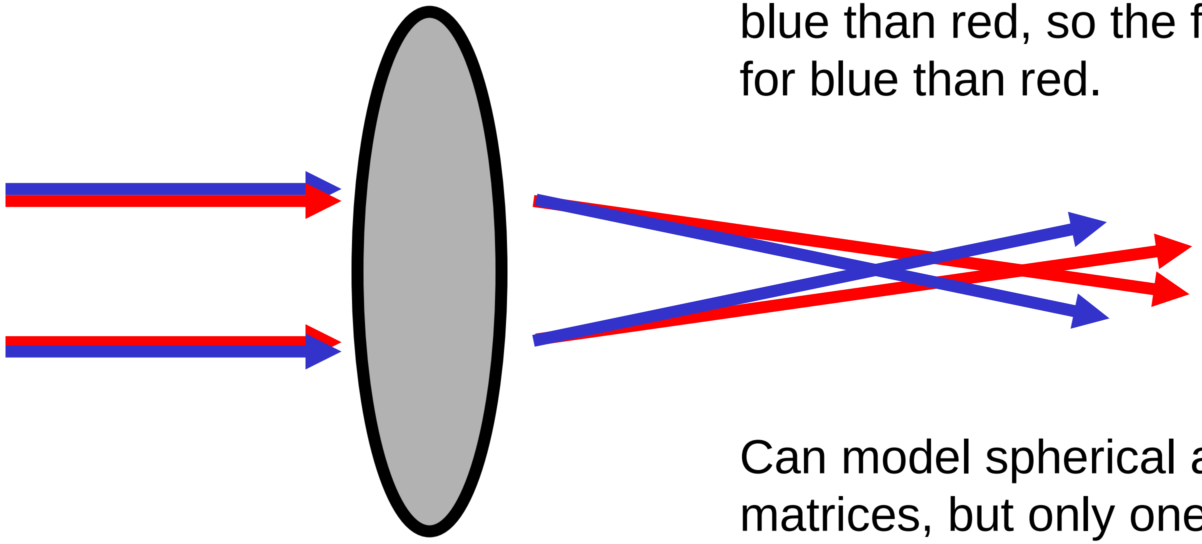
Most aberrations can't be modelled with ray matrices. Designers beat them with lenses of multiple elements, that is, several lenses in a row. Some zoom lenses can have as many as a dozen or more elements.

Chromatic Aberration

Because the lens material has a different refractive index for each wavelength, the lens will have a different focal length for each wavelength. Recall the lens-maker's formula:

$$1/f(\lambda) = (n(\lambda)-1)(1/R_1 - 1/R_2)$$

Here, the refractive index is larger for blue than red, so the focal length is less for blue than red.

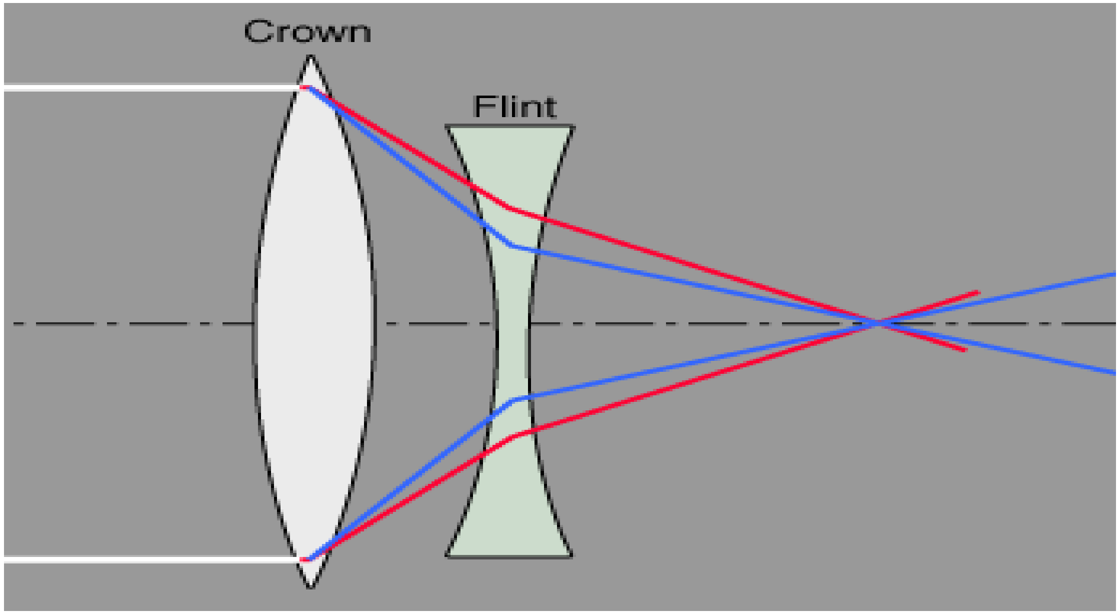


Can model spherical aberration using ray matrices, but only one colour at a time.

c.f. strong chromatic aberration in magnetic quadrupole lenses.

Chromatic aberration can be minimised using additional lenses

In an achromat, the second lens cancels the dispersion of the first.

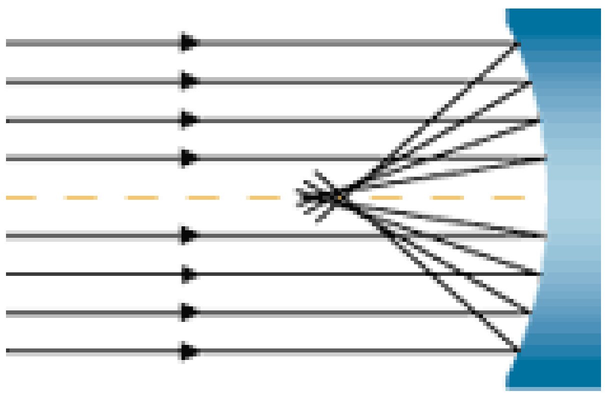


Achromats use two different materials, and one has a negative focal length.

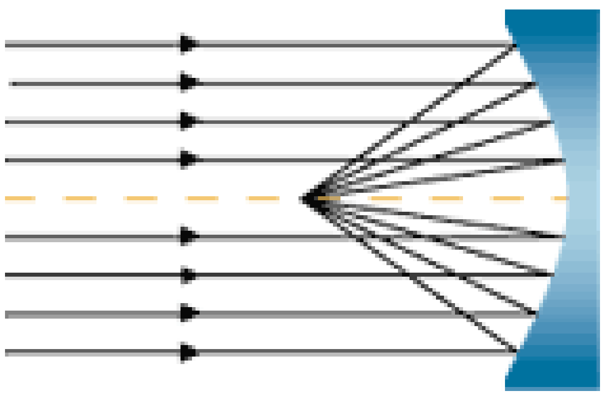
Spherical Aberration in Mirrors

For all rays to converge to a point a distance f away from a curved mirror, we require a paraboloidal surface.

Spherical surface



Paraboloidal surface

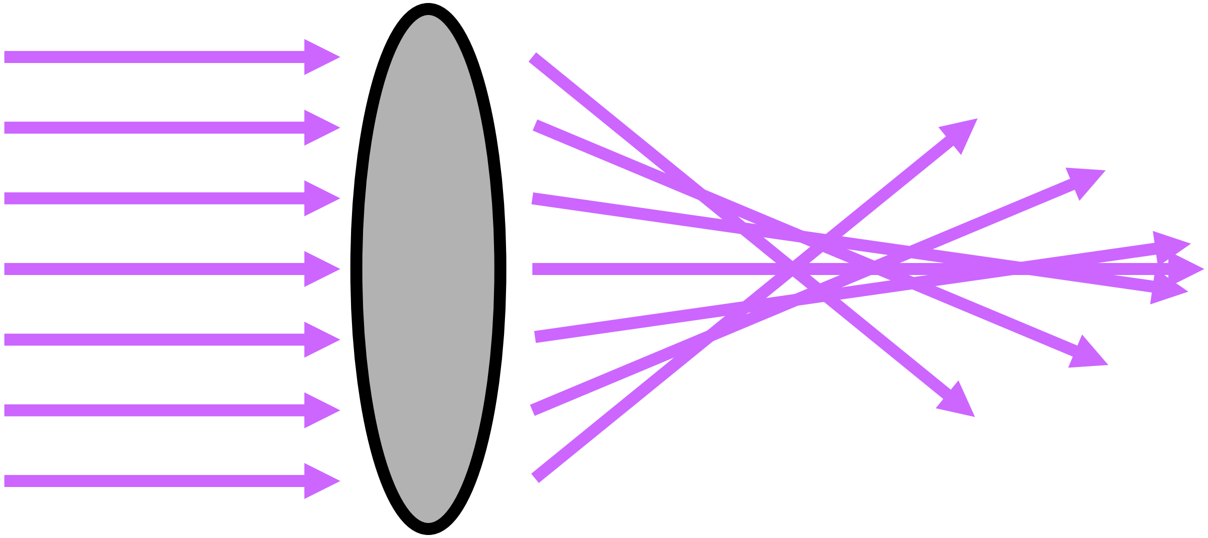


But this only works for rays with $\theta_{in} = 0$.

Spherical Aberration in Lenses

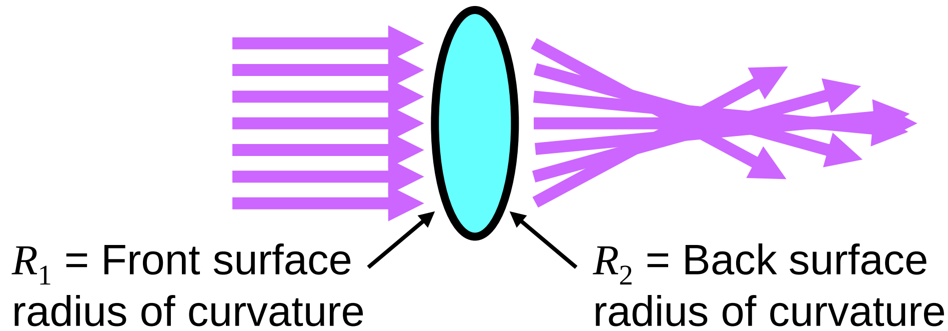
So we use spherical surfaces, which work better for a wider range of input angles.

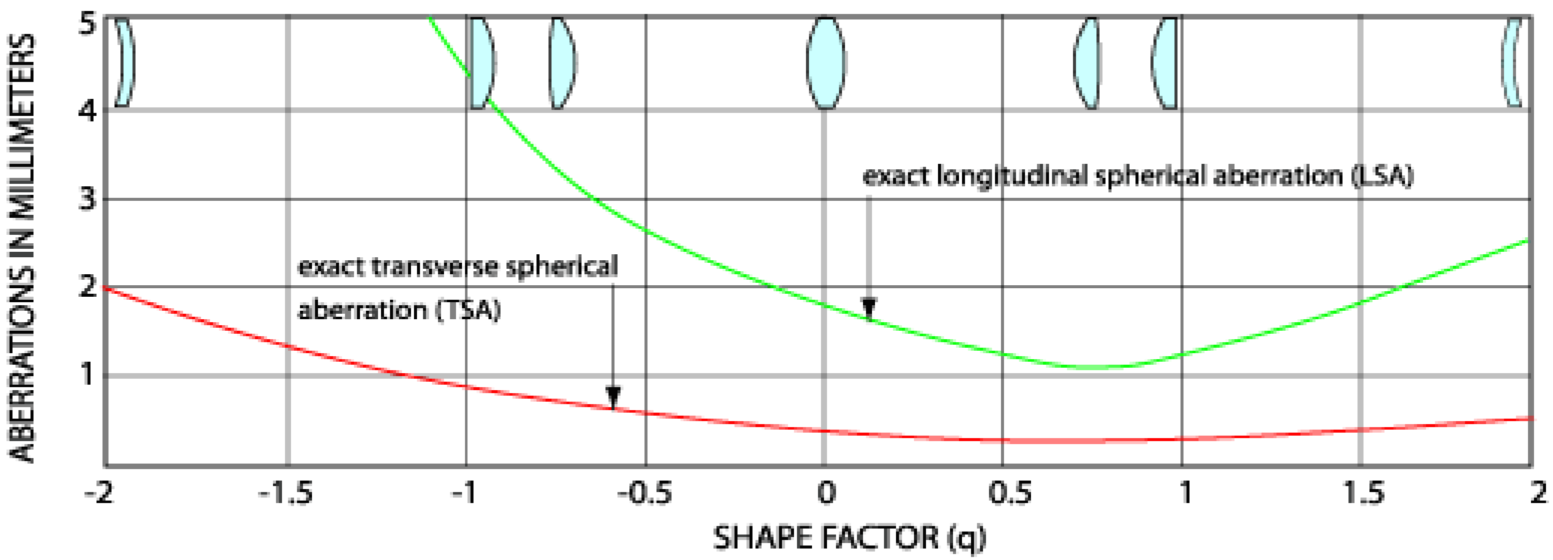
Nevertheless, off-axis rays see a different focal length, so lenses have spherical aberration, too.



Minimising spherical aberration in a focus

$$q \equiv \frac{R_2 + R_1}{R_2 - R_1}$$





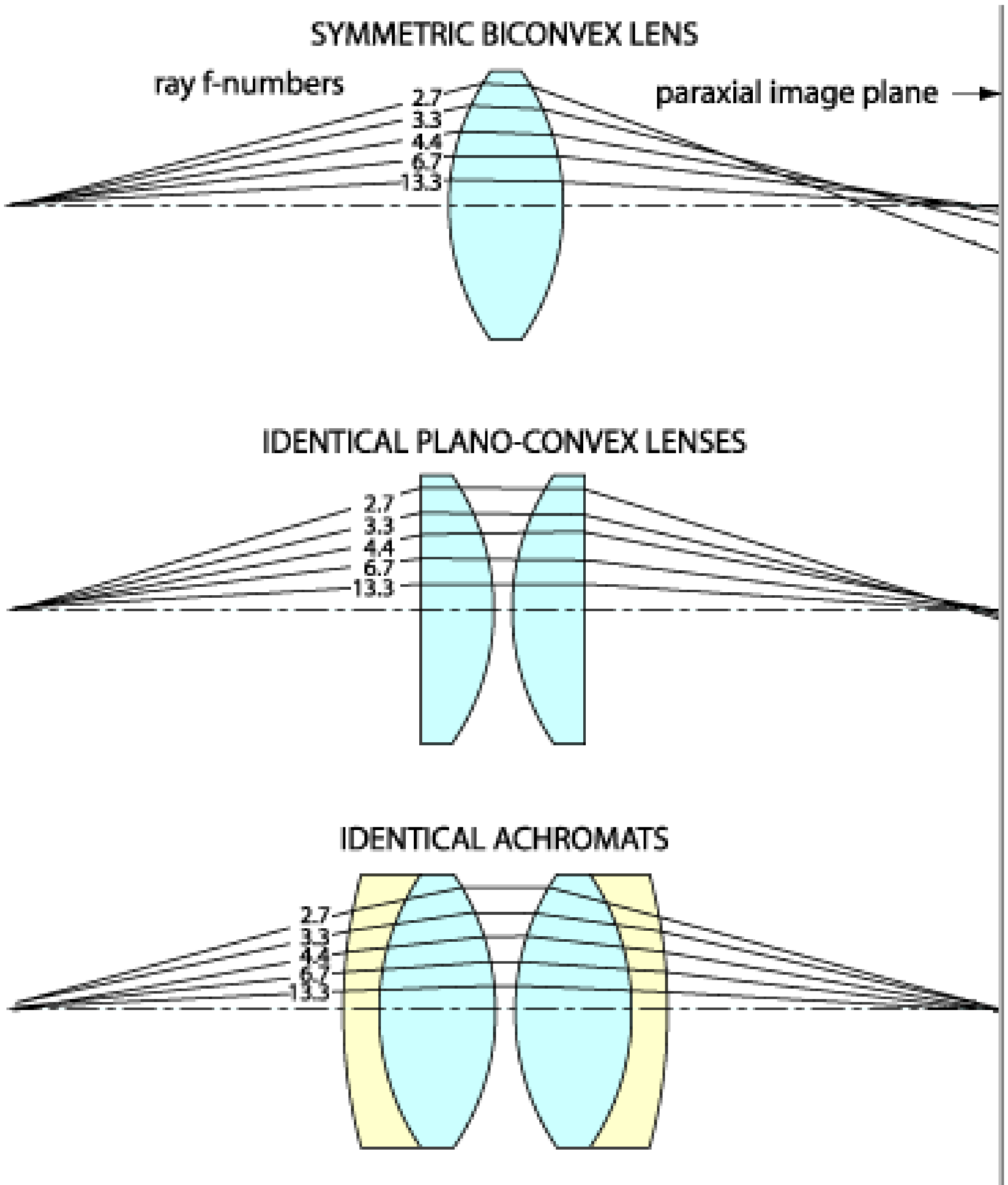
plano-convex lenses (with their flat surface facing the focus) are best for minimising spherical aberration when *focusing*.

One-to-one imaging works best with a *symmetrical lens* ($q = 0$).

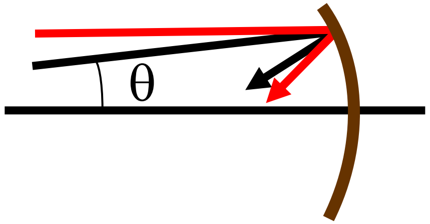
Spherical aberration can also be minimised using additional lenses

(general point here is that aberrations may be minimised by utilising symmetry.

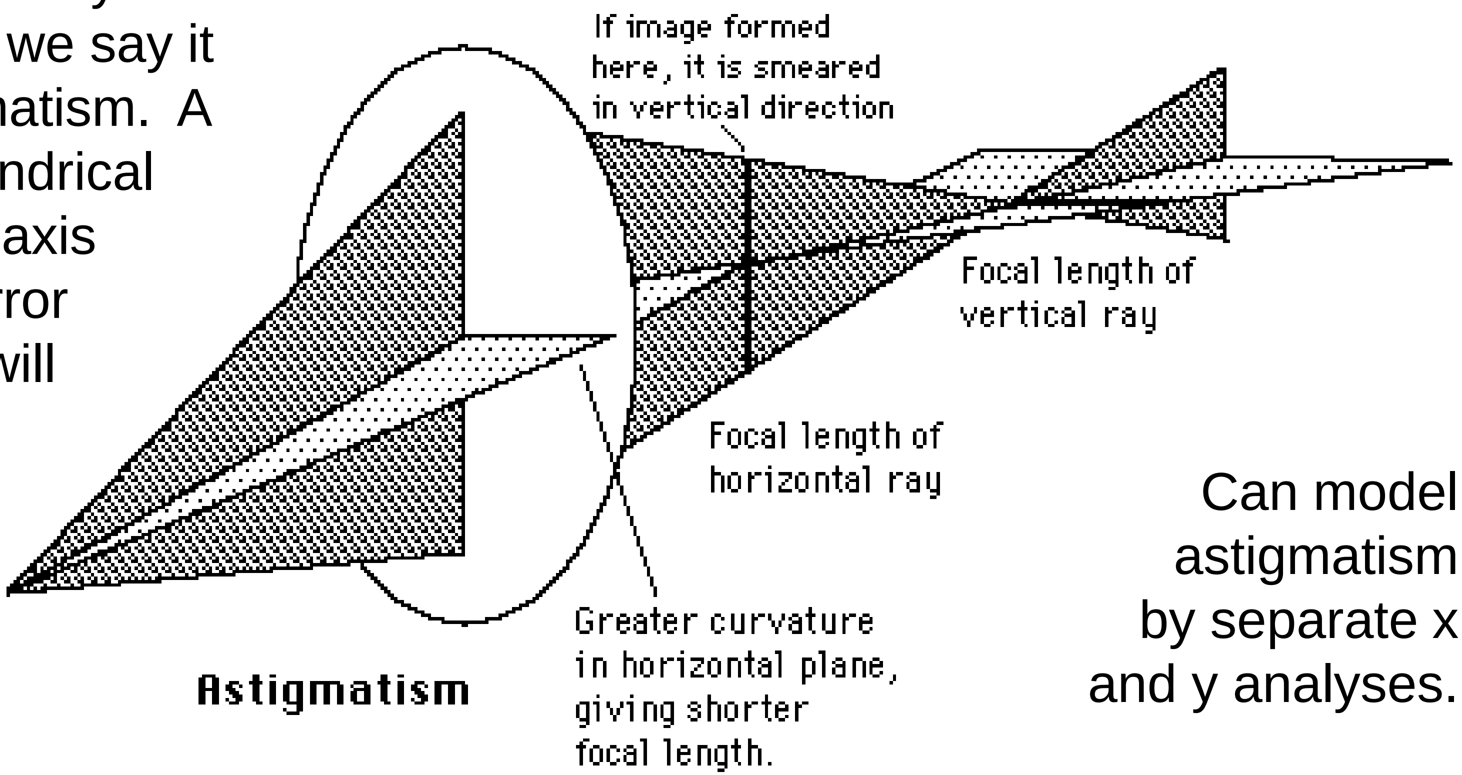
Also true in CP optics)



Astigmatism



When the optical system lacks perfect cylindrical symmetry, we say it has astigmatism. A simple cylindrical lens or off-axis curved-mirror reflection will cause this problem.

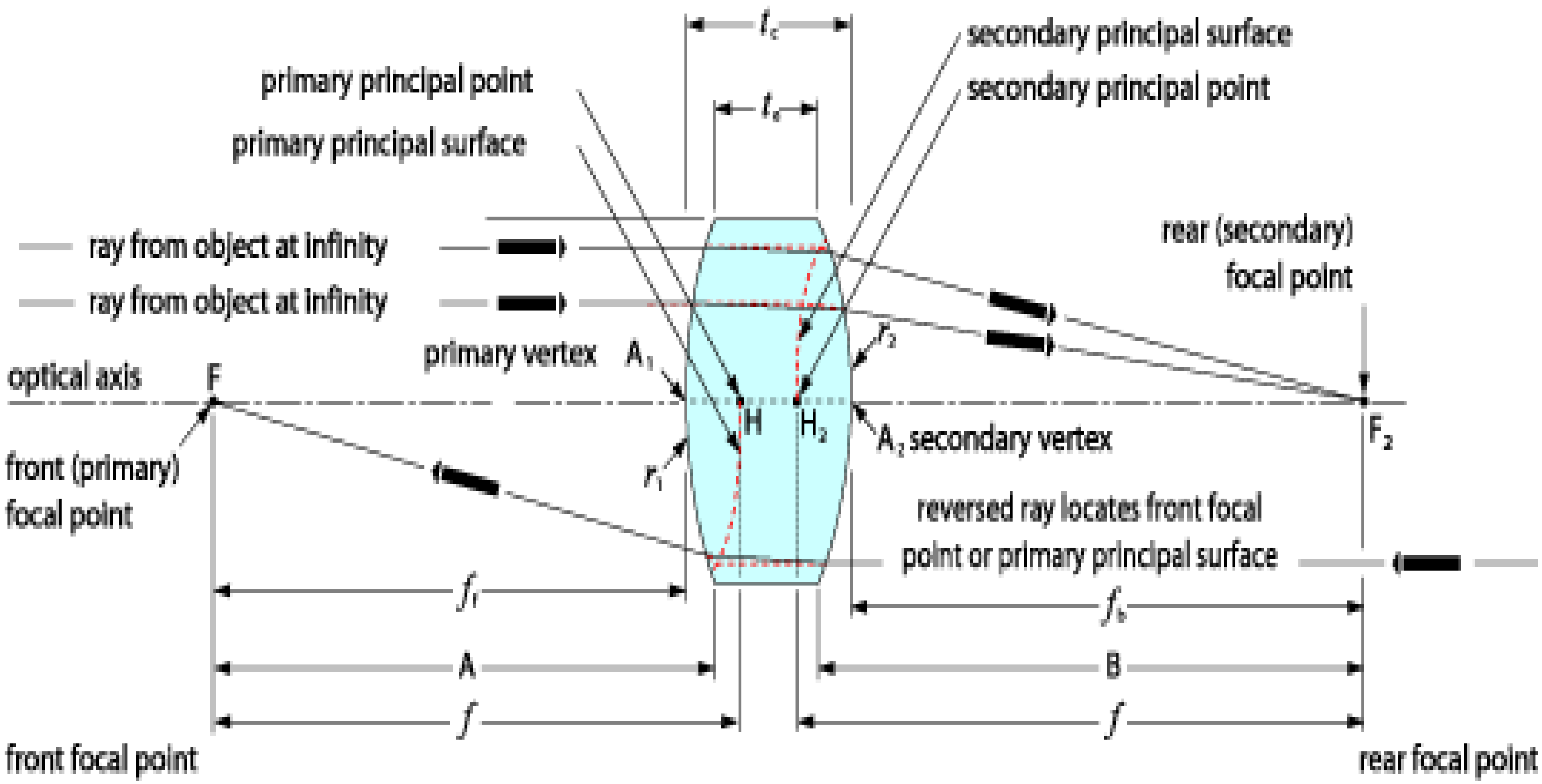


Can model astigmatism by separate x and y analyses.

<u>Cure</u> astigmatism with another cylindrical lens or off-axis curved mirror.

- We'll see that CP systems are <u>intrinsically astigmatic</u>

Geometrical Optics terms for lens system



Charged-Particle (CP) Optics

Here we use predominantly magnetic field configurations to bend and focus CP beams (positive ions or electrons).

- Bending performed by (predominantly) uniform-field dipole magnets
- Focusing performed (predominantly) by quadrupole magnets
- Higher-order corrections applied via sextupole, octupole, etc., magnets

Lorentz Force

$$\vec{F} = \underbrace{q\vec{E}}_{\text{Electric force}} + \underbrace{q\vec{v} \times \vec{B}}_{\text{Magnetic force}}$$

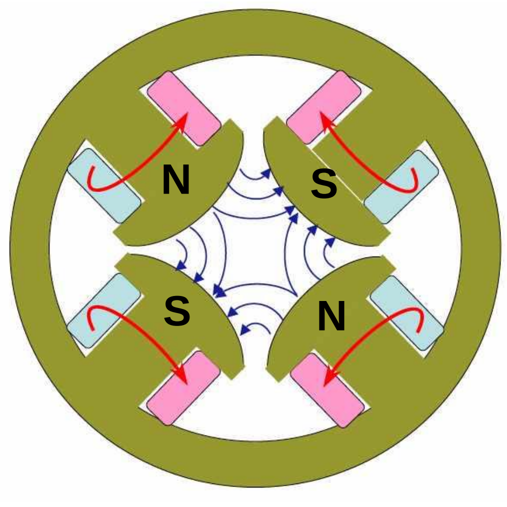
q = electric charge

B = magnetic induction

E = electric field

v = velocity

dipole bending magnet



quadrupole focusing magnet

quadrupole triplet in beam line

sextupole magnet

M^2 versus Emittance

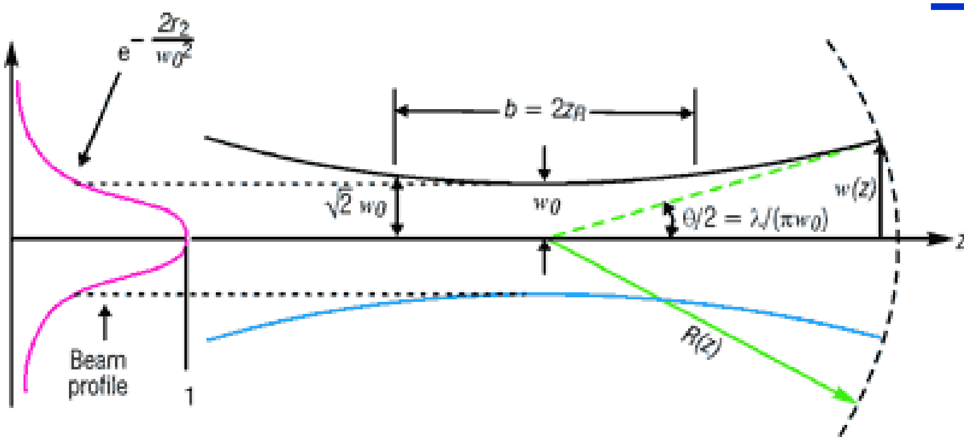
M^2 - deviation of real laser beam envelope from ideal TEM_{00} Gaussian envelope

- fundamental beam divergence a consequence of diffraction of finite aperture beam

Emittance – measure of finite phase space occupied by CP beam envelope (or an agreed fraction of it)

- normally a conserved quantity for beam transport systems (within well-defined conditions)

M^2 in real Gaussian beams



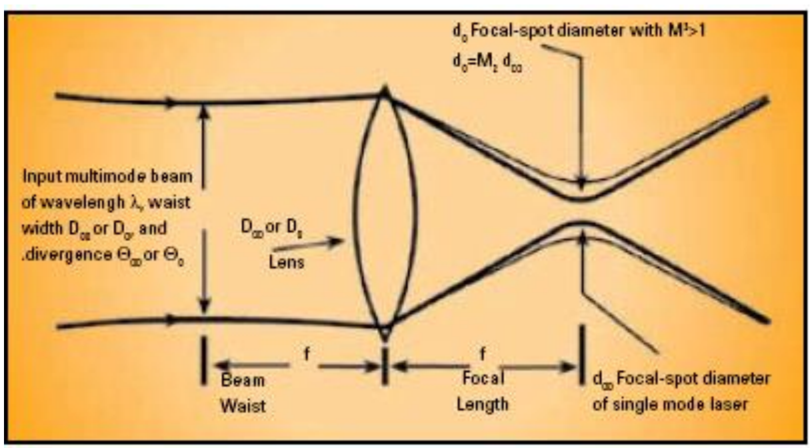
$$w(z) = w_0\sqrt{1+\left(M^2\frac{\lambda z}{\pi w_0^2}\right)^2}$$

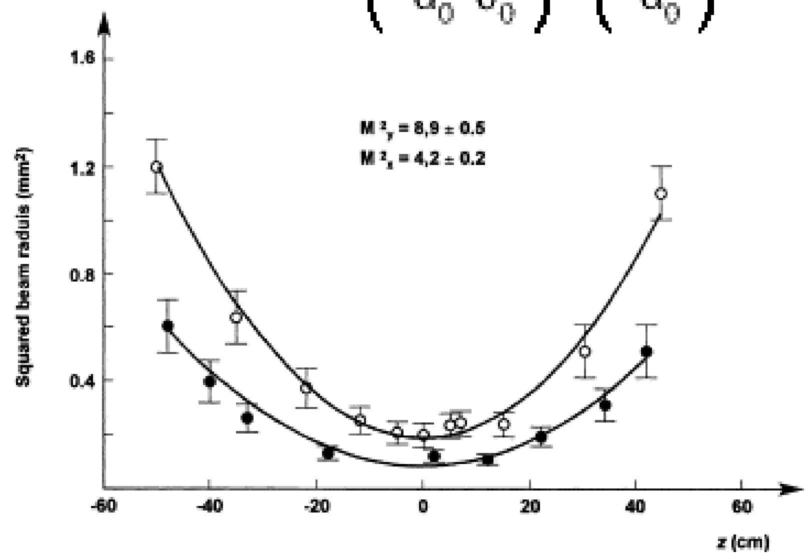
$$\theta = \frac{M^2\lambda}{\pi w_0}$$

$$M^2 = \left(\frac{D_m\cdot\Theta_m}{d_0\cdot\theta_0}\right) = \left(\frac{D_m}{d_0}\right)^2.$$

M-squared is a bit like emittance.

M-squared measurement is like a solenoid or quadrupole scan.

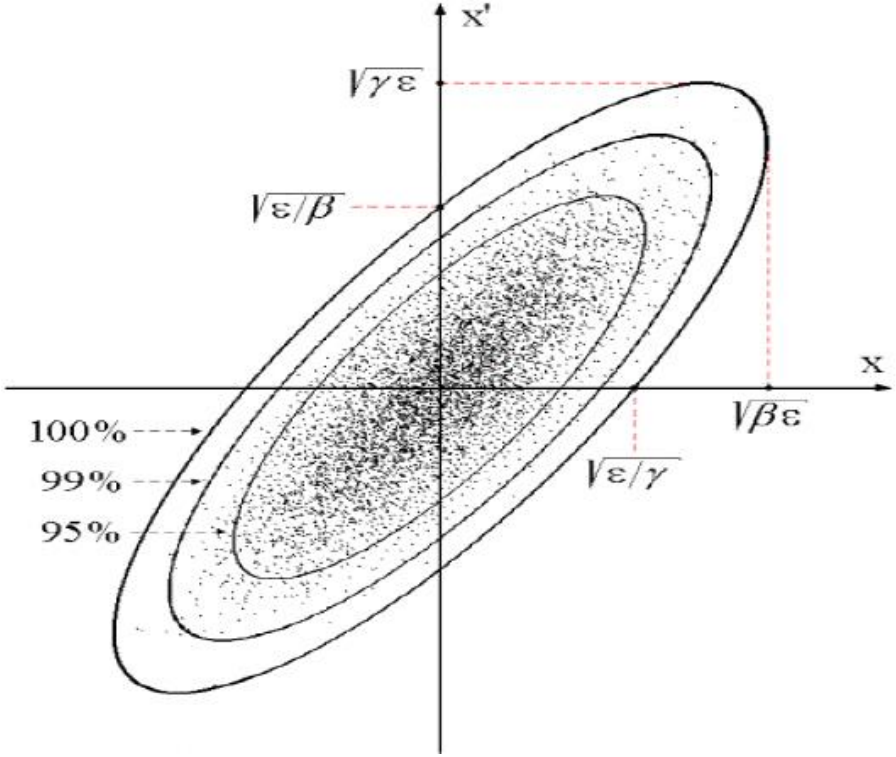




M^2 remains invariant through ABCD optical systems.

CP Beam Emittance, ε

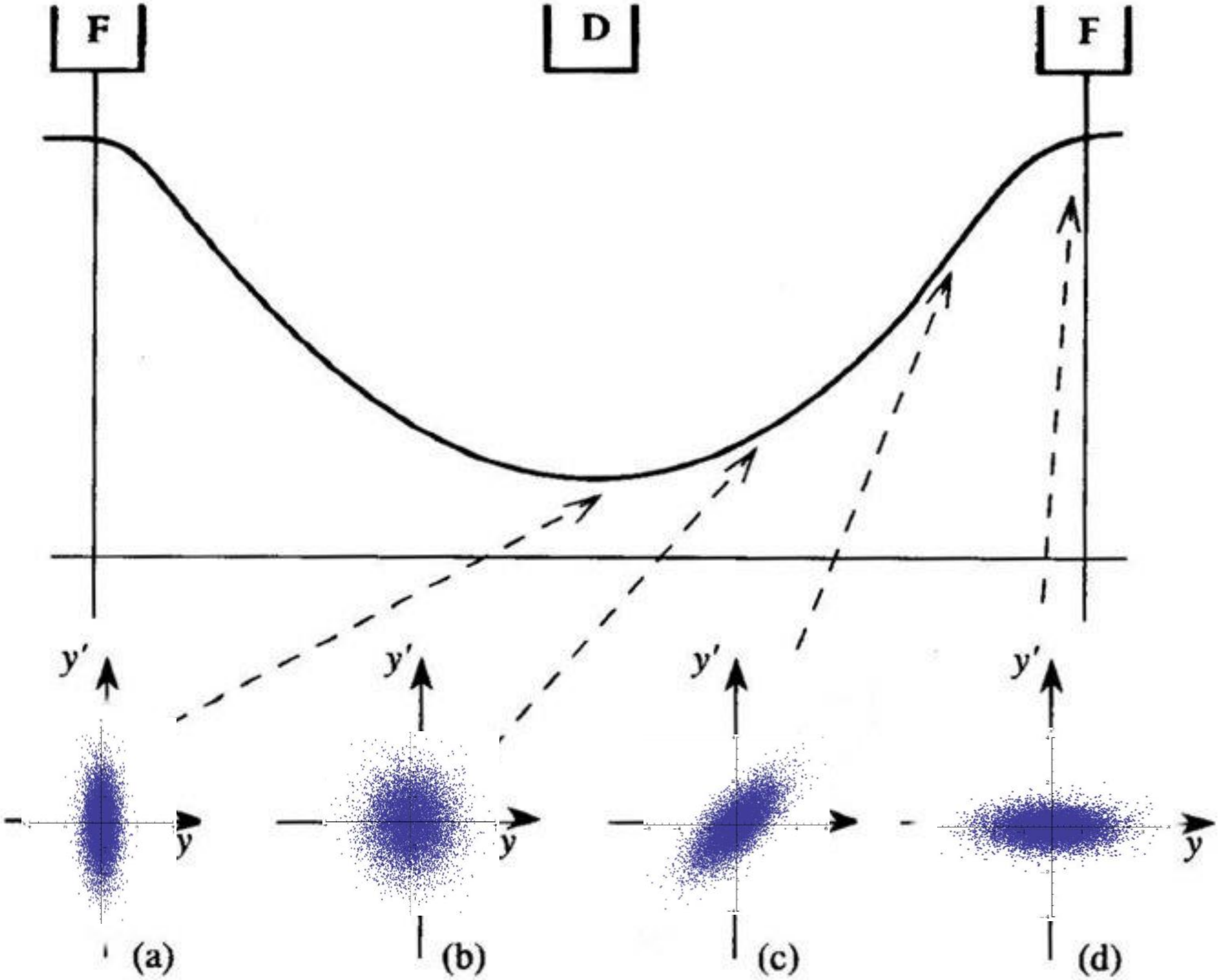
ε defined as 2D phase space area divided by π



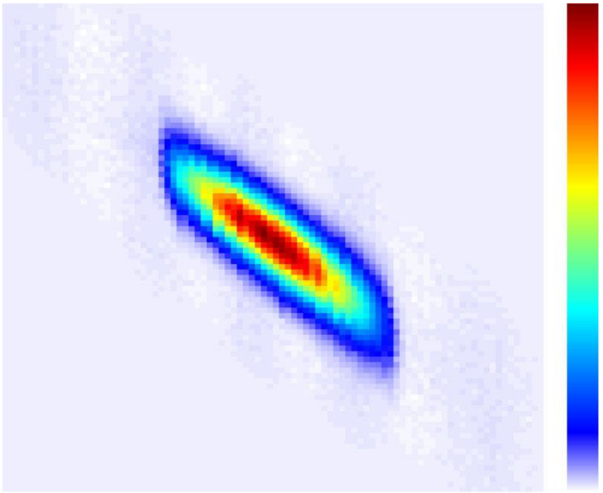
Liouville's Theorem:

- The volume occupied in phase space by a system of particles is constant.
- This is a general physics theorem in canonical co-ordinates, not limited to accelerators.
- Application of external forces or emission of radiation needs to be treated carefully [LT holds for fixed E & conservative forces]
- Can define normalised emittance, $\varepsilon_n = \varepsilon / \beta\gamma$

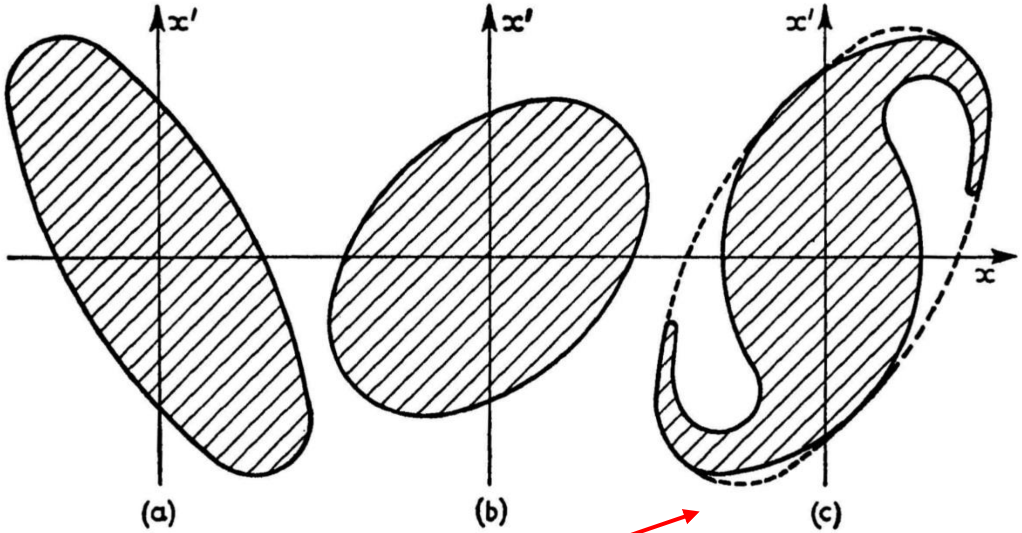
As beam propagates, shape of the phase ellipse will change



How the conserved phase space appears at different points in a FODO cell. The development of a constant-emittance beam in phase space at (a) a narrow waist, (b) and (c) places where the beam is diverging, and (d) at a broad maximum at the centre of an F lens.



measured H phase space
using solenoid scan



N.B. Even when LT holds, non-linear forces can give rise to an increase in the 'effective' phase space area occupied by the beam, termed 'filamentation'.

Transport of charged particle beams

- Basic calculations (to 3rd order) can be carried out by matrix techniques quite similar to ABCD
- BUT - inherent astigmatism of magnetic-optical components demands that we use separate matrix elements for x (H) and y (V) motion
- AND chromaticity of systems means we need to add 2 more components to our matrix, giving a 6 x 6 matrix.
- Matrix components are (x, θ, y, ϕ, ℓ, δ)

Extra co-ordinates are:

ℓ = bunch length of particle beam (strictly the difference in path length between the arbitrary trajectory and the central trajectory)

$\delta = \delta p/p$ = fractional *momentum* width of particles in beam (which is same as fractional *energy* width $\delta E/E$ for ultra-relativistic particles like electrons – but *not* in general for ions/protons)

Program TRANSPORT (K L Brown et al) is archetypal beam transport program, and almost all others use similar notation. Beam rep by 6 X 6 ‘R-matrix’:

$$\begin{bmatrix} x \\ \theta \\ y \\ \phi \\ \ell \\ \delta \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} & R_{16} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} & R_{26} \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} & R_{36} \\ & & & R_{44} & R_{45} & R_{46} \\ & etc & & & R_{55} & R_{56} \\ & & & & & R_{66} \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \\ y_0 \\ \phi_0 \\ \ell_0 \\ \delta \end{bmatrix}$$

Due to symmetries in system, many matrix elements are always zero

$$\begin{bmatrix} x \\ \theta \\ y \\ \phi \\ \ell \\ \delta \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & 0 & 0 & 0 & R_{16} \\ R_{21} & R_{22} & 0 & 0 & 0 & R_{26} \\ 0 & 0 & R_{33} & R_{34} & 0 & 0 \\ 0 & 0 & R_{43} & R_{44} & 0 & 0 \\ R_{51} & R_{52} & 0 & 0 & 1 & R_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \\ y_0 \\ \phi_0 \\ \ell_0 \\ \delta \end{bmatrix}$$

horiz (x) plane matrix

vert (y) plane matrix

chromaticity (ℓ, δ) matrix

…assuming here (as is normal) that all *bends* are in *horizontal* plane.

Transport of a ray though a system of beam line elements is as before for L optics:

$$x_n = (R_n R_{n-1} \dots R_0)\, x_0$$

Complete system is represented by one matrix:

$$R_{system} = R_n R_{n-1} \dots R_0$$

…and, as for L optics, optical imaging conditions can be specified by setting certain matrix elements to zero. e.g.:

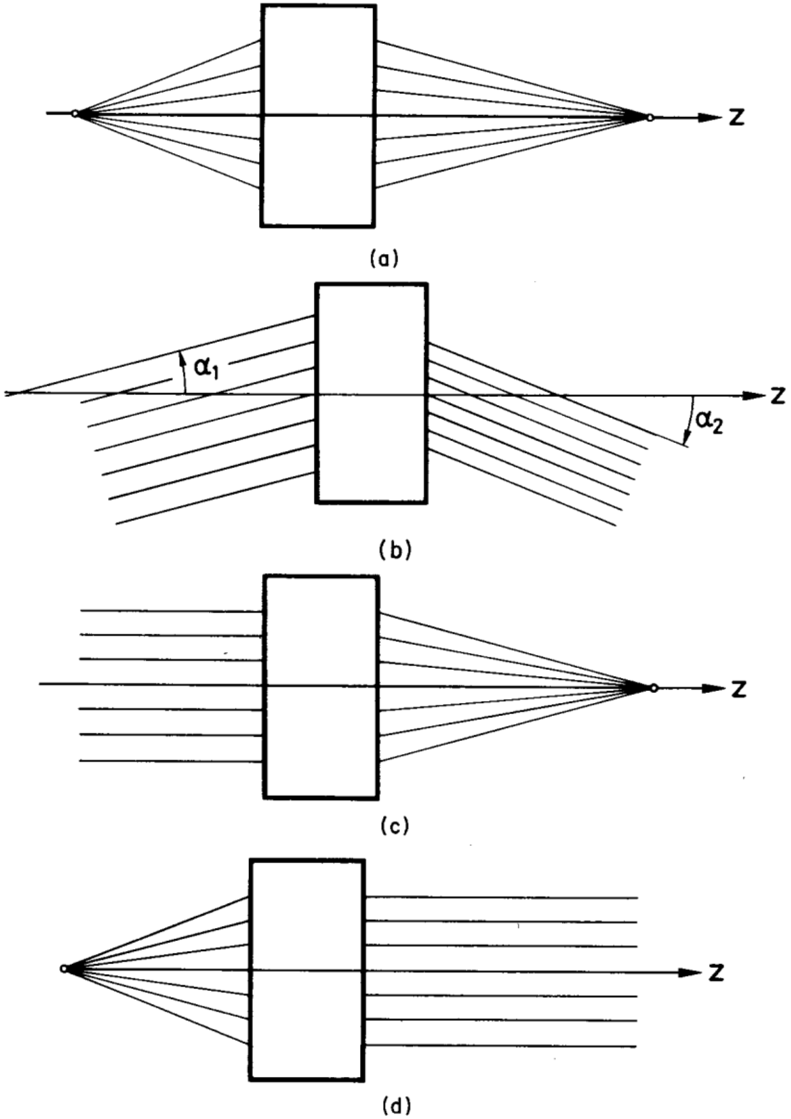
$R_{11} = 0 \Rightarrow$ H plane parallel-to-point focusing

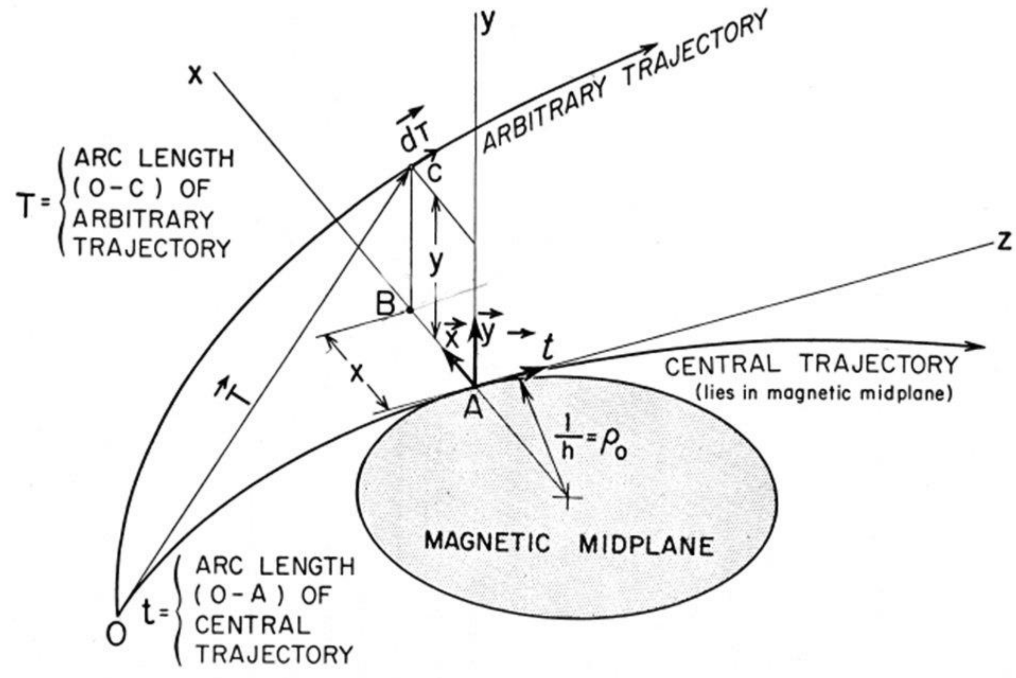
$R_{12} = 0 \Rightarrow$ H plane point-to-point imaging

$R_{21} = 0 \Rightarrow$ H plane parallel-to-parallel transformation

$R_{22} = 0 \Rightarrow$ H plane point-to-parallel transformation

… with corresponding conditions for V plane (R_{33} to R_{44})





Curvilinear TRANSPORT Coordinate System

(a) Schematic representation of an optical system with vanishing $(x|a)$. Such a system is also referred to as point-to-point focusing. (b) Schematic representation of an optical system with $(a|x) = 0$. Such a system is also referred to as parallel-to-parallel focusing (it is also called a telescope). (c) Schematic representation of an optical system with $(x|x) = 0$. Such a system is also referred to as parallel-to-point focusing. (d) Schematic representation of an optical system with $(a|a) = 0$. Such a system is also referred to as point-to-parallel focusing.

In bend (H) plane also have chromaticity conditions:

$R_{16} = 0 \Rightarrow$ achromatic and $R_{26} = 0 \Rightarrow$ angle-achromatic

examples follow

Now we can also impose conditions on bunch length:
e.g.

$R_{56} = 0 \Rightarrow$ isochronous condition (bunch length does NOT depend on particle momentum)….

plus if $R_{51} = 0$ and $R_{52} = 0$ also, this isochronous condition will be maintained down the transport system

This should ideally be maintained to 2nd order ($T_{566} = 0$, etc).

Important in many modern accelerator and free-electron laser applications – but often difficult to achieve (see later).

e.g. TRANSPORT matrices of a Drift and a Quadrupole

DRIFT space matrix

The first-order R matrix for a drift space is as follows:

$$
\begin{pmatrix}
1 & L & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & L & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

where

L = the length of the drift space.

First-order quadrupole matrix

$$
\begin{bmatrix}
\cos k_q L & \frac{1}{k_q}\sin k_q L & 0 & 0 & 0 & 0 \\
-k_q \sin k_q L & \cos k_q L & 0 & 0 & 0 & 0 \\
0 & 0 & \cosh k_q L & \frac{1}{k_q}\sinh k_q L & 0 & 0 \\
0 & 0 & k_q \sinh k_q L & \cosh k_q L & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

focusing plane of quadrupole

defocusing plane of quadrupole

no path length dependence
(to first order)

These elements are for a quadrupole which focuses in the horizontal (x) plane (B positive). A vertically (y-plane) focusing quadrupole (B negative) has the first two diagonal submatrices interchanged.

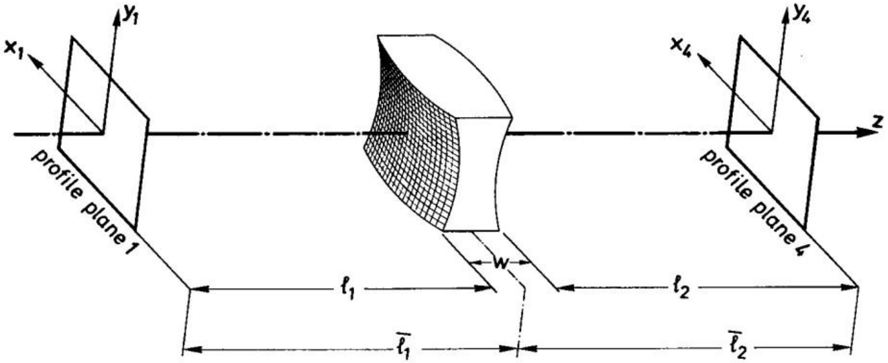
Definitions:
- L = the effective length of the quadrupole
- a = the radius of the aperture
- B_0 = the field at radius a
- $k_q^2 = (B_0/a)(1/B\rho_0)$, where $(B\rho_0)$ = the magnetic rigidity (momentum) of the central trajectory.

Combine drifts and quadrupoles as before to make doublets and triplets:

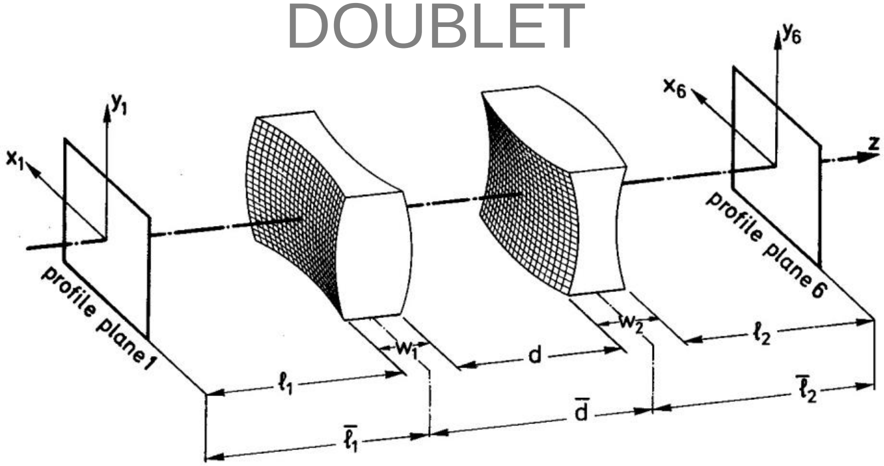
N.B. Essential in CPO because quadrupole is highly astigmatic – need combinations to mimic rotationally-symmetric LO lenses.

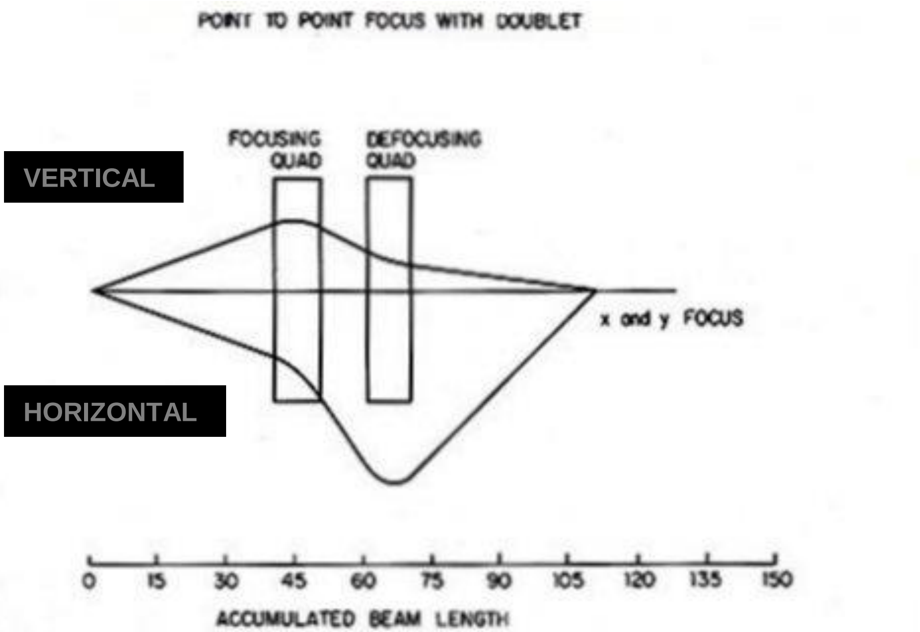
optics of a quadrupole singlet & doublet

SINGLET



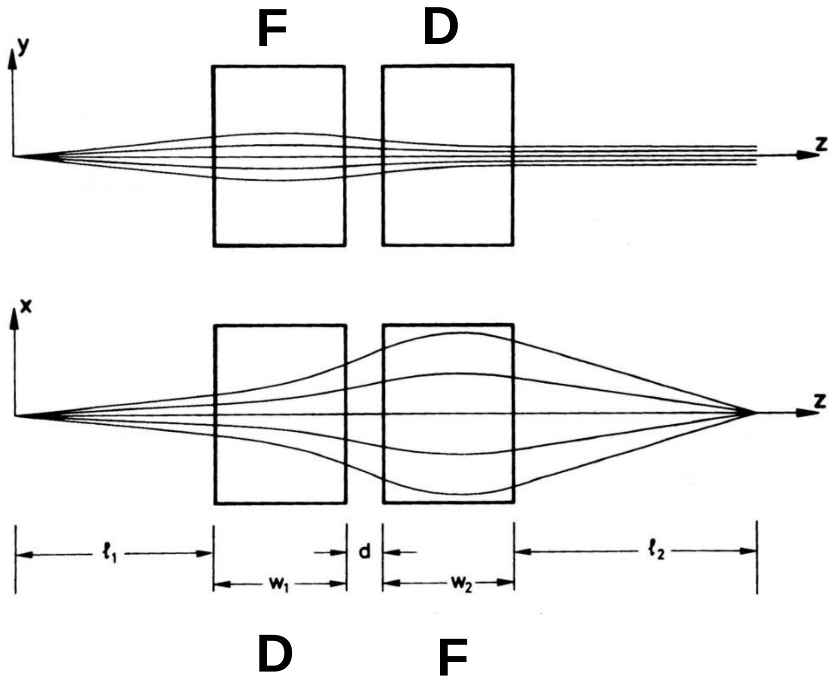
DOUBLET





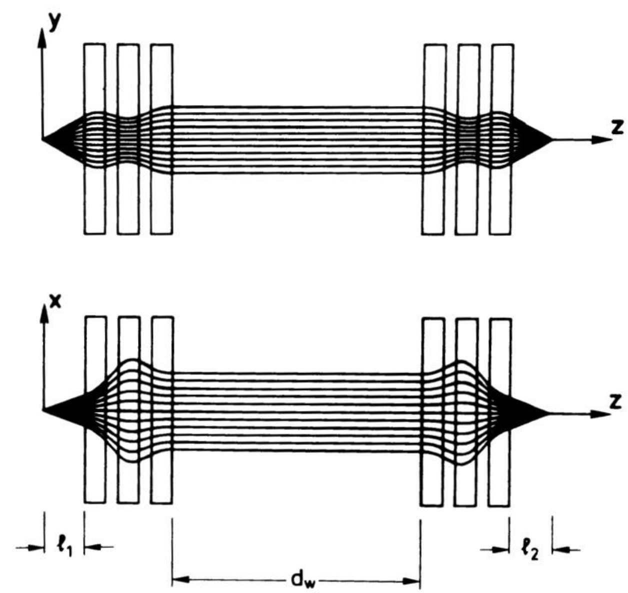
Point-to-point focusing with a quadrupole doublet. The two trajectories shown are in the horizontal and vertical planes respectively.

Imagine designing laser optics with lenses like these!

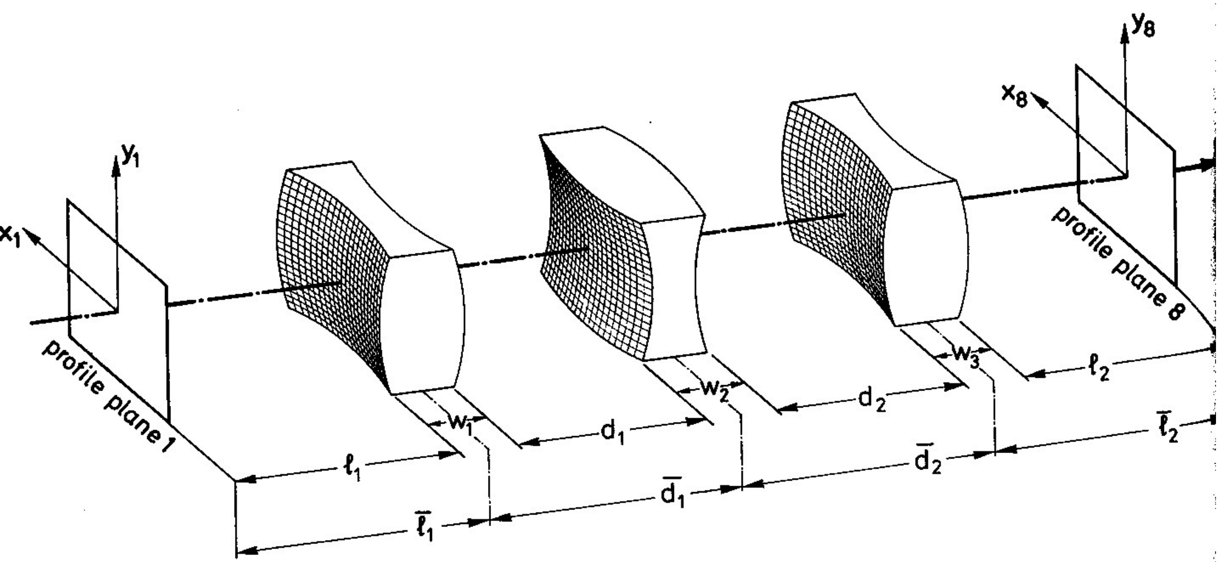


FD and DF planes of a
stigmatic quadrupole doublet

point-to-point
quadrupole sextet



A point-to-parallel-to-point focusing quadrupole sextet showing particle trajectories that all start at the center of the object. Note that this sextet is a DFDDFD system.

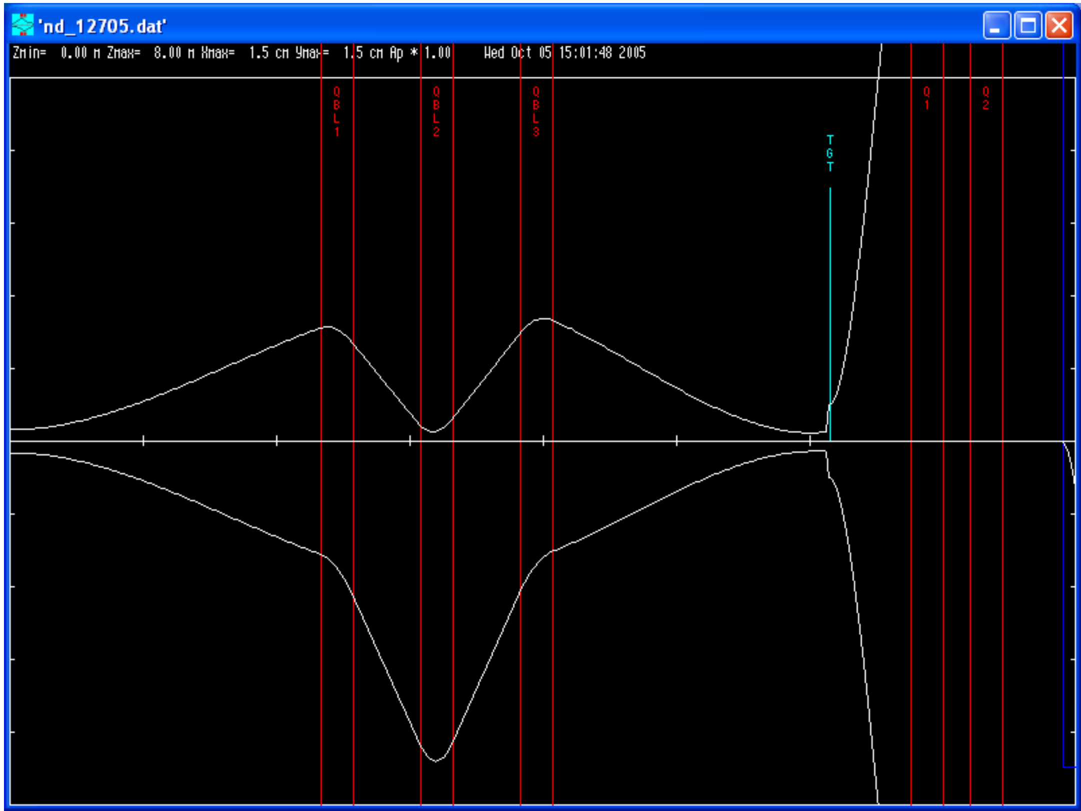


focusing with a quadrupole triplet

TRANSPORT design calculation of a quadrupole triplet focusing on a target (tgt).
horiz.(x) and vert.(y) envelopes of the phase ellipse (σ-matrix).

This symmetrical triplet ½f - d - ½f corresponds to an optical lens.

This indicates the complexity of beam transport with magnetic elements!



TRANSPORT bending magnet (dipole) matrix (first-order only):

First-order wedge bending magnet matrix

$$
\begin{bmatrix}
\cos k_x L & \frac{1}{k_x}\sin k_x L & 0 & 0 & 0 & \frac{h}{k_x^2}\left(1-\cos k_x L\right) \\
-k_x \sin k_x L & \cos k_x L & 0 & 0 & 0 & \frac{h}{k_x}\sin k_x L \\
0 & 0 & \cos k_y L & \frac{1}{k_y}\sin k_y L & 0 & 0 \\
0 & 0 & -k_y \sin k_y L & \cos k_y L & 0 & 0 \\
-\frac{h}{k_x}\sin k_x L & -\frac{h}{k_x^2}\left(1-\cos k_x L\right) & 0 & 0 & 1 & -\frac{h^2}{k_x^3}\left(k_x L - \sin k_x L\right) \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

Definitions: $h = 1/\rho_0$, $k_x^2 = (1-n)h^2$, $k_y^2 = nh^2$

$\alpha = hL$ = the angle of bend

L = path length of the central trajectory.

No time to go into details of dipoles here ….

Examples of systems,

and comparison with L optics

What follows are some arbitrarily chosen examples to illustrate the similarities and differences between CP and L optics.

The references at the end contain many more examples.

Schematic layouts of achromatic magnet systems

Achromatic systems have broad momentum acceptance, or allow selection of a narrow momentum/energy spread

Achromatic Energy Selection System (in Proton Therapy Facility)

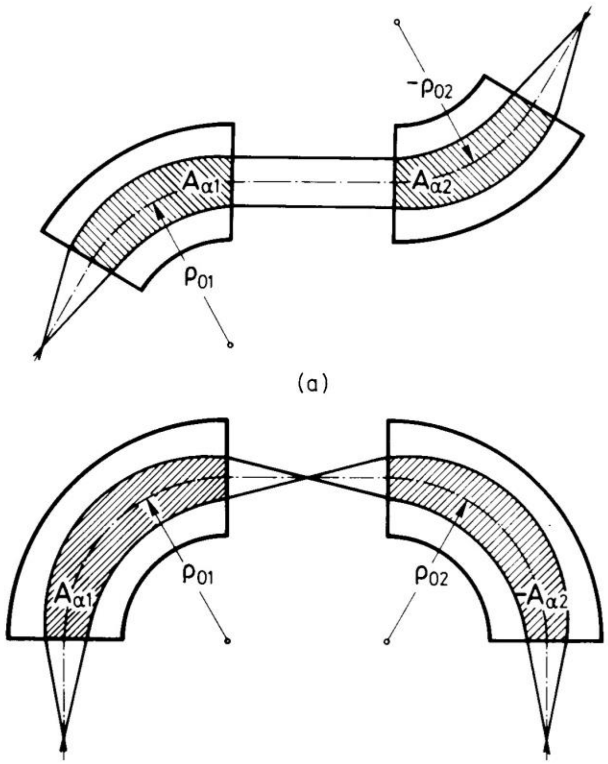
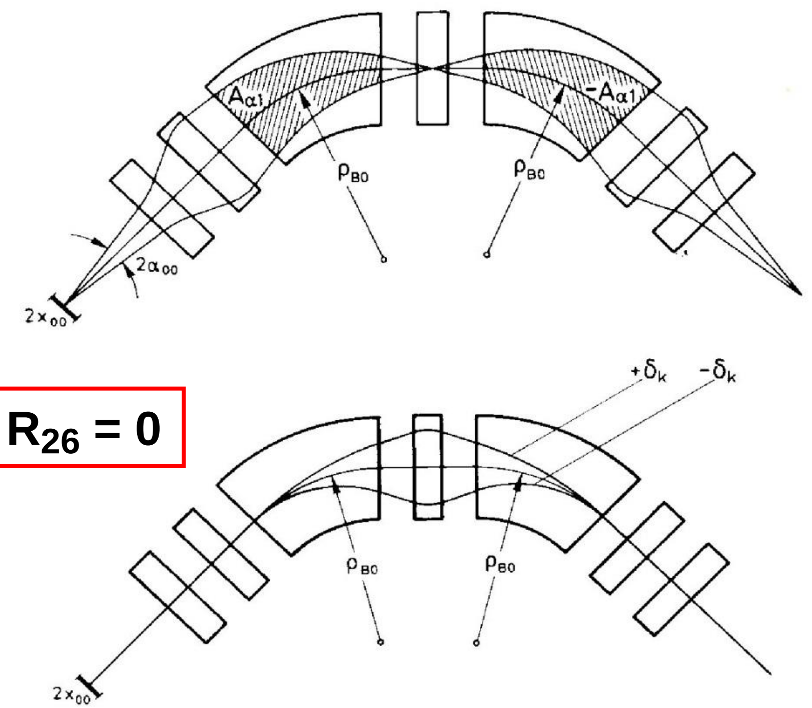


Fig. 9.6. Two double sector field arrangements that both cause achromatic beam deflections. Note that there are opposite signs for the radii of deflection in (a) and opposite signs for the area A_α in (b).

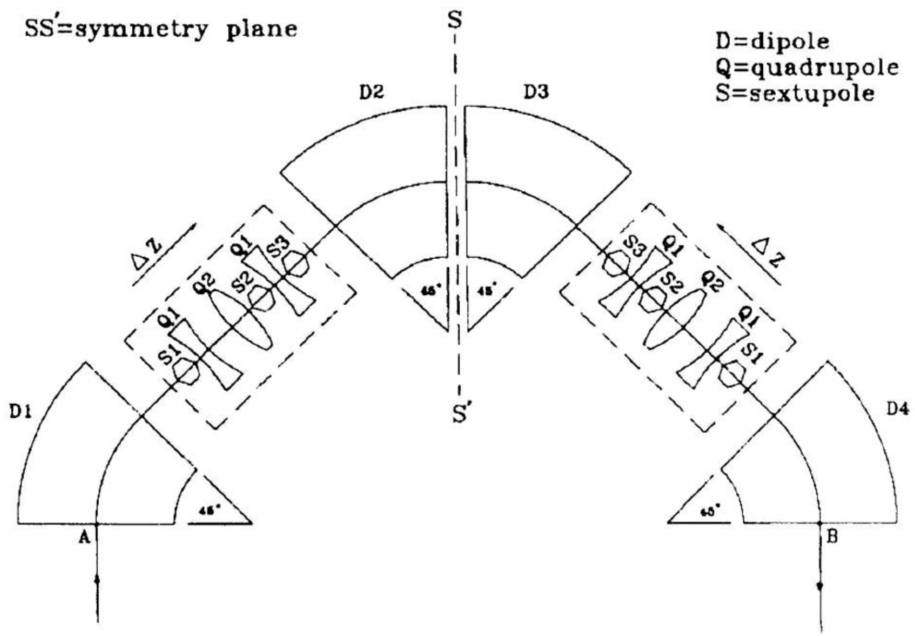


$R_{16} = R_{26} = 0$

An achromatic beam deflector consisting of two sector fields preceded and followed by quadrupole doublets. Note that the areas $A_{\alpha 1}$ and $-A_{\alpha 1}$ are of equal size. The quadrupole in the middle between the sector fields can be adjusted so that the overall system is not only dispersion free $(x|\delta_K) = (x|\delta_m) = 0$ but achromatic $(a|\delta_K) = (a|\delta_m) = 0$; i.e., particles of different rigidities $(\delta_K \neq 0)$ are parallel in the middle of the center quadrupole and coincide at the end of the systems if they coincided at the beginning.

Isochronous Systems

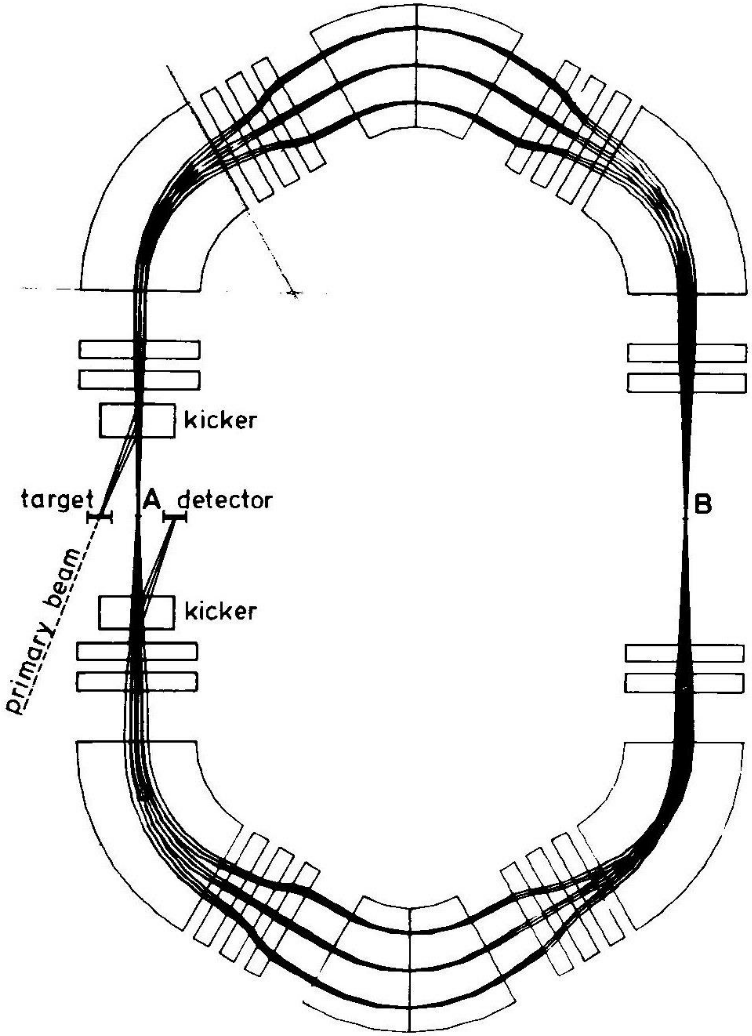
There is no point in generating an ultrashort electron (or CP) bunch if we cannot deliver this bunch unaltered to a target! Systems required need to be isochronous to at least 2nd order in the particle co-ordinates.



(sextupoles are required to cancel out the chromatic aberrations produced by the quadrupoles)

adjustably isochronous 180-deg transport system for free-electron lasers (Gillespie, 1996). Dipoles D2 and D3 can be separated if required.

Adjustability of the isochronism may be required, for example, in a recirculating accelerator such as an energy-recovery linac (ERL)
e.g. Jefferson Lab ERL in Virginia.



An isochronous time-of-flight mass spectrometer is shown that is nothing more than an accelerator storage ring with the quadrupole excitations chosen such that after a 180° bend the matrix elements $(t|\delta_K)$, $(x|a)$, and $(y|b)$ vanish exactly and $(a|x)$ as well as $(b|y)$ vanish approximately. Particles with start simultaneously at the target thus arrive at the same time after a deflection of $n\pi$, with $n = 1, 2, 3, \ldots$, independent of x, y, a, b and δ_K. If introduced by the indicated first "kicker," they can be ejected from the ring by the second "kicker" after seveal turns. Introducing ac driven electrostatic deflectors at the points A and/or B, we can use such a ring also as a high-resolving mass separator, since charged particles can then pass only if the electrostatic fields go through zero. If this ac frequency is high, the achievable mass resolving powers can be very high.

Extreme example of isochronous system - TOF mass spectrometer (Wollnik, 1987)

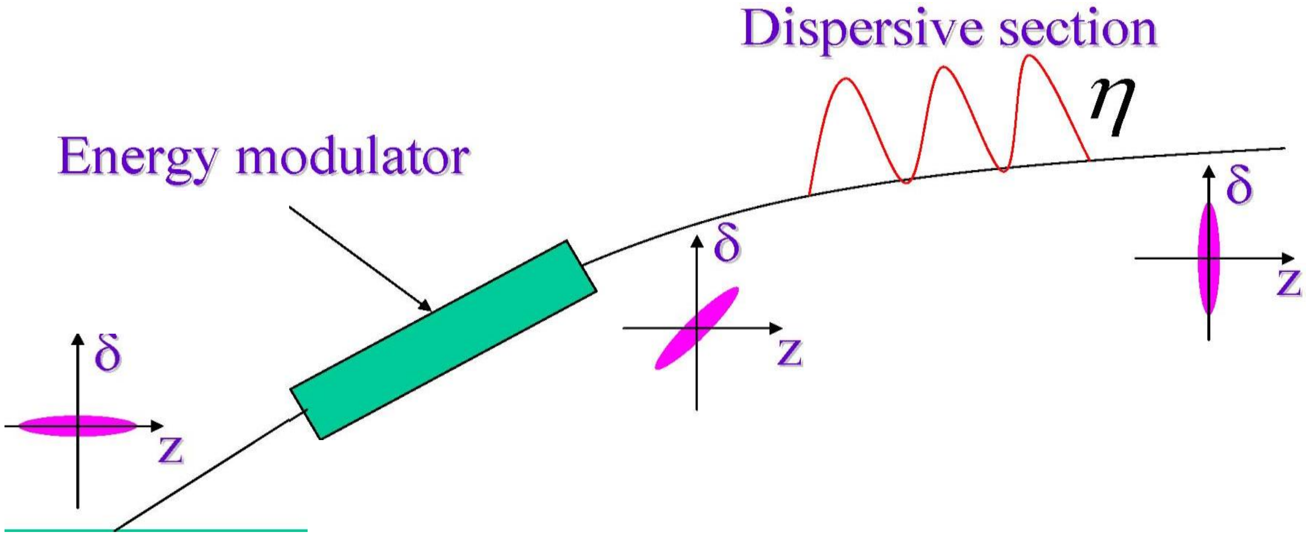
Non-isochronous systems

Comparison: electron bunch compressors with laser pulse compressors/expanders

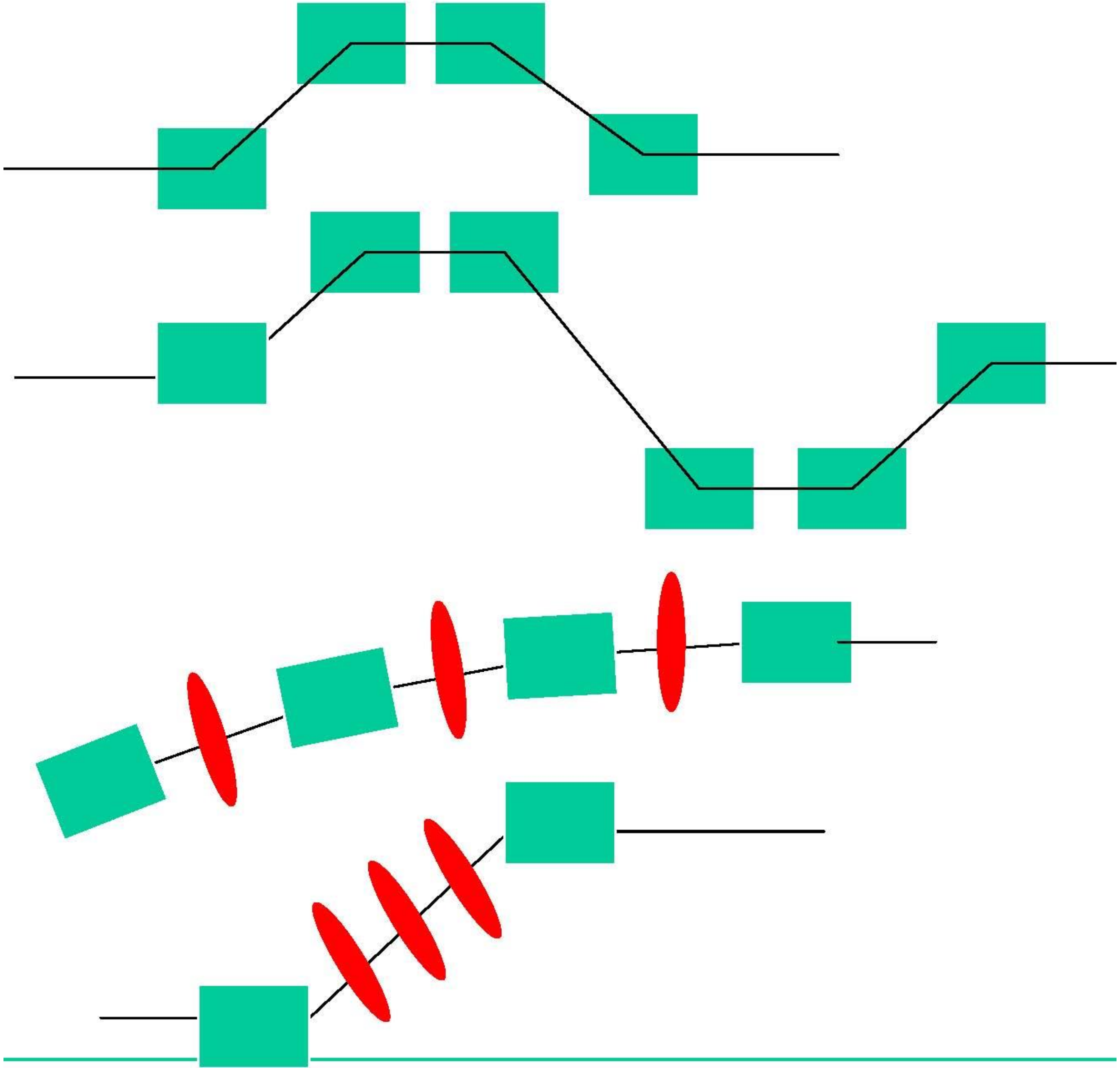
Problem: Coherent X-ray sources (single-pass FELs), linear colliders and laser plasma wakefield accelerators all require high peak currents/very short electron bunches, so we need bunch compression techniques.

Magnetic electron bunch compression – Principle:

- Energy modulator: RF structure, laser, wakefield
- Non-isochronous section of beam line
- In practice, may need multi-stage compression



Types of magnetic bunch compressor



chicane
R_{56}< 0 T_{566}> 0

S-chicane
R56< 0 T566> 0

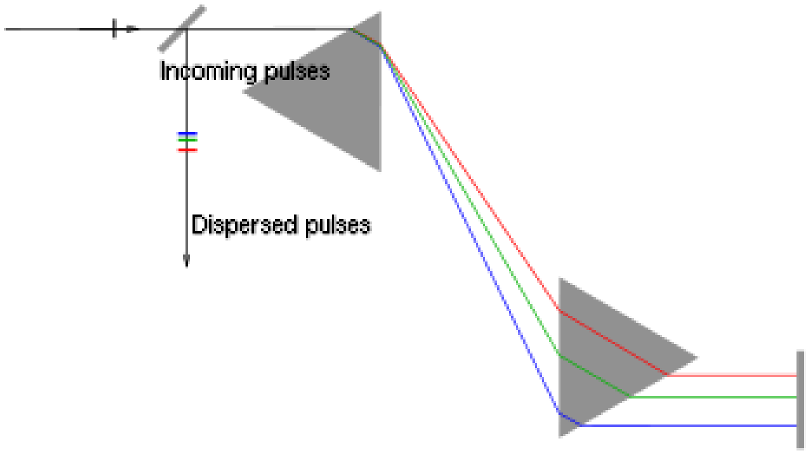
FODO arc
R56 > 0 T566> 0

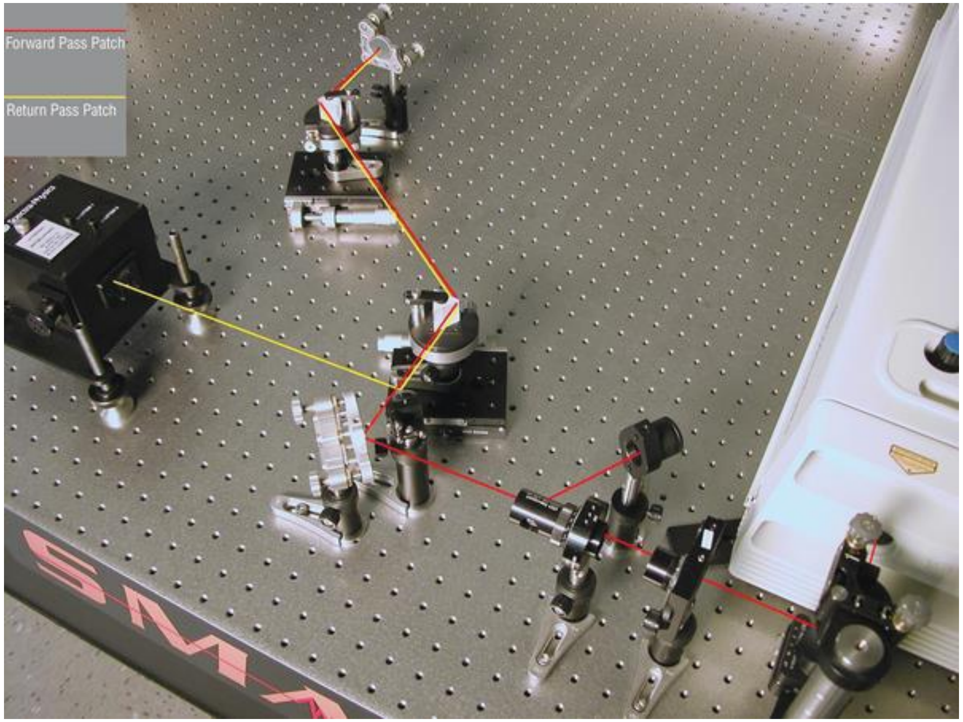
Dog-leg
R56 > 0 T566> 0

Compare laser beam expanders/compressors

Equivalents of magnetic dipoles are optical prisms or diffraction gratings.

prisms

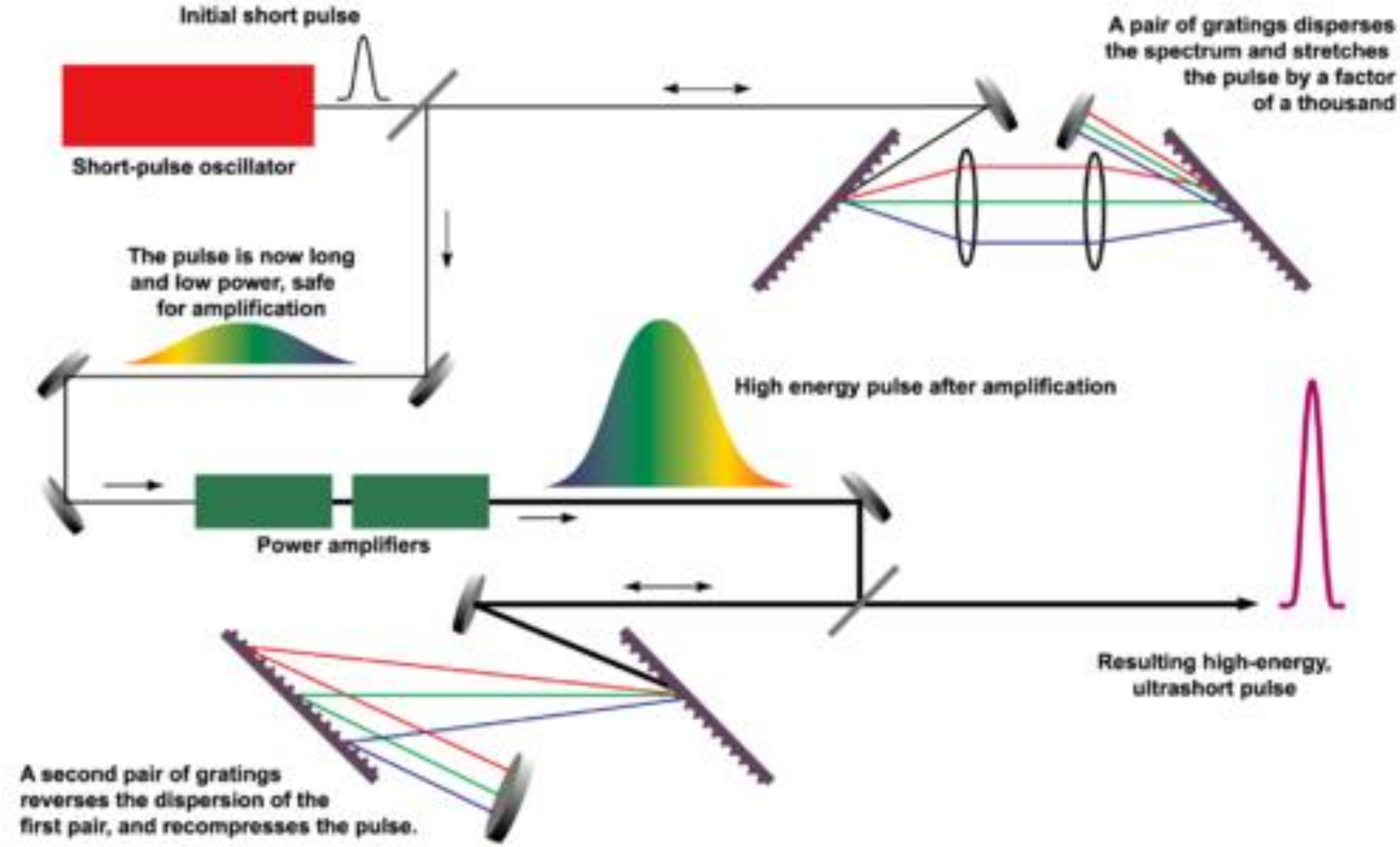




Newport Corporation

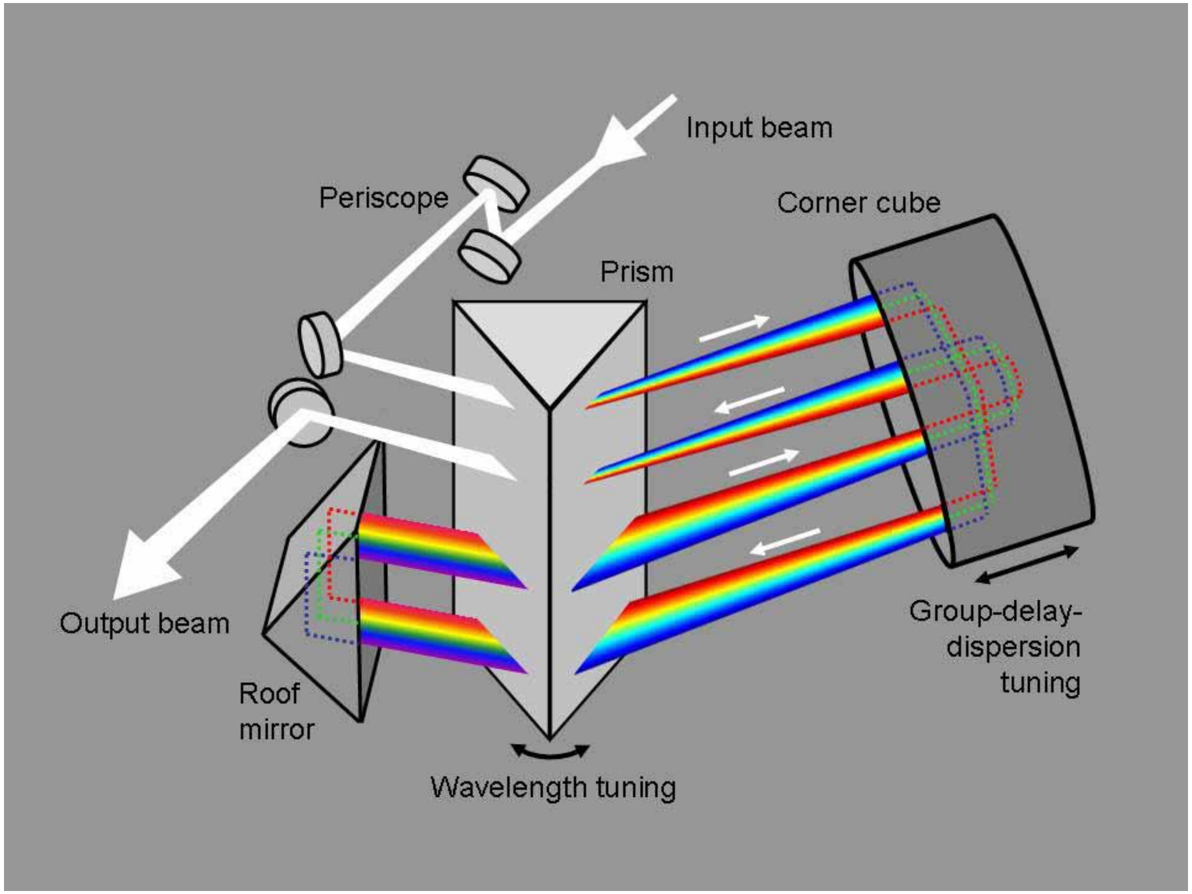
Prism compressor. The red lines represent rays of longer wavelengths and the blue lines those of shorter wavelengths. The spacing of the red, green, and blue wavelength components after the compressor is drawn to scale. This setup has a negative dispersion.

gratings



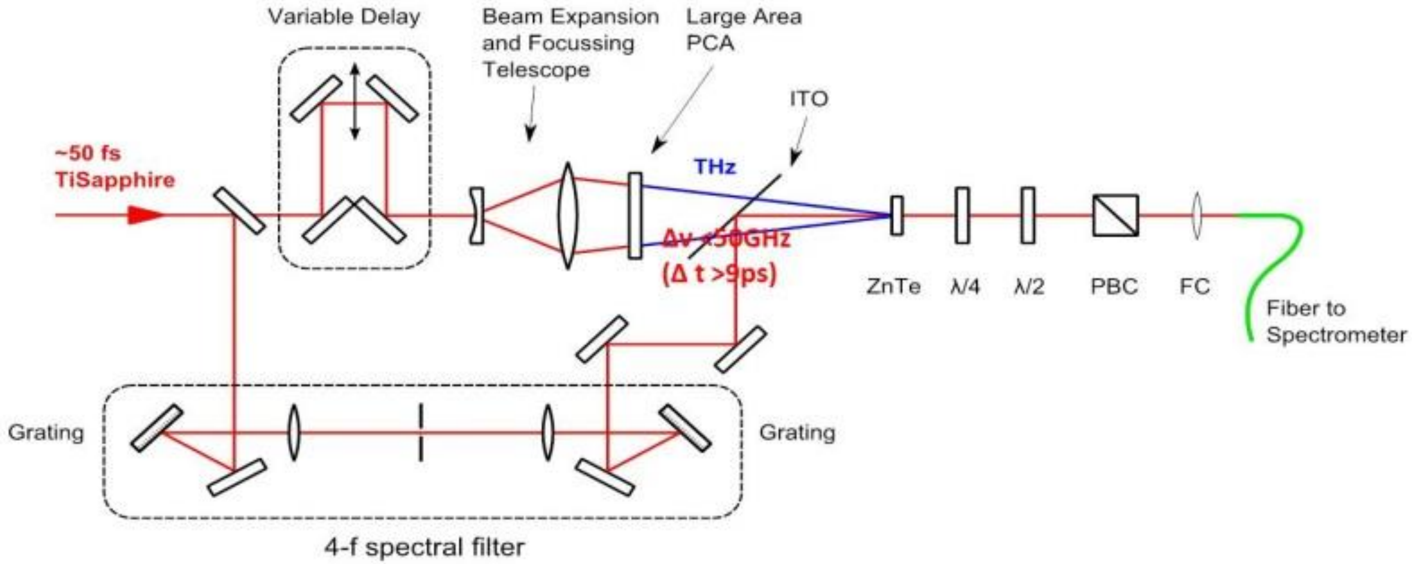
chirped pulse amplification using pairs of gratings.

CPA is a powerful method of producing high-power ultrashort laser pulses

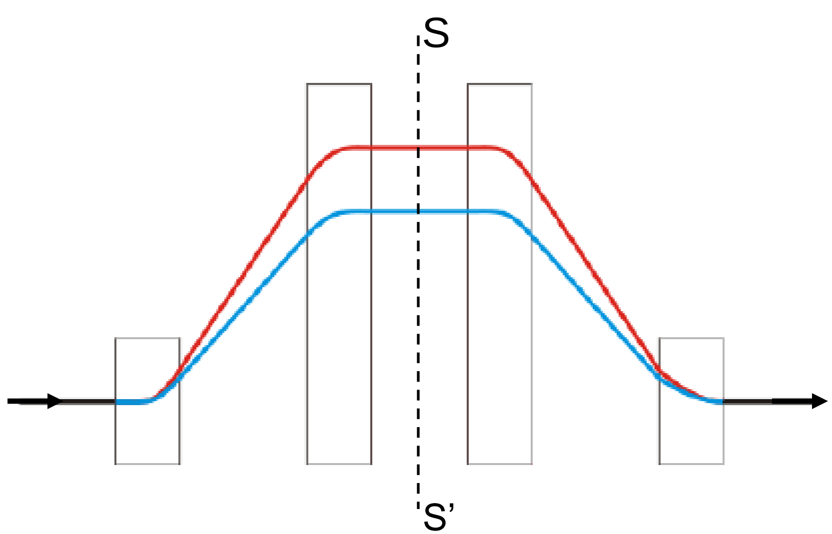


Swamp Optics

Single-prism pulse compressor (Trebino)



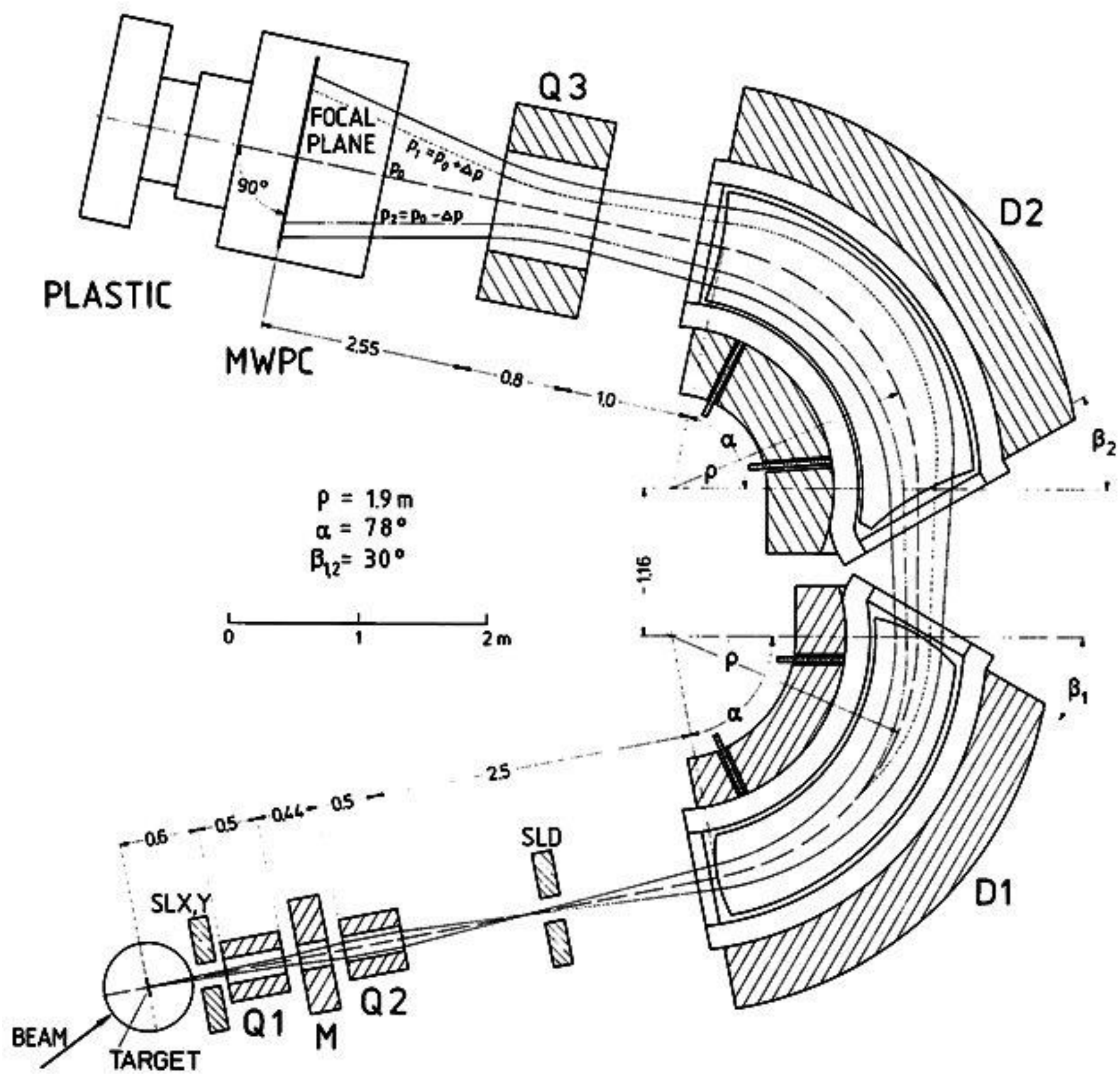
4-f spectral filter used to carve out narrow-band pulse (Dundee-Daresbury group)



4-dipole magnetic chicane can be used to insert mask at symmetry plane SS'
(extensively used in FEL accelerator injectors)

Examples of dispersive systems (charged particle spectrometers)

BIG KARL
Spectrometer
(Jülich, KFZ)

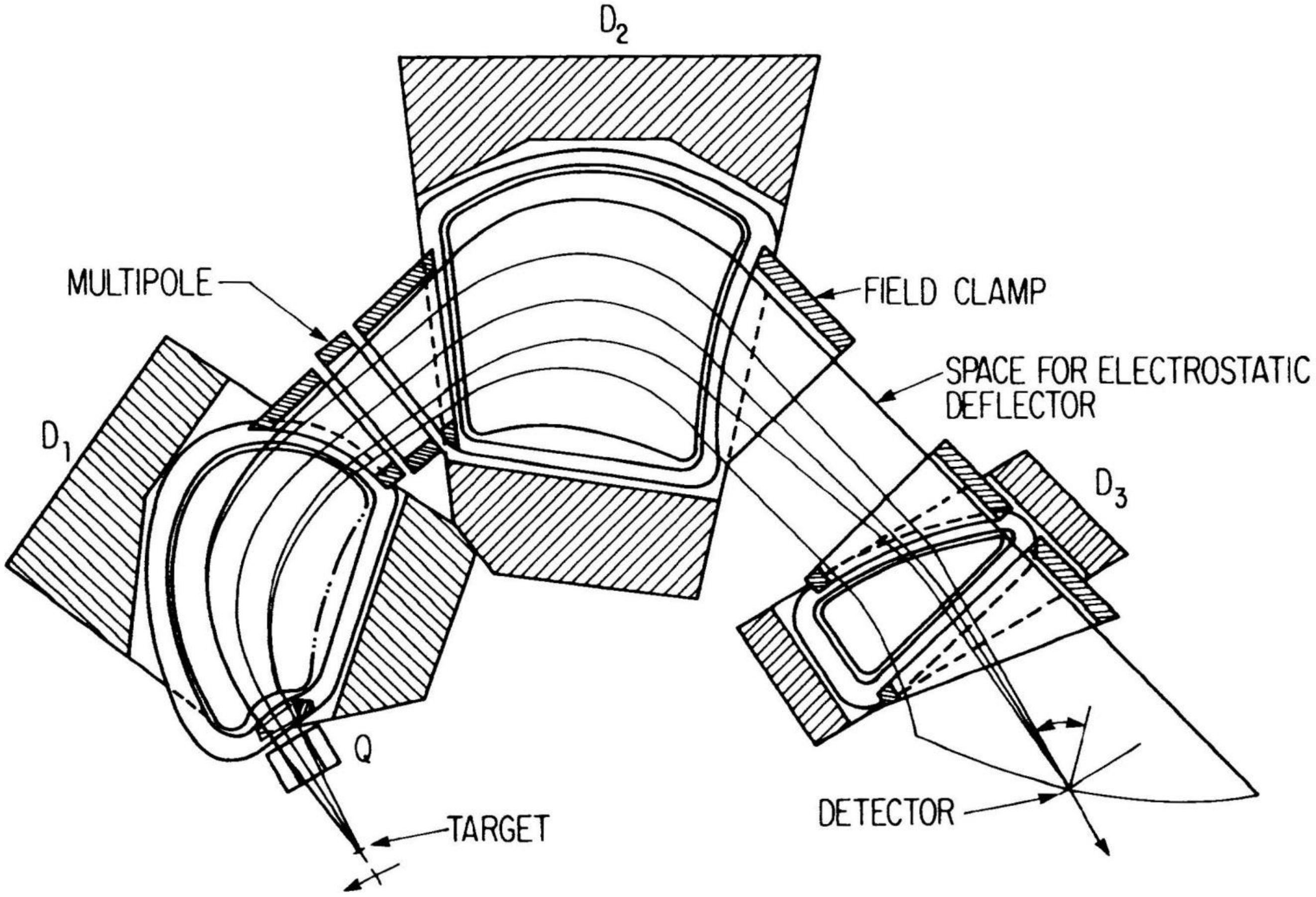


Bending radius ρ_0 = 1.98 m
B_{max} = 1.7 T
gap = 6cm
weight = ~ 50 tons (D1)
~ 70 tons (D2)

Arrangement of the magnetic elements of the QQDDQ spectrometer BIG KARL. The central ray (optical axis) is shown as dashed curve. The outermost rays with the extreme radial distances are drawn as full lines. Four channels in the inner yokes allow NMR probes to be moved into the gaps of the dipoles for radial field measurements. The multipole element between Q1 and Q2 allows the correlation of vertical aberration.

Resolv. power: $p/\Delta p$ = 20600
Dispersion = -2.0 to 26 cm/%
Magnification M_x = 0.63 – 1.26
Magnification M_y = 25.4 – 1.94
Large range: E_{min}/E_{max} = 1.14
Solid angle: < 12.5 msr

Q3D spectrometer (Enge and Kowalski, MIT)

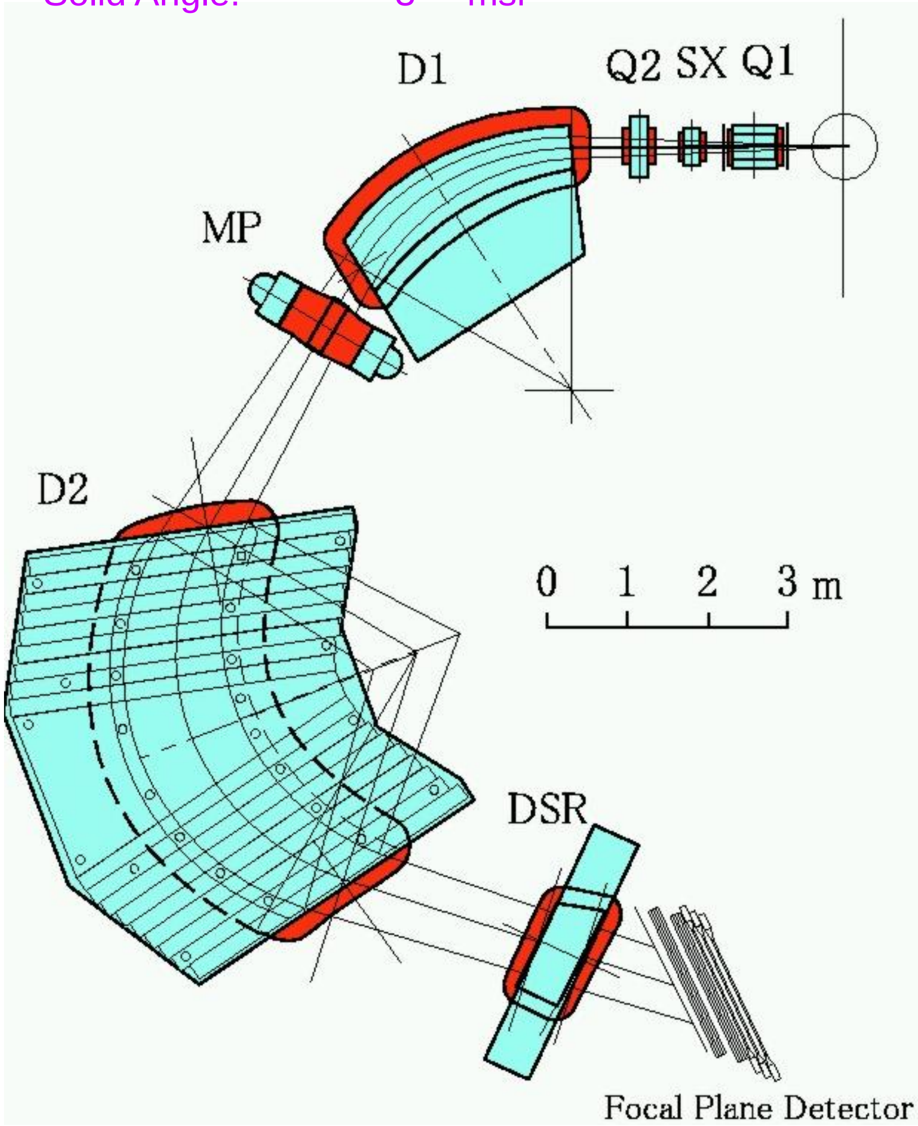


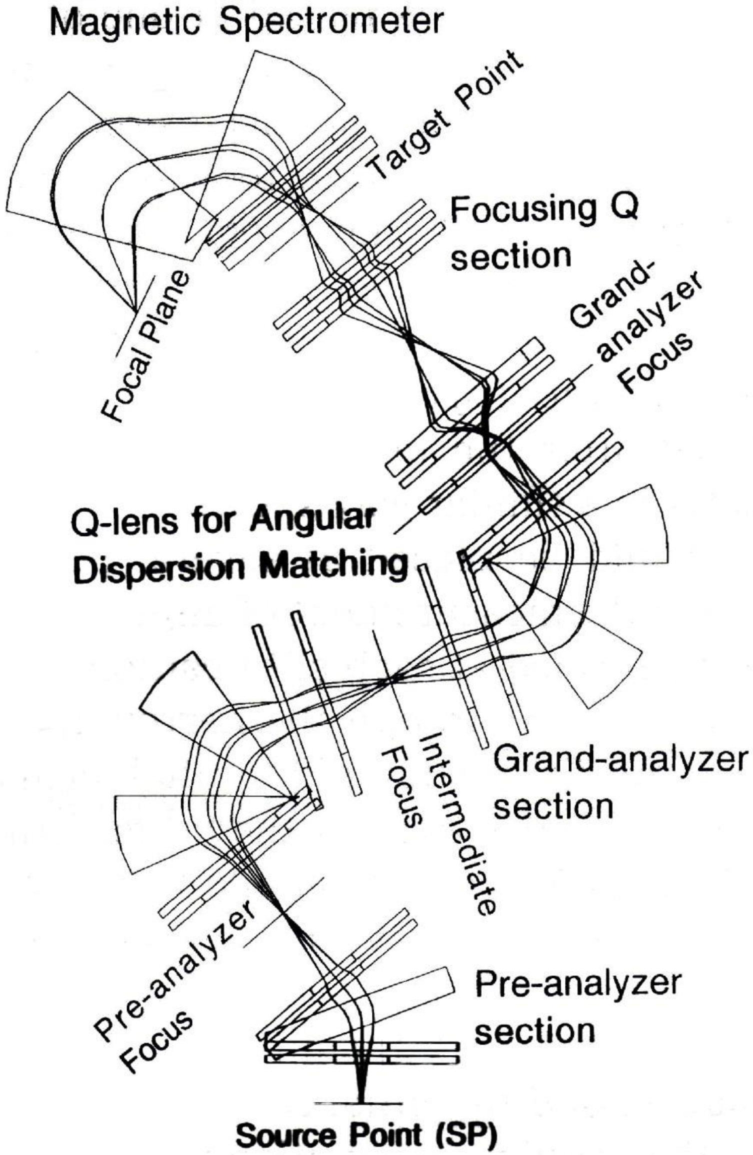
A Q3D Spectrometer used for nuclear physics. Second-order corrections are made through curved entrance and exit faces of the bending magnets.

Grand Raiden High-Resolution Spectrometer
(Osaka, Japan)

Max. Magn. Rigidity: 5.1 Tm
Bending Radius: 3.0 m
Solid Angle: 3 msr

Beam Line/Spectrometer fully matched





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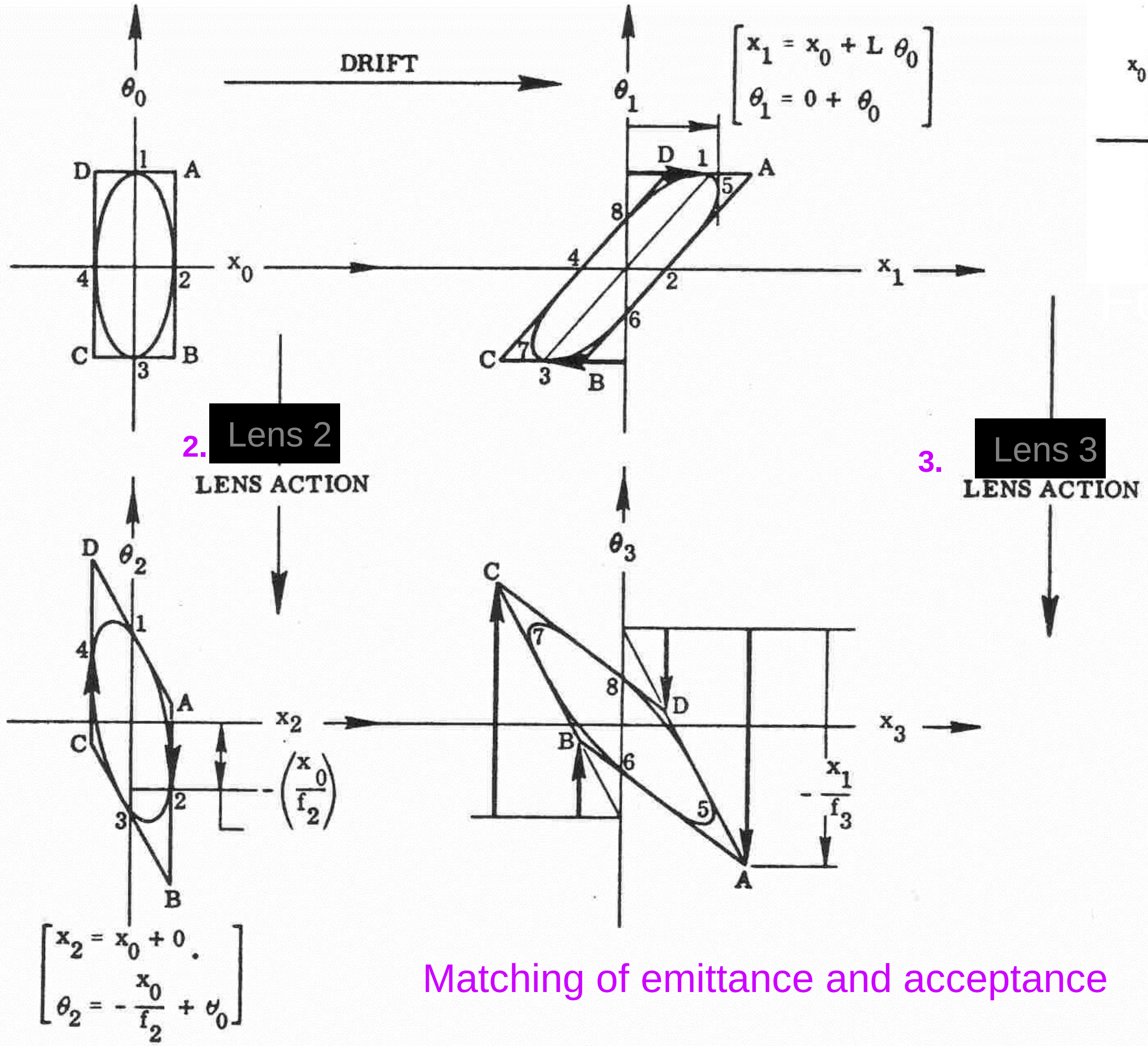
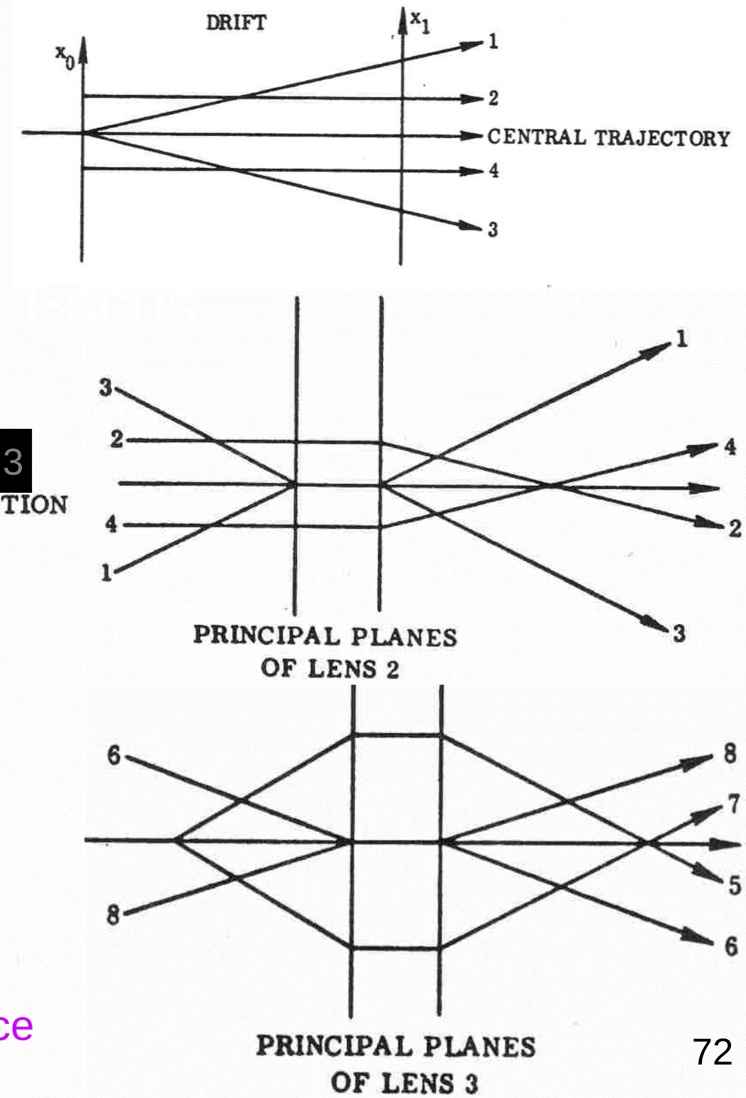
END

The transport of rays and phase ellipses

in a drift and a focusing quadrupole lens

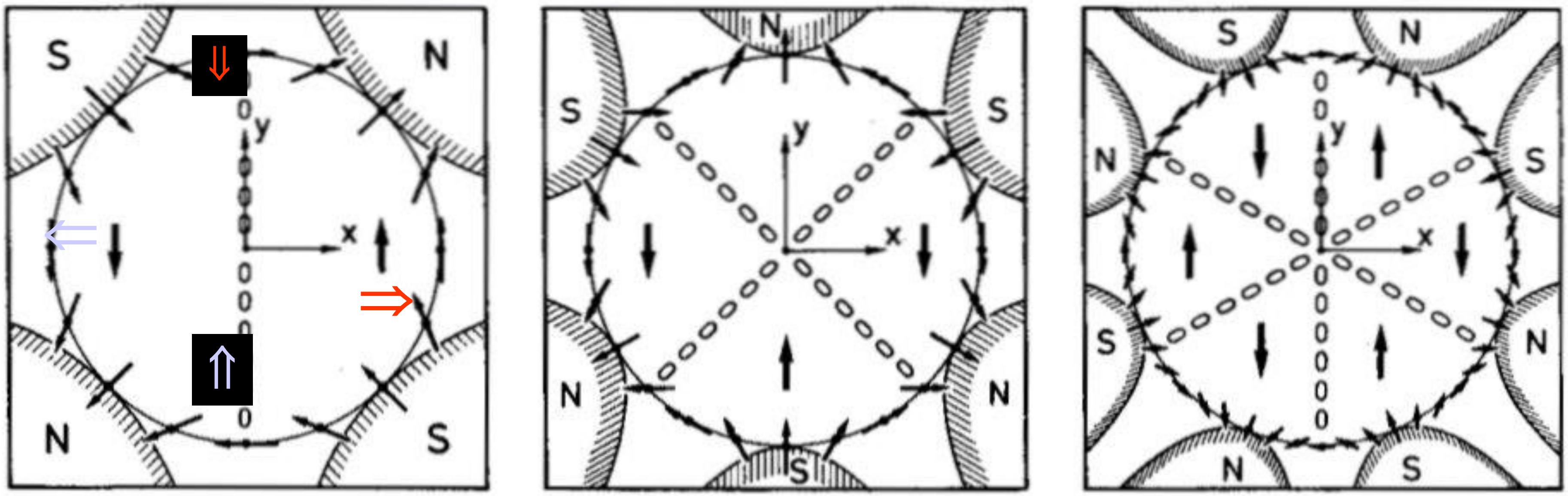




Matching of emittance and acceptance

Forces on ions in magnetic multipoles

Quadrupole Hexapole Octopole



Pole arrangements of magnetic quadrupoles, hexapoles, and octopoles are indicated. Also shown is a circle of radius r_0 along which the magnetic flux density is constant, and its direction varies as indicated. Finally, strings of zeros indicate lines along which B_y, the y component of the magnetic flux density vanishes. These lines separate regions in which B_y is parallel or antiparallel to the y axis.