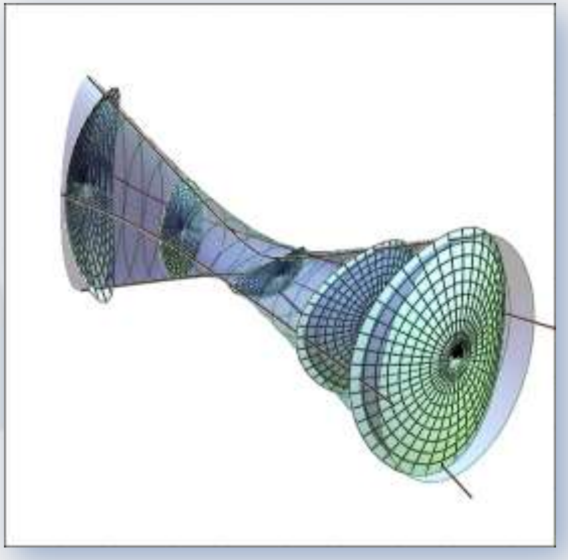
Laser beam characterization

LA^{3}NET – Workshop Aachen 2103

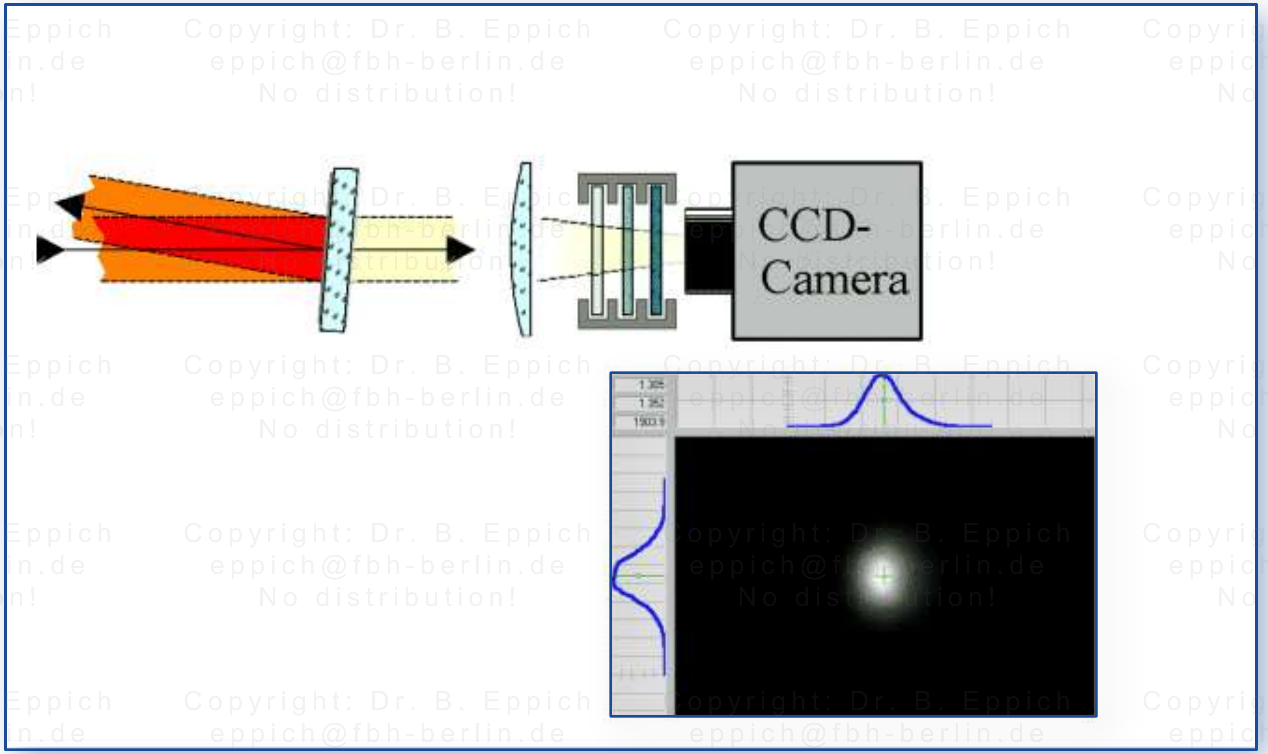


Dr. Bernd Eppich

Content

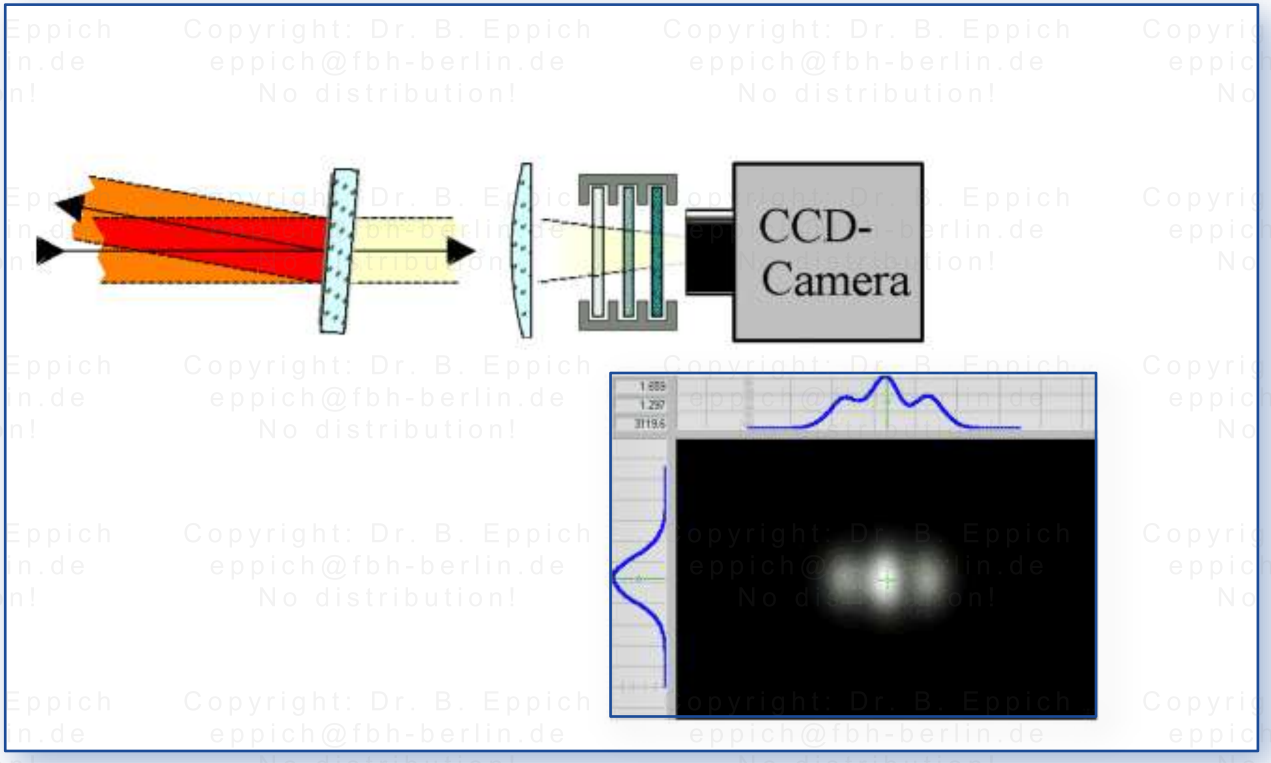
- Motivation
- ISO standard
- Beams and optical systems
- Beam diameter definitions
- Beam classification
- The ten second order moments
- Measurement of the ten second order moments

Evolution of beam profiles



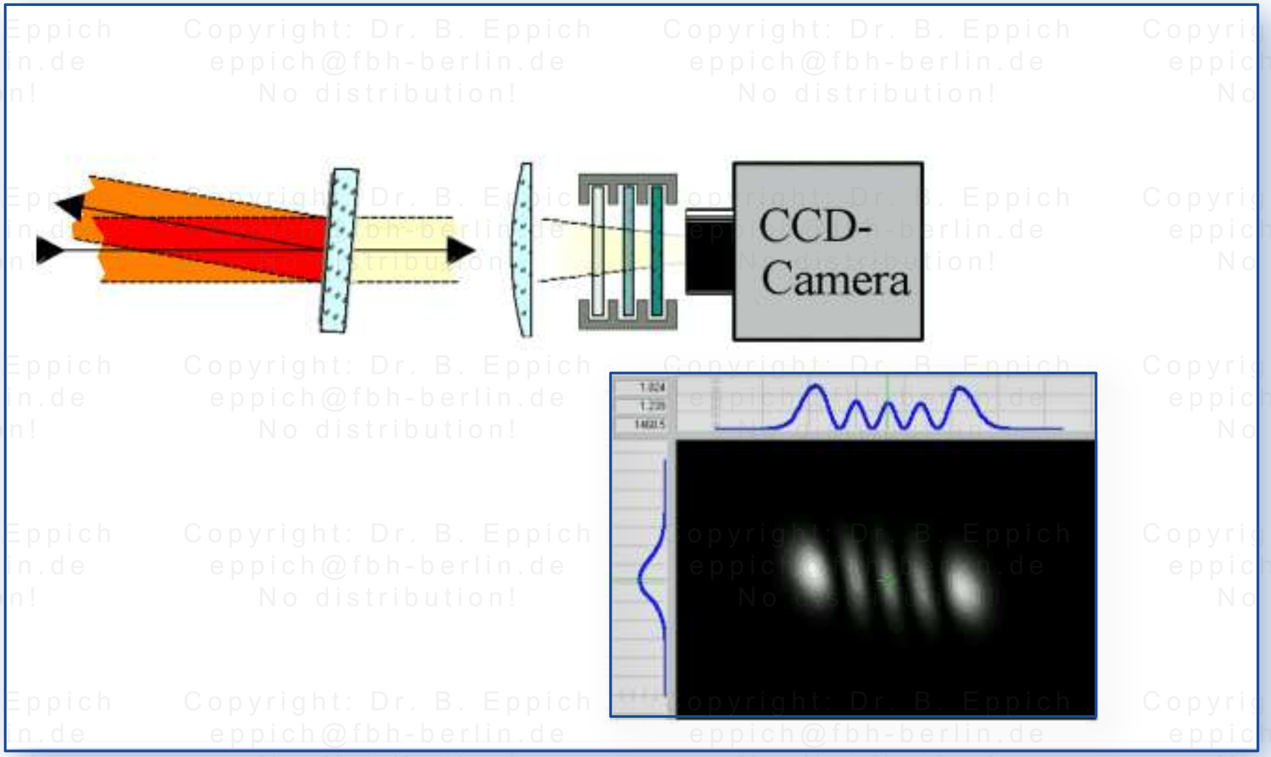
- Nearly no change in structure
- Almost circular profile

Evolution of beam profiles



- Apparent variation in structures
- Elliptical (non-circular) profile
- Same azimuthal orientation

Evolution of beam profiles

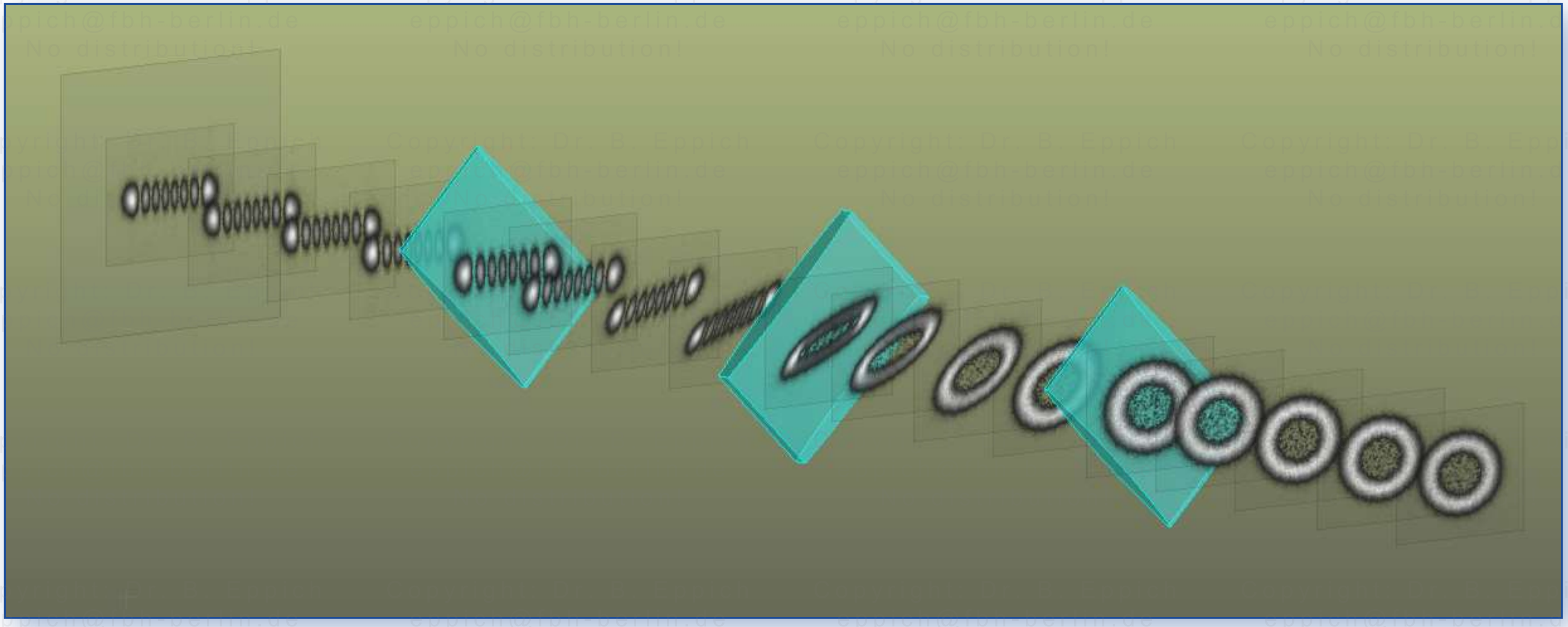


- Apparent variation in structures

Aim: Description and prediction of beam propagation

- Azimuthal orientation **changes**

Beam forming



Aim: Design of optical systems

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Available ISO documents regarding laser beam characterization:

ISO 11145	**-**	**Terms and definitions**
ISO 11146	**-**	**Beam widths, divergence, propagation parameter**
ISO 11554	**-**	**Power, Energy, temporal characteristics**
ISO 11670	**-**	**Beam positional stability**
ISO 12005	**-**	**Polarization**
ISO 13694	**-**	**Power density distribution**
ISO 13695	**-**	**Spectral characteristics**
ISO 15367	**-**	**Wavefront, Shack-Hartmann detectors**

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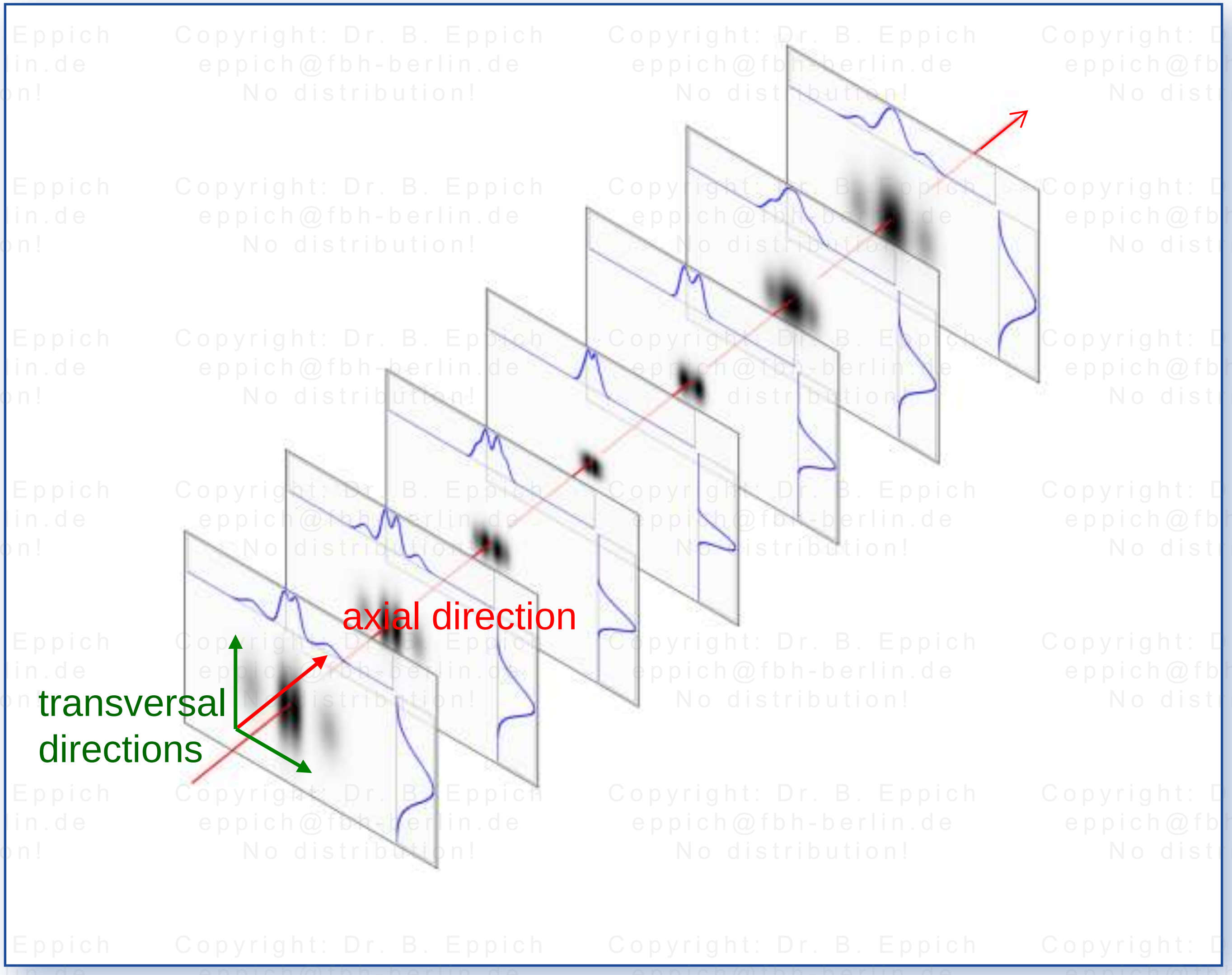
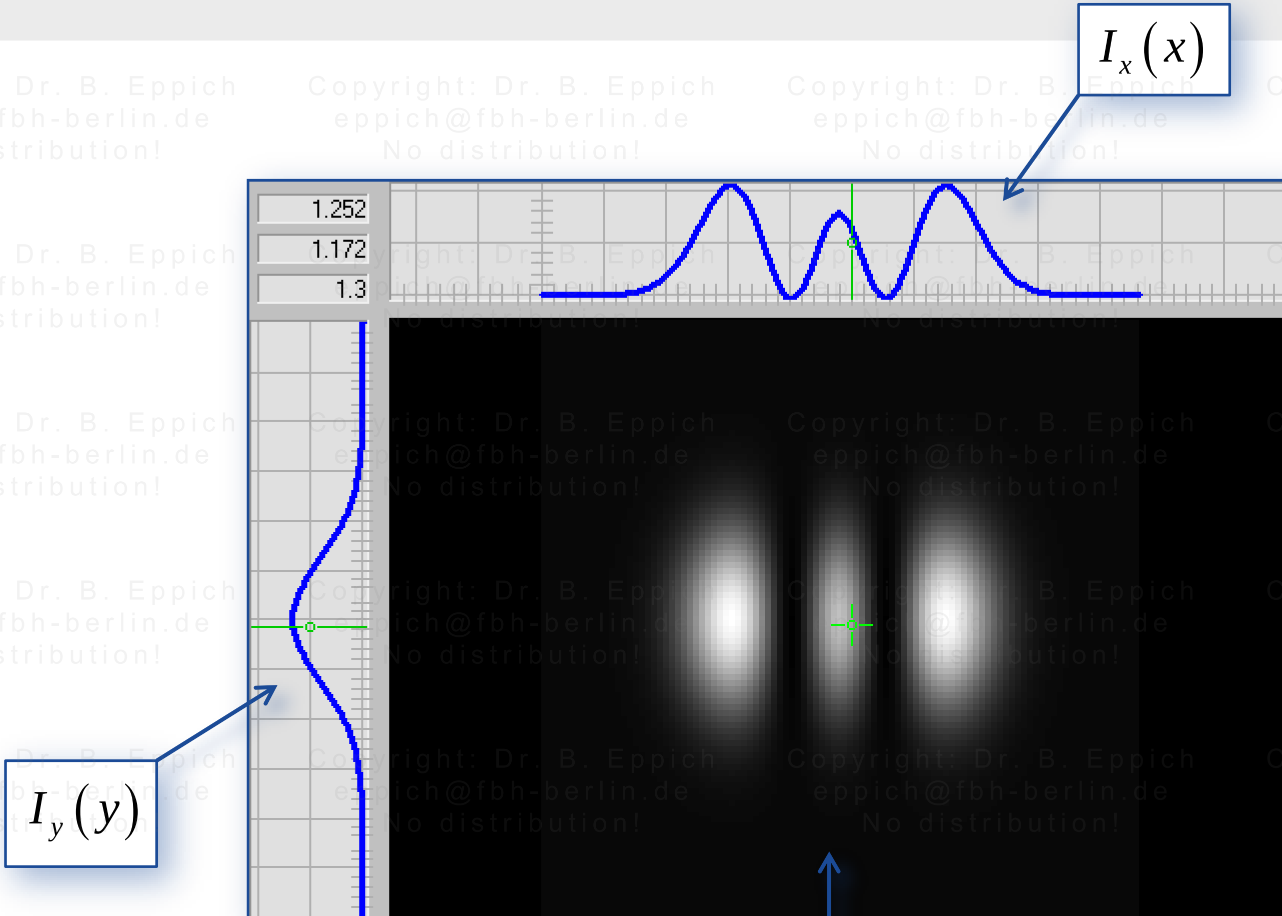
Two-dimensional beam profiles

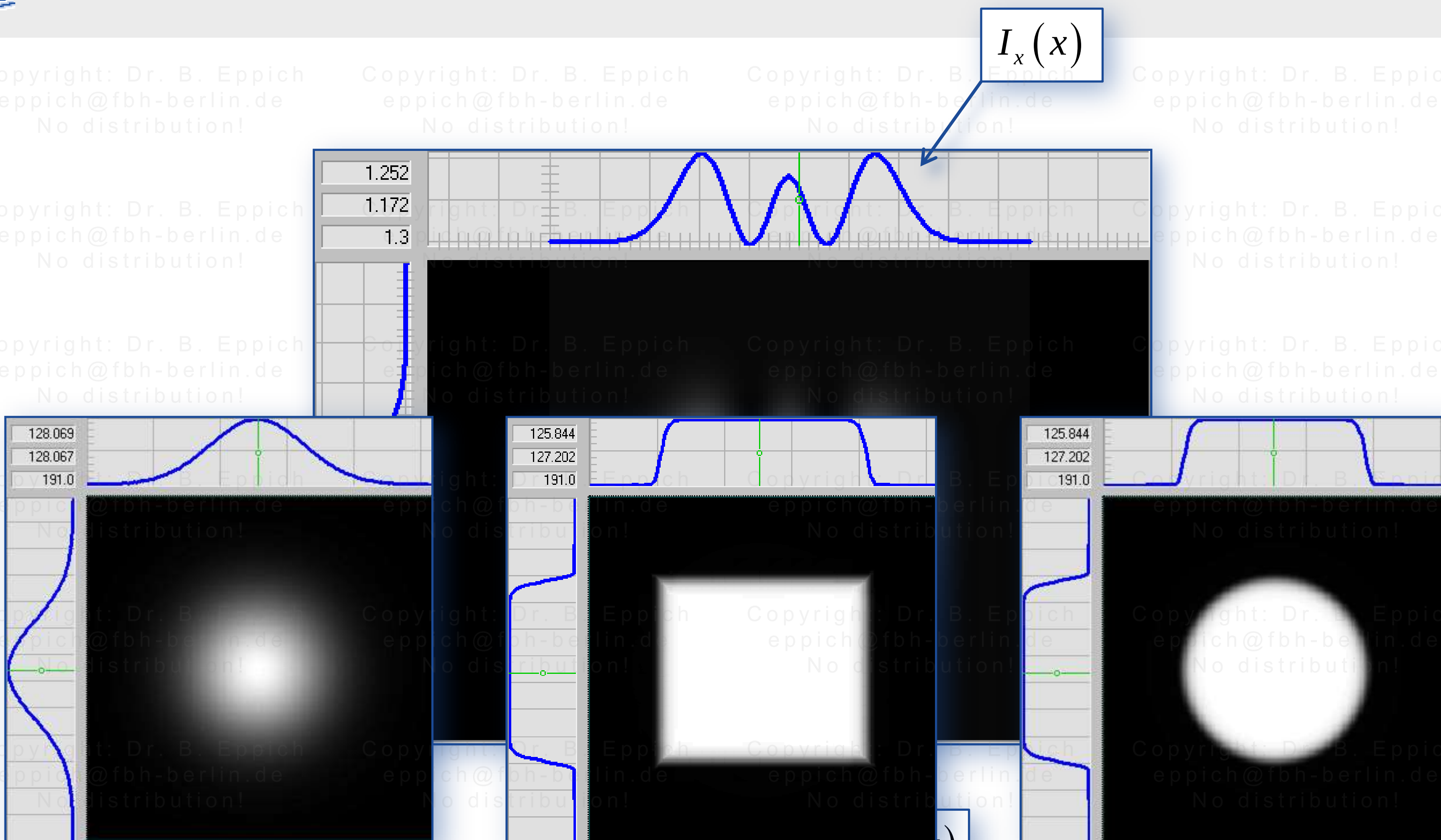


Separable beam profiles



$$I(x,y) = I_x(x) \cdot I_y(y)$$

Separable beam profiles

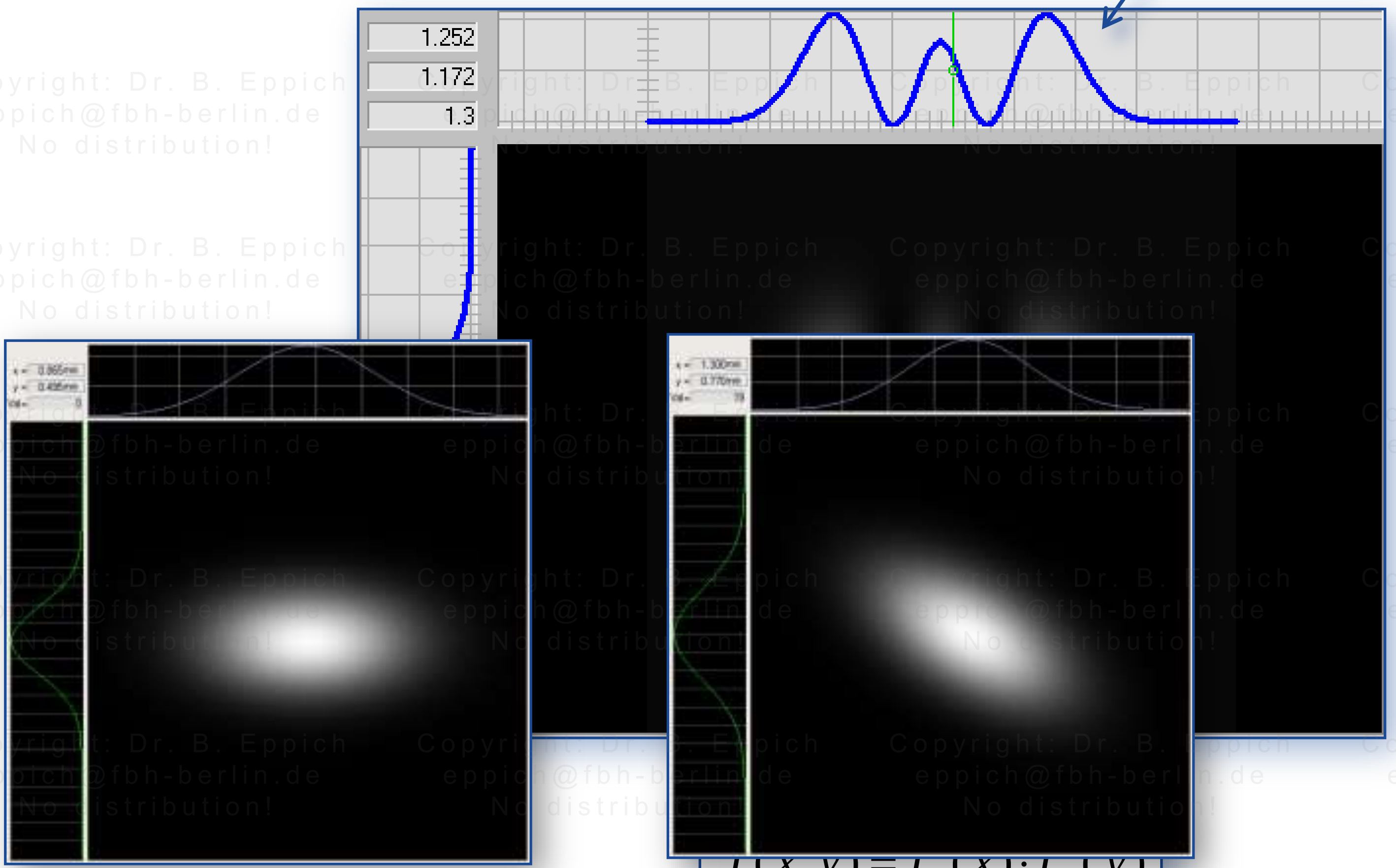


$I(x,y) = I_x(x) \cdot I_y(y)$

separable | (almost) separable | non-separable

Separable beam profiles

$I_x(x)$

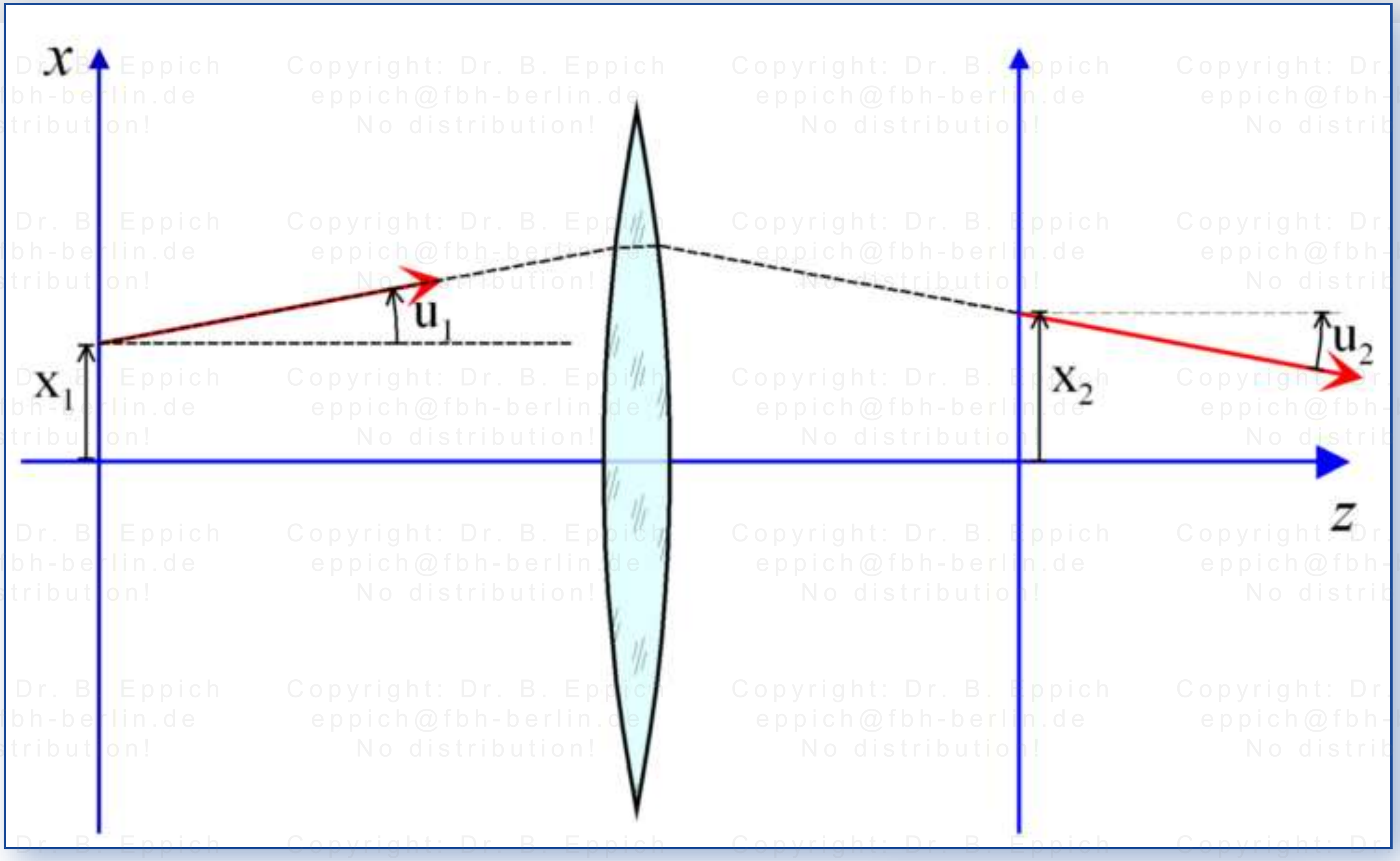


$I(x,y)=I_x(x)\cdot I_y(y)$

separable

Separable
but not in x-y-direction

Geometrical matrix optics, one-dimensional systems



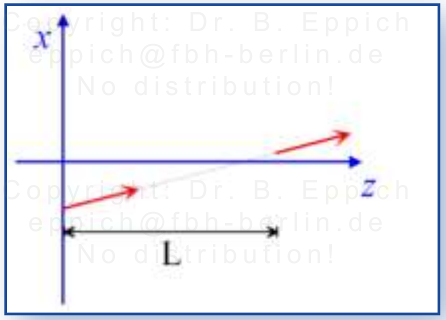
$$x_2 = A \cdot x_1 + B \cdot u_1$$

$$u_2 = C \cdot x_1 + D \cdot u_1$$

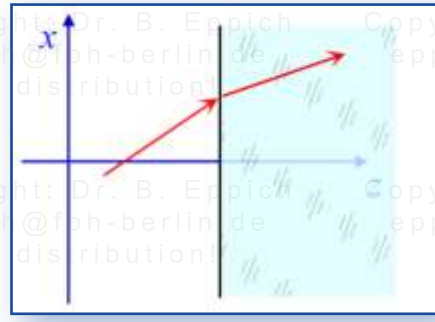
(Paraxial approximation!)

$$\begin{pmatrix} x_2 \\ u_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ u_1 \end{pmatrix}$$

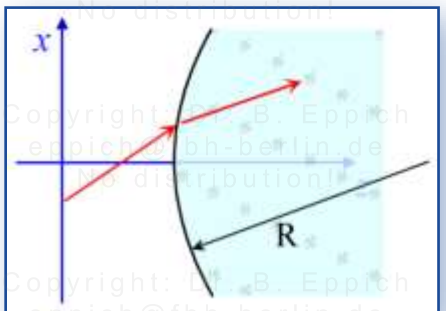
Geometrical matrix optics, one-dimensional systems



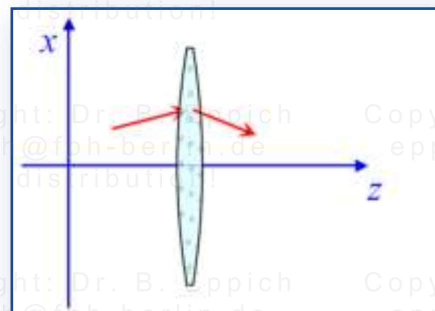
$$\mathbf{S} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$



$$\mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{pmatrix}$$



$$\mathbf{S} = \begin{pmatrix} 1 & 0 \\ \frac{n_1 - n_2}{n_2 \cdot R} & \frac{n_1}{n_2} \end{pmatrix}$$



$$\mathbf{S} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

...and others.

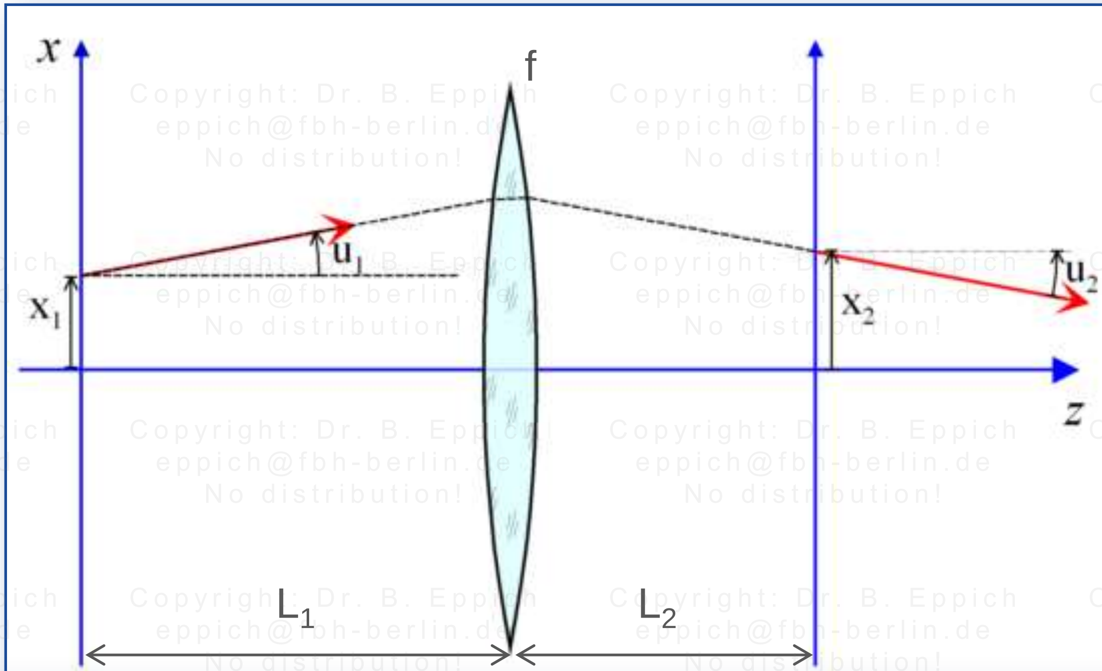
Restriction:

$$\det(\mathbf{S}) = AD - BC = \frac{n_1}{n_2}$$

Geometrical matrix optics, one-dimensional systems

Cascading systems: $\mathbf{S}_{tot} = \mathbf{S}_N \cdot \ldots \cdot \mathbf{S}_2 \cdot \mathbf{S}_1 \cdot \mathbf{S}_0$

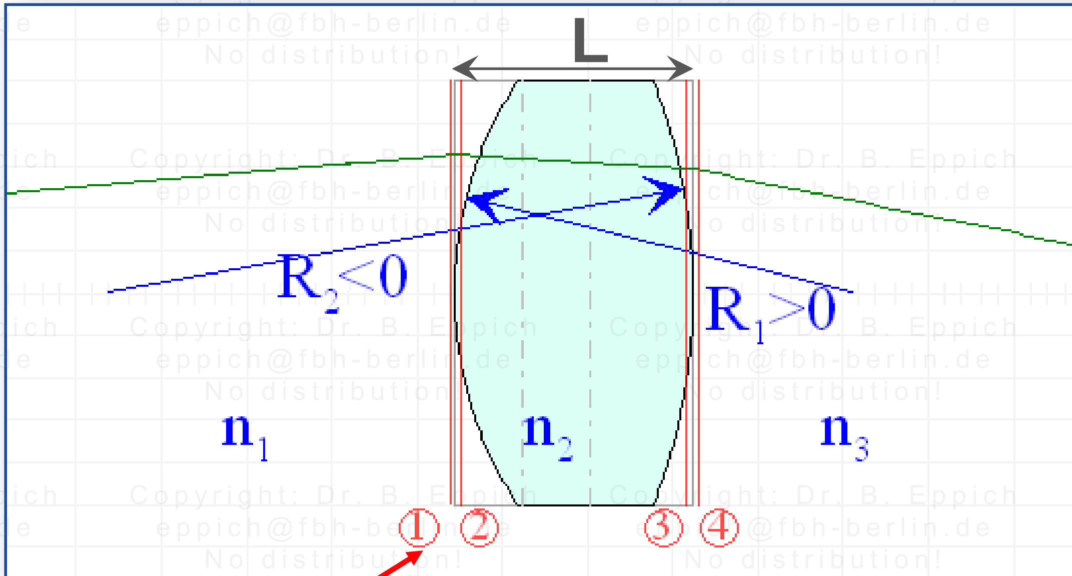
Example:

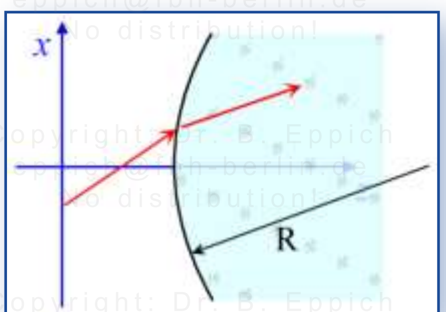


$$\mathbf{S}_{tot} = \begin{pmatrix} 1 & L_2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{L_1}{f} & L_1 + L_2 - \frac{L_1 L_2}{f} \\ -\frac{1}{f} & 1-\frac{L_2}{f} \end{pmatrix}$$

Geometrical matrix optics, one-dimensional systems

Thick Lens

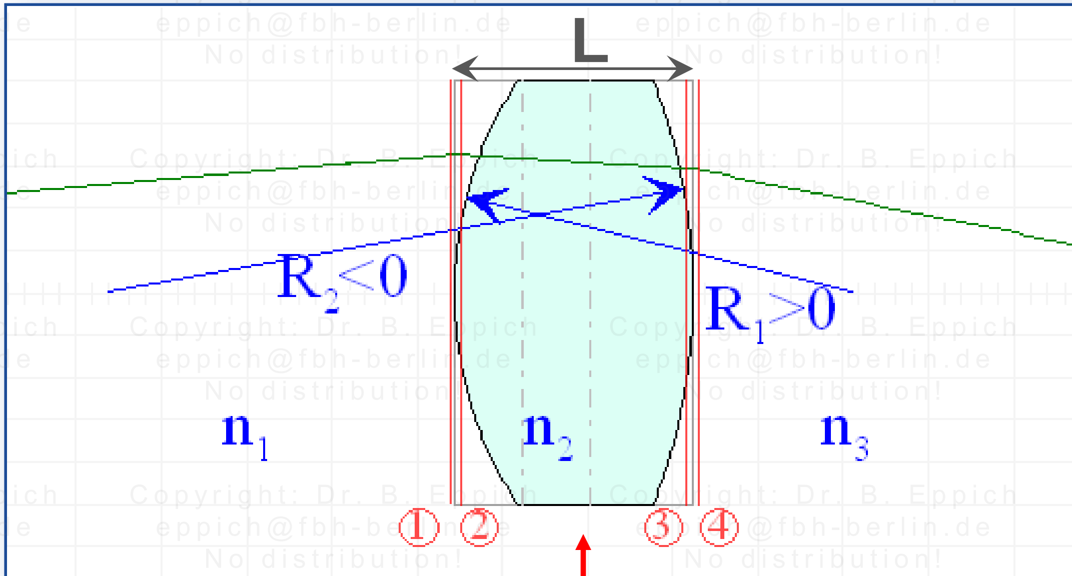


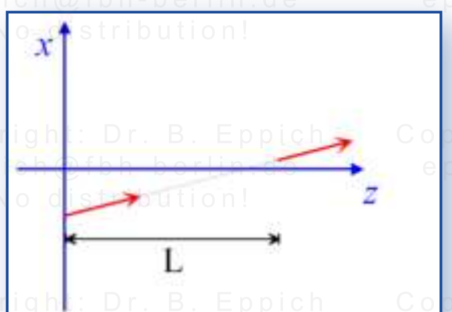


$$\mathbf{S} = \begin{pmatrix} 1 & 0 \\ \dfrac{n_1 - n_2}{n_2 \cdot R_1} & \dfrac{n_1}{n_2} \end{pmatrix}$$

Geometrical matrix optics, one-dimensional systems

Thick Lens

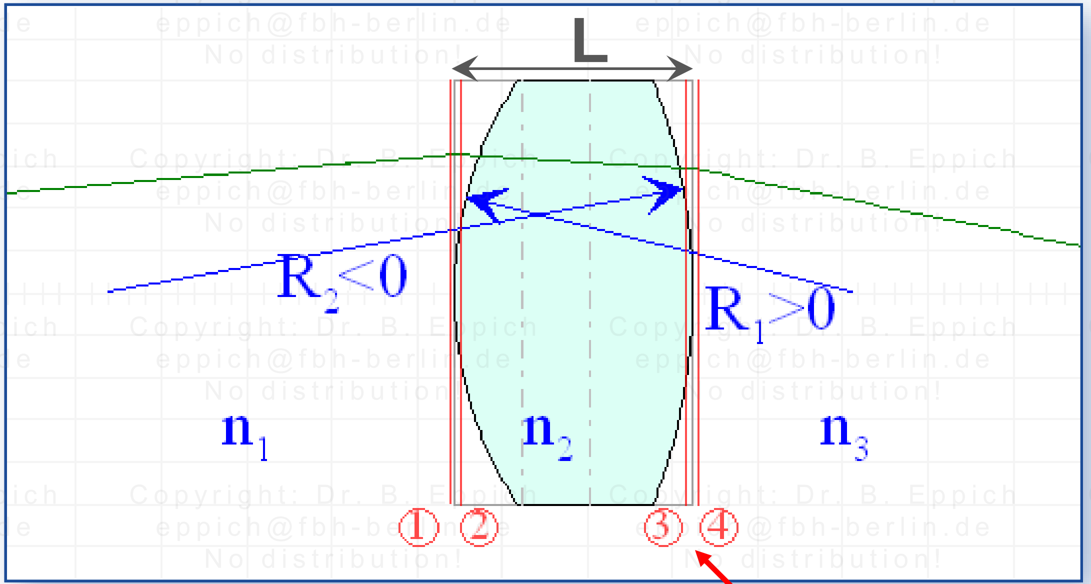


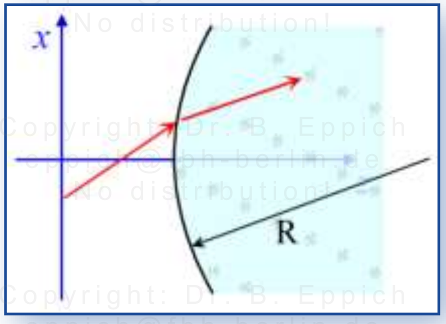


$$\mathbf{S} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

Geometrical matrix optics, one-dimensional systems

Thick Lens

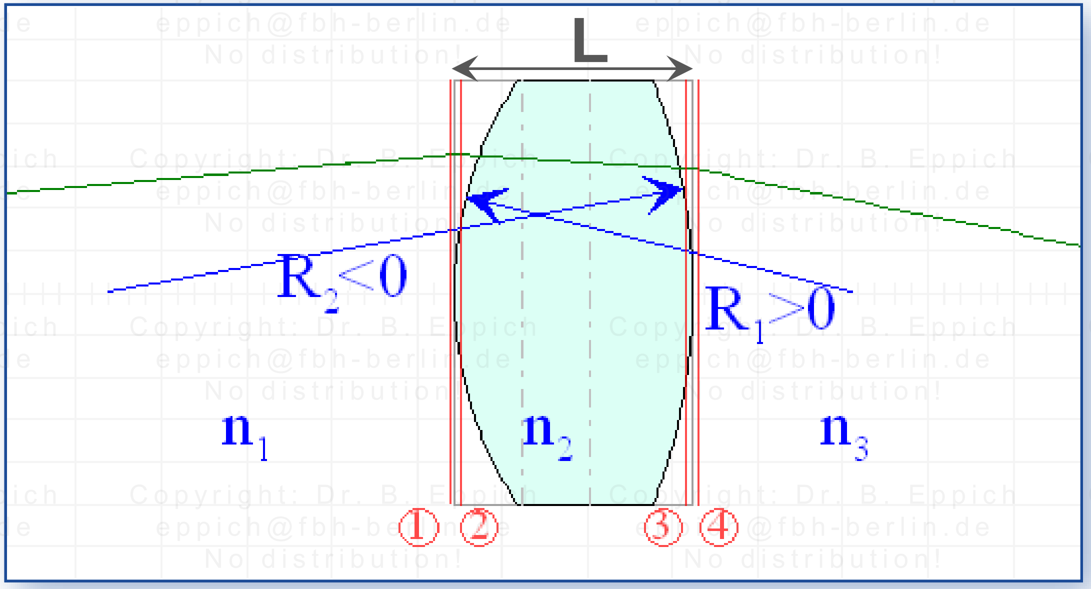




$$S = \begin{pmatrix} 1 & 0 \\ \dfrac{n_2 - n_3}{n_3 \cdot R_2} & \dfrac{n_2}{n_3} \end{pmatrix}$$

Geometrical matrix optics, one-dimensional systems

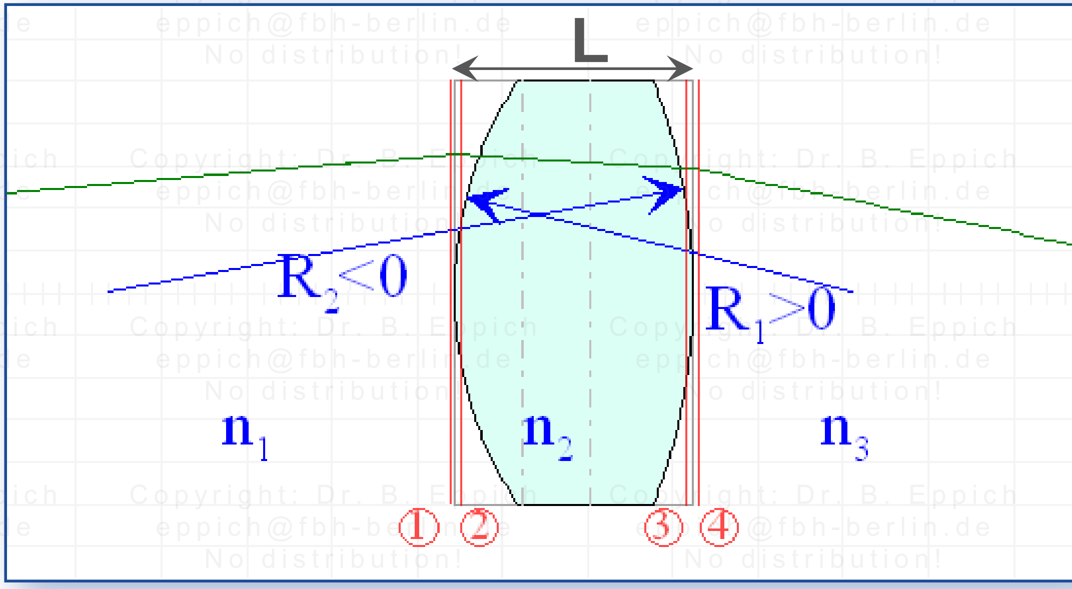
Thick Lens



$$\mathbf{S}=\begin{pmatrix}1 & 0\\ \dfrac{n_2-n_3}{n_3\cdot R_2} & \dfrac{n_2}{n_3}\end{pmatrix}\cdot\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ \dfrac{n_1-n_2}{n_2\cdot R_1} & \dfrac{n_1}{n_2}\end{pmatrix}$$

Geometrical matrix optics, one-dimensional systems

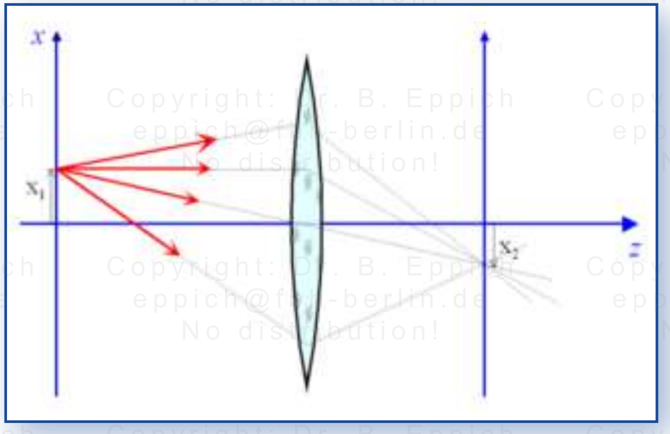
Thick Lens



$$\mathbf{S}=\begin{pmatrix} 1+\dfrac{1-n}{n}\dfrac{L}{R_1} & \dfrac{L}{n} \\ (n-1)\left(\dfrac{1}{R_1}-\dfrac{1}{R_2}\right)-(n-1)^2\dfrac{L}{nR_1R_2} & 1+\dfrac{1-n}{n}\dfrac{L}{R_2} \end{pmatrix}$$

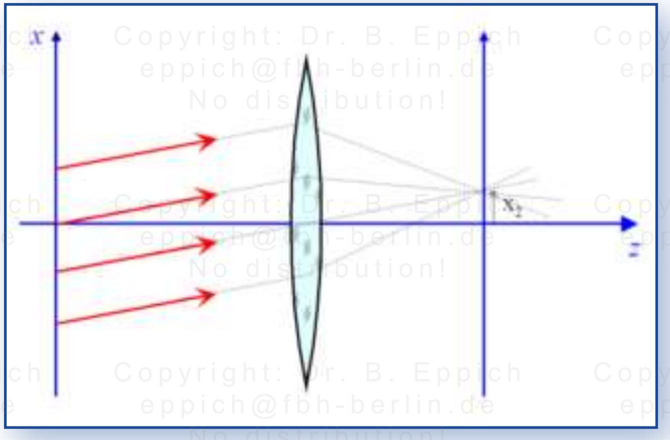
Geometrical matrix optics, one-dimensional systems

Imaging



$$\mathbf{S} = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$$

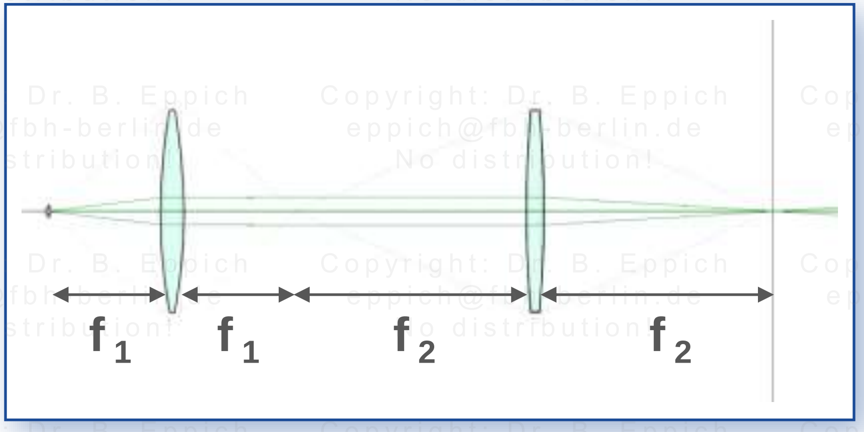
Fourier transformer, far field imaging



$$\mathbf{S} = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$$

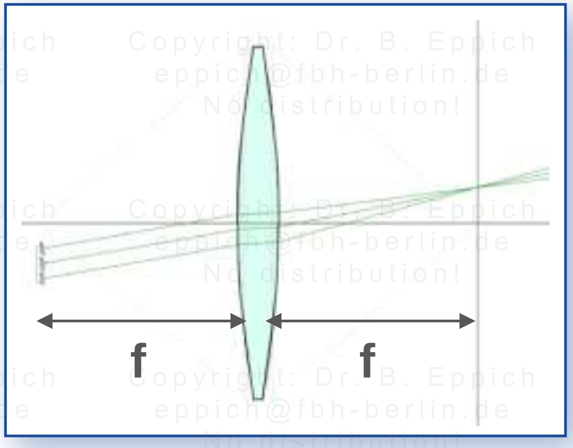
Geometrical matrix optics, one-dimensional systems

Telescope, magnifier



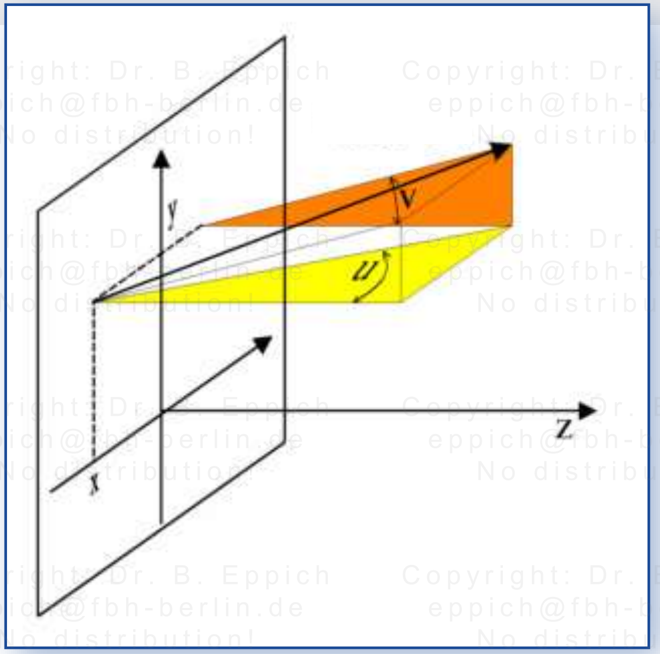
$$\mathbf{S} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

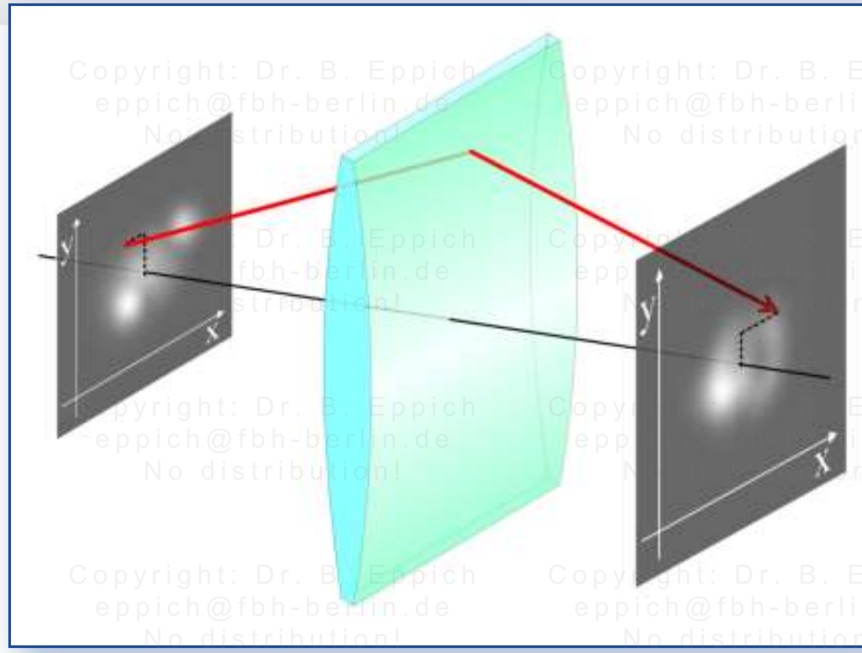
Fourier transformer, far field imaging



$$\mathbf{S} = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

Geometrical matrix optics, two-dimensional systems

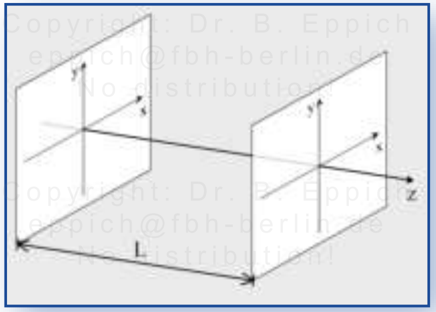




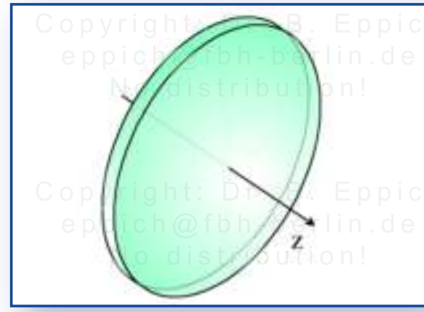
- **two spatial coordinates**
- **two angular coordinates**

$$\begin{pmatrix} x_2 \\ y_2 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} A_{xx} & A_{xy} & B_{xx} & B_{xy} \\ A_{yx} & A_{yy} & B_{yx} & B_{yy} \\ C_{xx} & C_{xy} & D_{xx} & D_{xy} \\ C_{yx} & C_{yy} & D_{yx} & D_{yy} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ y_1 \\ u_1 \\ v_1 \end{pmatrix}$$

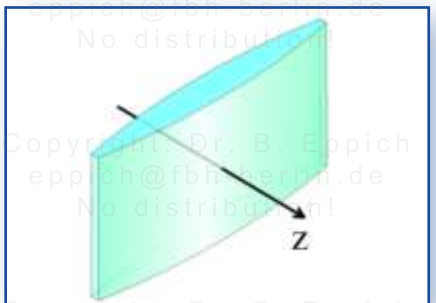
Geometrical matrix optics, two-dimensional systems



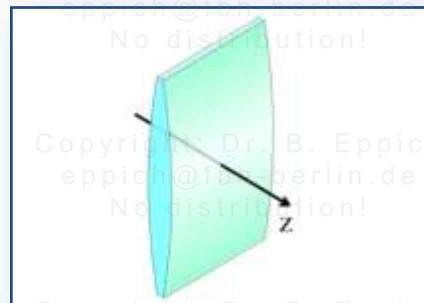
$$\mathbf{S} = \begin{pmatrix} 1 & 0 & L & 0 \\ 0 & 1 & 0 & L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



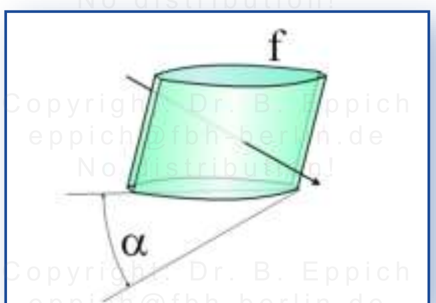
$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1/f & 0 & 1 & 0 \\ 0 & -1/f & 0 & 1 \end{pmatrix}$$



$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1/f & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1/f & 0 & 1 \end{pmatrix}$$

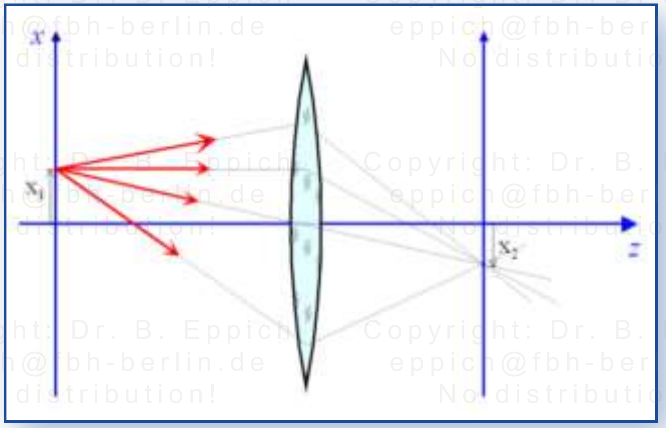


$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\cos\alpha^2/f & \cos\alpha\sin\alpha/f & 1 & 0 \\ \cos\alpha\sin\alpha/f & -\sin\alpha^2/f & 0 & 1 \end{pmatrix}$$

... and others.

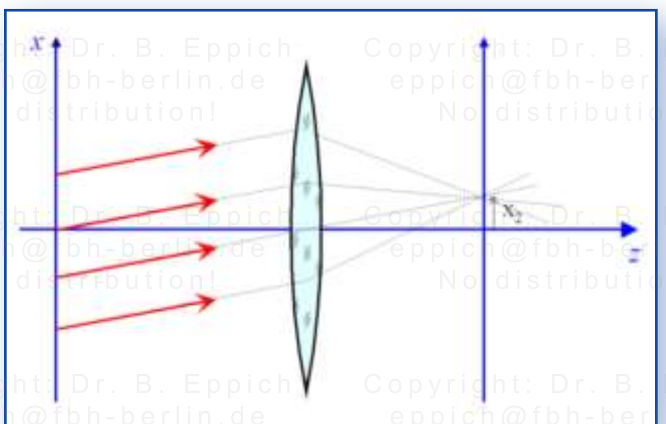
Geometrical matrix optics, two-dimensional systems

Imaging



$$\mathbf{S} = \begin{pmatrix} A_{xx} & 0 & 0 & 0 \\ 0 & A_{yy} & 0 & 0 \\ C_{xx} & C_{xy} & D_{xx} & D_{xy} \\ C_{yx} & C_{yy} & D_{yx} & D_{yy} \end{pmatrix}$$

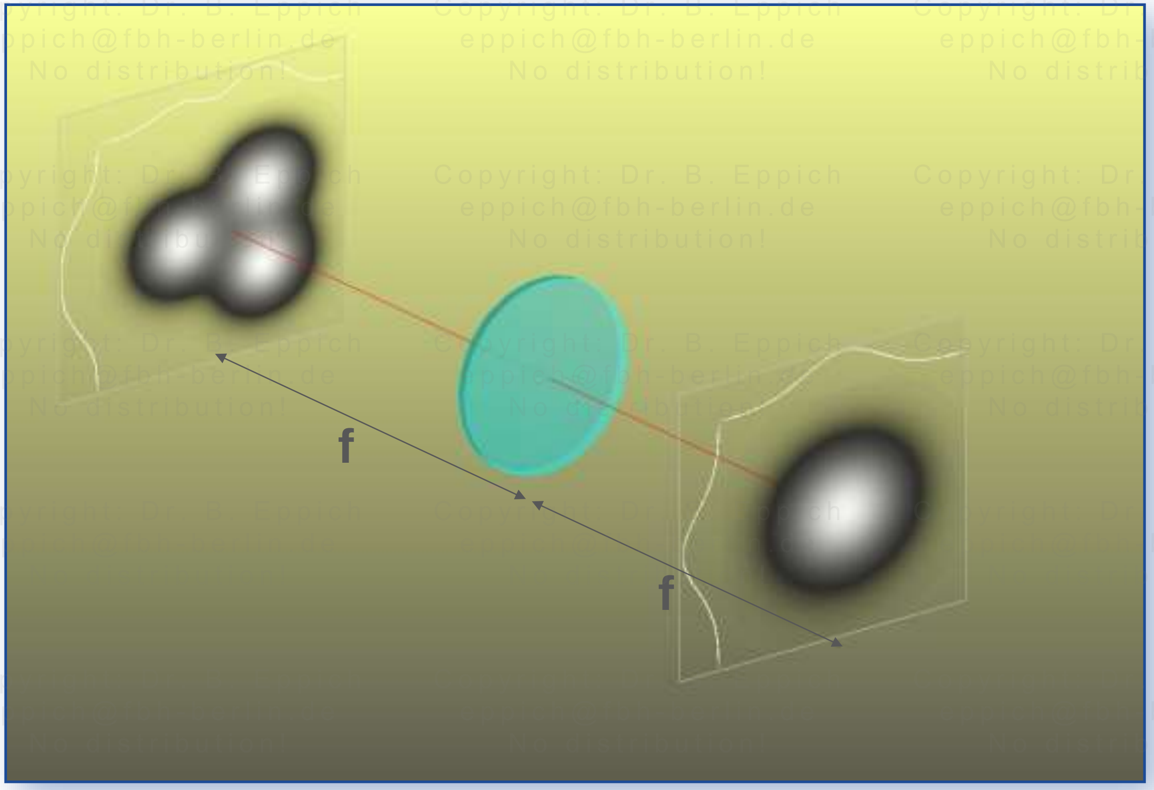
Fourier transformer, far field imaging



$$\mathbf{S} = \begin{pmatrix} 0 & 0 & B_{xx} & 0 \\ 0 & 0 & 0 & B_{yy} \\ C_{xx} & C_{xy} & D_{xx} & D_{xy} \\ C_{yx} & C_{yy} & D_{yx} & D_{yy} \end{pmatrix}$$

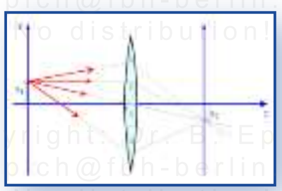
Geometrical matrix optics, two-dimensional systems

Anamorphotic systems

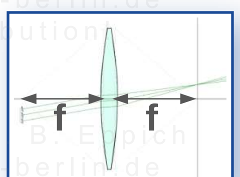


$$\mathbf{S}=\begin{pmatrix} 0 & 0 & f & 0 \\ 0 & 0 & 0 & f \\ -1/f & 0 & 0 & 0 \\ 0 & -1/f & 0 & 0 \end{pmatrix}$$

Fourier transformer, far field imaging



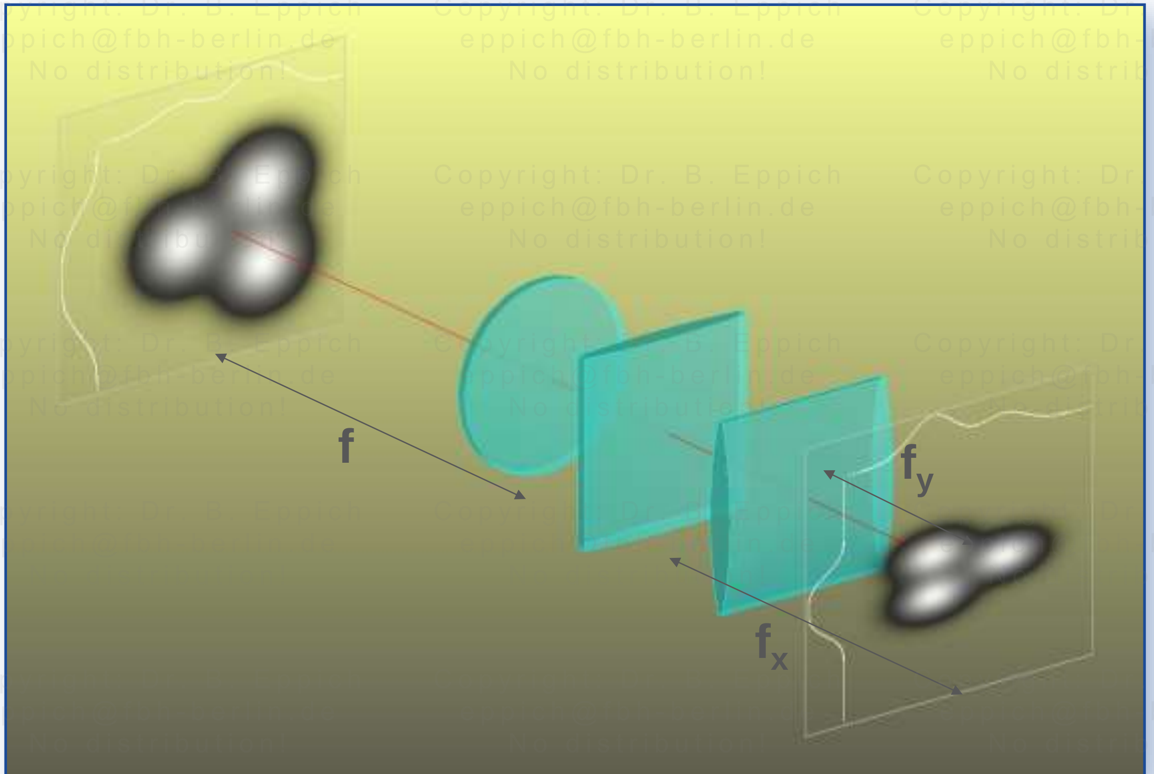
$$\mathbf{S}=\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$$



$$\mathbf{S}=\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

Geometrical matrix optics, two-dimensional systems

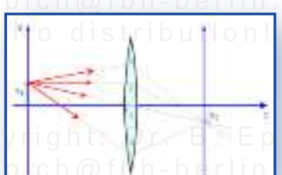
Anamorphotic systems



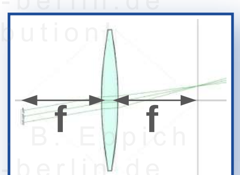
$$\mathbf{S} = \begin{pmatrix} A_{xx} & 0 & 0 & 0 \\ 0 & A_{yy} & 0 & 0 \\ C_{xx} & 0 & D_{xx} & 0 \\ 0 & C_{yy} & 0 & D_{yy} \end{pmatrix}$$

$$A_{xx} \neq A_{yy}$$

Anamorphotic relay imaging



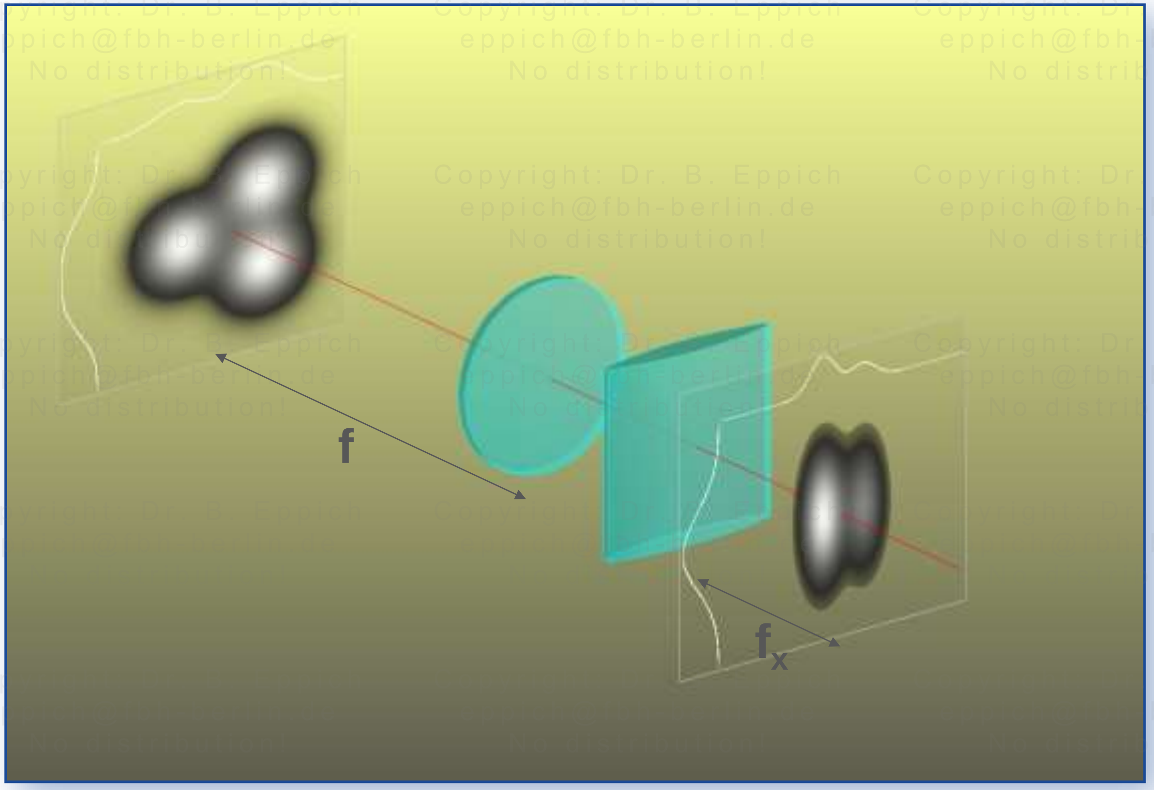
$$\mathbf{S} = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$$



$$\mathbf{S} = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

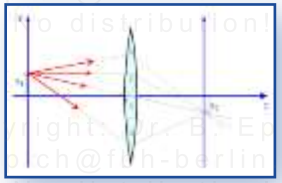
Geometrical matrix optics, two-dimensional systems

Anamorphotic systems

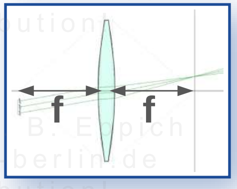


$$\mathbf{S} = \begin{pmatrix} A_{xx} & 0 & 0 & 0 \\ 0 & 0 & 0 & f \\ C_{xx} & 0 & D_{xx} & 0 \\ 0 & -1/f & 0 & 0 \end{pmatrix}$$

Horizontal imaging, vertical Fourier transform



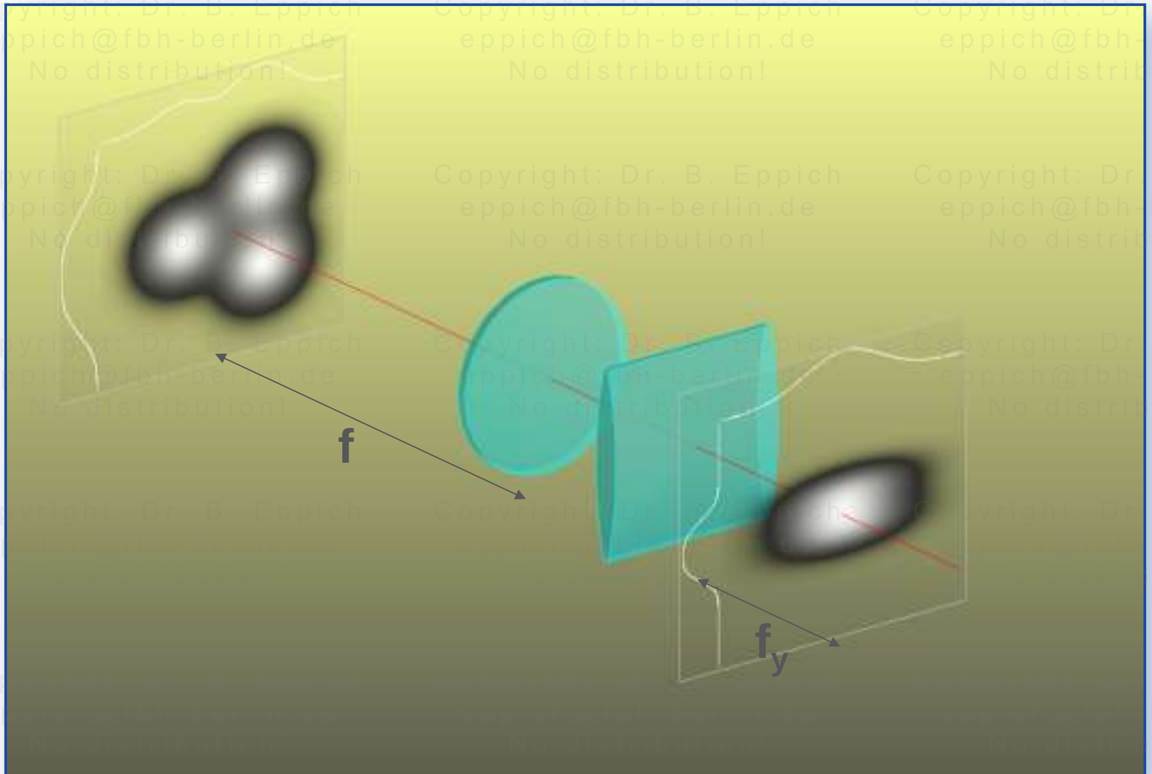
$$\mathbf{S} = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$$



$$\mathbf{S} = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

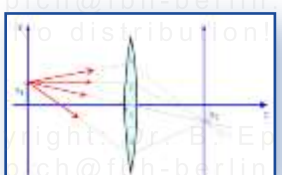
Geometrical matrix optics, two-dimensional systems

Anamorphotic systems

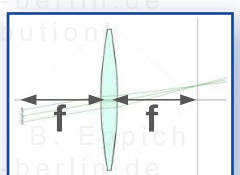


$$\mathbf{S} = \begin{pmatrix} 0 & 0 & f & 0 \\ 0 & A_{yy} & 0 & 0 \\ -1/f & 0 & 0 & 0 \\ 0 & C_{yy} & 0 & D_{yy} \end{pmatrix}$$

Horizontal Fourier transform, vertical imaging



$$\mathbf{S} = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$$



$$\mathbf{S} = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

Geometrical matrix optics, two-dimensional systems

$$\mathbf{S} = \begin{pmatrix} A_{xx} & A_{xy} & B_{xx} & B_{xy} \\ A_{yx} & A_{yy} & B_{yx} & B_{yy} \\ C_{xx} & C_{xy} & D_{xx} & D_{xy} \\ C_{yx} & C_{yy} & D_{yx} & D_{yy} \end{pmatrix}$$

Restriction:

$$\mathbf{S} \cdot \mathbf{J} \cdot \mathbf{S}^{\mathrm{T}} = \mathbf{J} \quad \text{with} \quad \mathbf{J} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

(symplecticity)

→ **only ten independent parameters!**

Separable optical systems

$$\mathbf{S} = \begin{pmatrix} A_{xx} & 0 & B_{xx} & 0 \\ 0 & A_{yy} & 0 & B_{yy} \\ C_{xx} & 0 & D_{xx} & 0 \\ 0 & C_{yy} & 0 & D_{yy} \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ y_2 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} A_{xx} & 0 & B_{xx} & 0 \\ 0 & A_{yy} & 0 & B_{yy} \\ C_{xx} & 0 & D_{xx} & 0 \\ 0 & C_{yy} & 0 & D_{yy} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ y_1 \\ u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} A_{xx}\, x_1 + B_{xx}\, u_1 \\ A_{yy}\, y_1 + B_{yy}\, v_1 \\ C_{xx}\, x_1 + D_{xx}\, u_1 \\ C_{yy}\, y_1 + D_{yy}\, v_1 \end{pmatrix}$$

Separable optical systems

$$\mathbf{S} = \begin{pmatrix} A_{xx} & 0 & B_{xx} & 0 \\ 0 & A_{yy} & 0 & B_{yy} \\ C_{xx} & 0 & D_{xx} & 0 \\ 0 & C_{yy} & 0 & D_{yy} \end{pmatrix}$$

$$\mathbf{S}_x = \begin{pmatrix} A_{xx} & B_{xx} \\ C_{xx} & D_{xx} \end{pmatrix}$$

$$\mathbf{S}_y = \begin{pmatrix} A_{yy} & B_{yy} \\ C_{yy} & D_{yy} \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ y_2 \\ u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} A_{xx} & 0 & B_{xx} & 0 \\ 0 & A_{yy} & 0 & B_{yy} \\ C_{xx} & 0 & D_{xx} & 0 \\ 0 & C_{yy} & 0 & D_{yy} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ y_1 \\ u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} A_{xx}\, x_1 + B_{xx}\, u_1 \\ A_{yy}\, y_1 + B_{yy}\, v_1 \\ C_{xx}\, x_1 + D_{xx}\, u_1 \\ C_{yy}\, y_1 + D_{yy}\, v_1 \end{pmatrix}$$

Separable optical systems

$$\mathbf{S}=\begin{pmatrix} A_{xx} & 0 & B_{xx} & 0 \\ 0 & A_{yy} & 0 & B_{yy} \\ C_{xx} & 0 & D_{xx} & 0 \\ 0 & C_{yy} & 0 & D_{yy} \end{pmatrix}$$

$$\mathbf{S}_x=\begin{pmatrix} A_{xx} & B_{xx} \\ C_{xx} & D_{xx} \end{pmatrix}$$

$$\mathbf{S}_y=\begin{pmatrix} A_{yy} & B_{yy} \\ C_{yy} & D_{yy} \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ u_2 \end{pmatrix}=\begin{pmatrix} A_{xx} & B_{xx} \\ C_{xx} & D_{xx} \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ u_1 \end{pmatrix}=\begin{pmatrix} A_{xx}\,x_1+B_{xx}\,u_1 \\ C_{xx}\,x_1+D_{xx}\,u_1 \end{pmatrix}$$

$$\begin{pmatrix} y_2 \\ v_2 \end{pmatrix}=\begin{pmatrix} A_{yy} & B_{yy} \\ C_{yy} & D_{yy} \end{pmatrix}\cdot\begin{pmatrix} y_1 \\ v_1 \end{pmatrix}=\begin{pmatrix} A_{yy}\,y_1+B_{yy}\,v_1 \\ C_{yy}\,y_1+D_{yy}\,v_1 \end{pmatrix}$$

Separable optical systems

$$\mathbf{S} = \begin{pmatrix} A_{xx} & 0 & B_{xx} & 0 \\ 0 & A_{yy} & 0 & B_{yy} \\ C_{xx} & 0 & D_{xx} & 0 \\ 0 & C_{yy} & 0 & D_{yy} \end{pmatrix}$$

$$\mathbf{S}_x = \begin{pmatrix} A_{xx} & B_{xx} \\ C_{xx} & D_{xx} \end{pmatrix}$$

$$\mathbf{S}_y = \begin{pmatrix} A_{yy} & B_{yy} \\ C_{yy} & D_{yy} \end{pmatrix}$$

$$\mathbf{S} \cdot \mathbf{J} \cdot \mathbf{S}^{\mathrm{T}} = \mathbf{J}$$

→ $$A_{xx} D_{xx} - B_{xx} C_{xx} = 1$$

→ $$A_{yy} D_{yy} - B_{yy} C_{yy} = 1$$

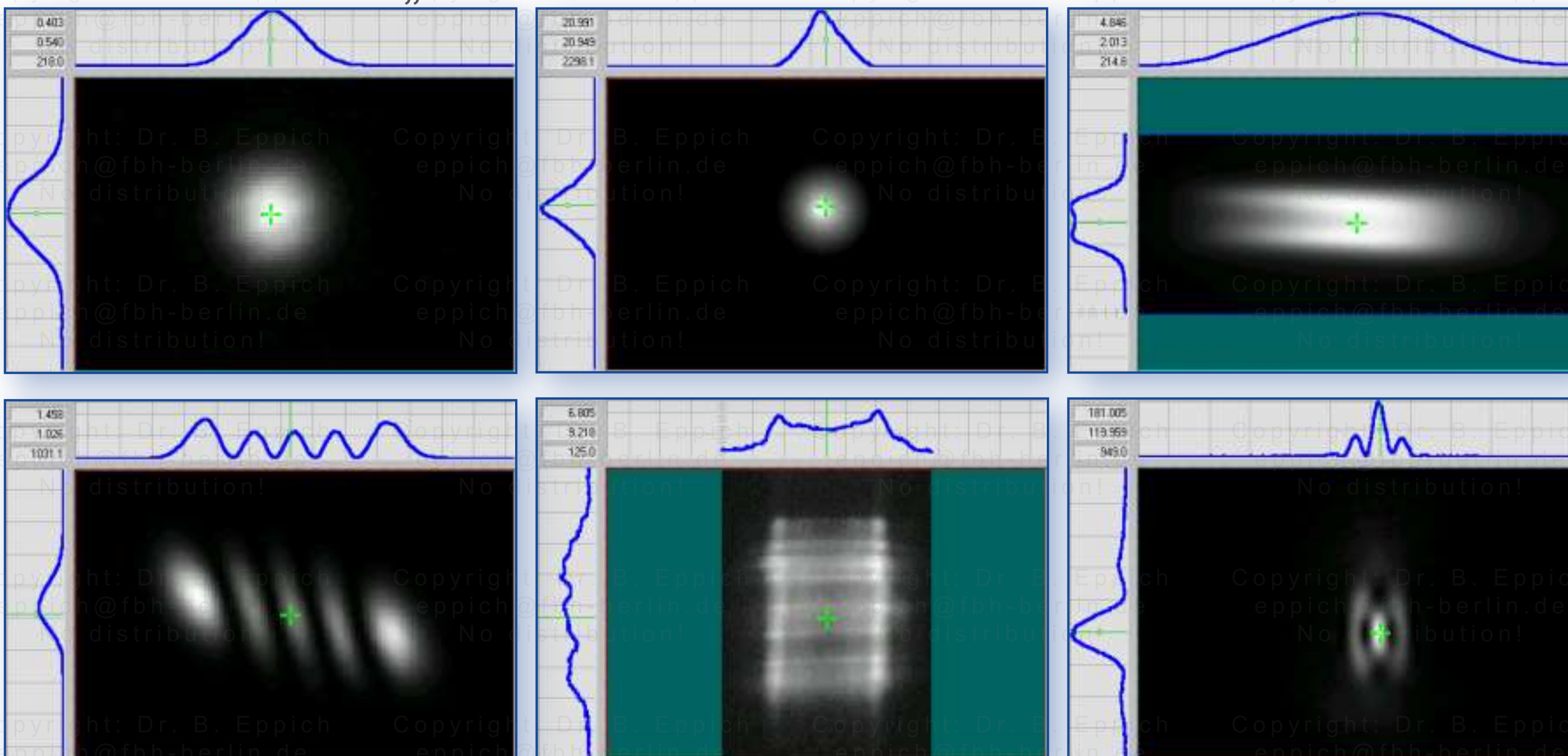
Content

- Motivation
- ISO standard
- Beams and optical systems
- Beam diameter definitions
- Beam classification
- The ten second order moments
- Measurement of the ten second order moments

„Conventional“ beam characterization

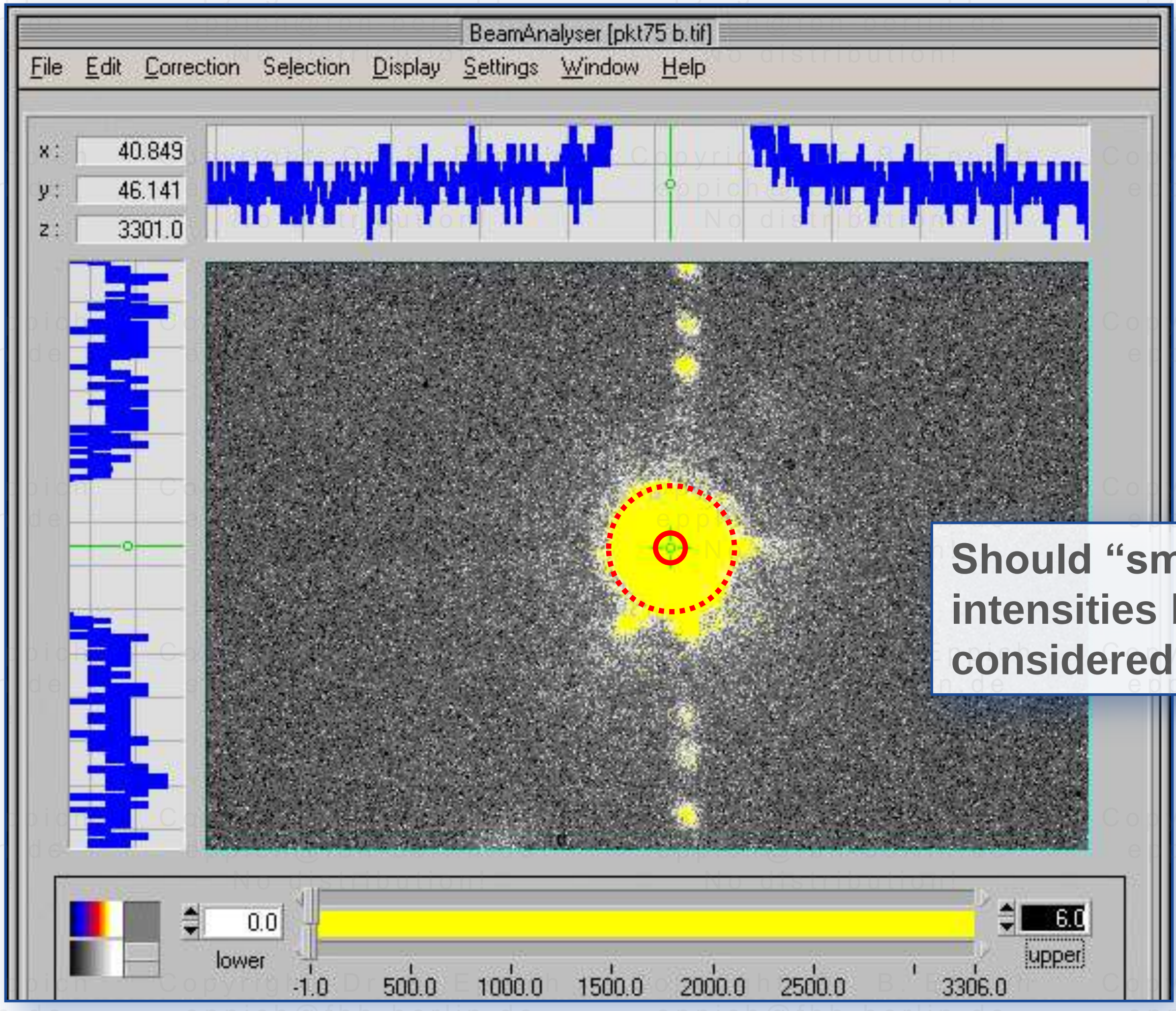
- **Restriction to transversal beam profile extent**
- **Neglecting the beam profile structure**

How to define a beam „diameter“ ?



Beam profile example

Low relative intensities



Should "small" intensities be considered?

Requirements

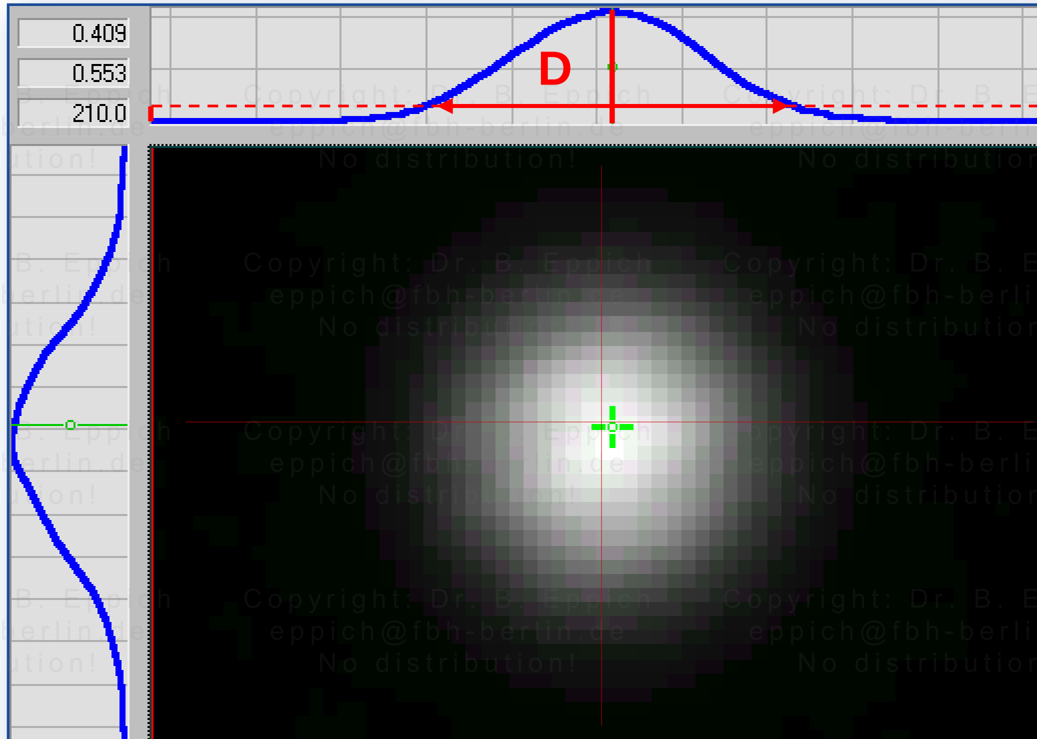
Requirements for a „suitable“ beam diameter definition:

- **Meaningful**
- **Useful for applications**
- **Accurately measurable**
- **Simple propagation law**

„Historical“ beam diameter definitions

1/e²-intensity-threshold, for Gaussian beams: definition

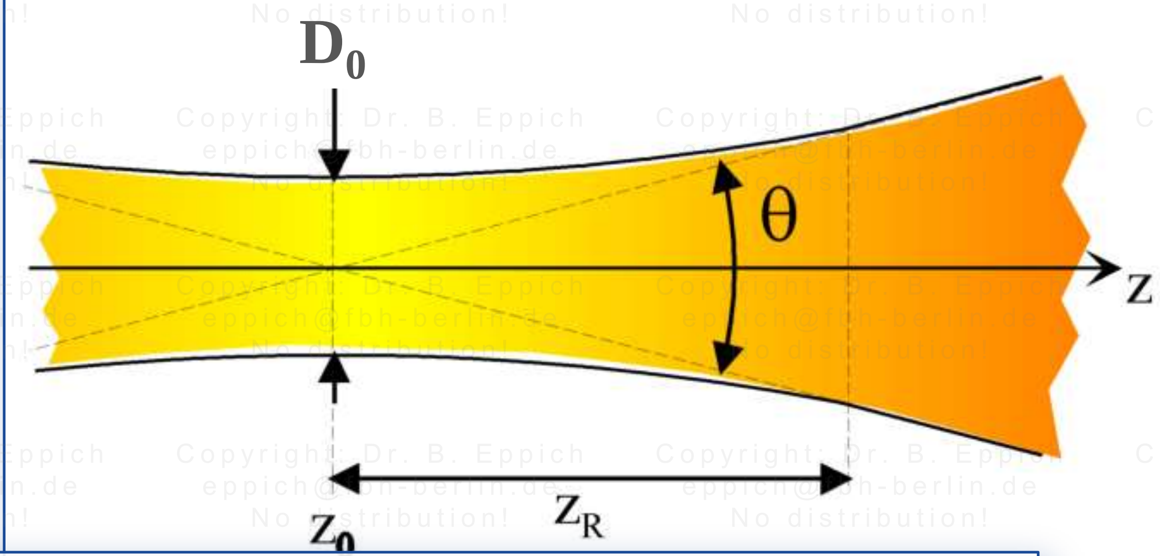
$$I(x,y) = I_0\, e^{-2\frac{x^2+y^2}{w^2}}$$



$$D = 2\,w$$

„Historical“ beam diameter definitions

$1/e^2$-intensity-threshold, for Gaussian beams: free space propagation



Valid only for Gaussian beams!

$$D(z) = D_0 \sqrt{1 + \left(\frac{z - z_0}{z_R}\right)^2} = \sqrt{D_0^2 + \theta^2 (z - z_0)^2}$$

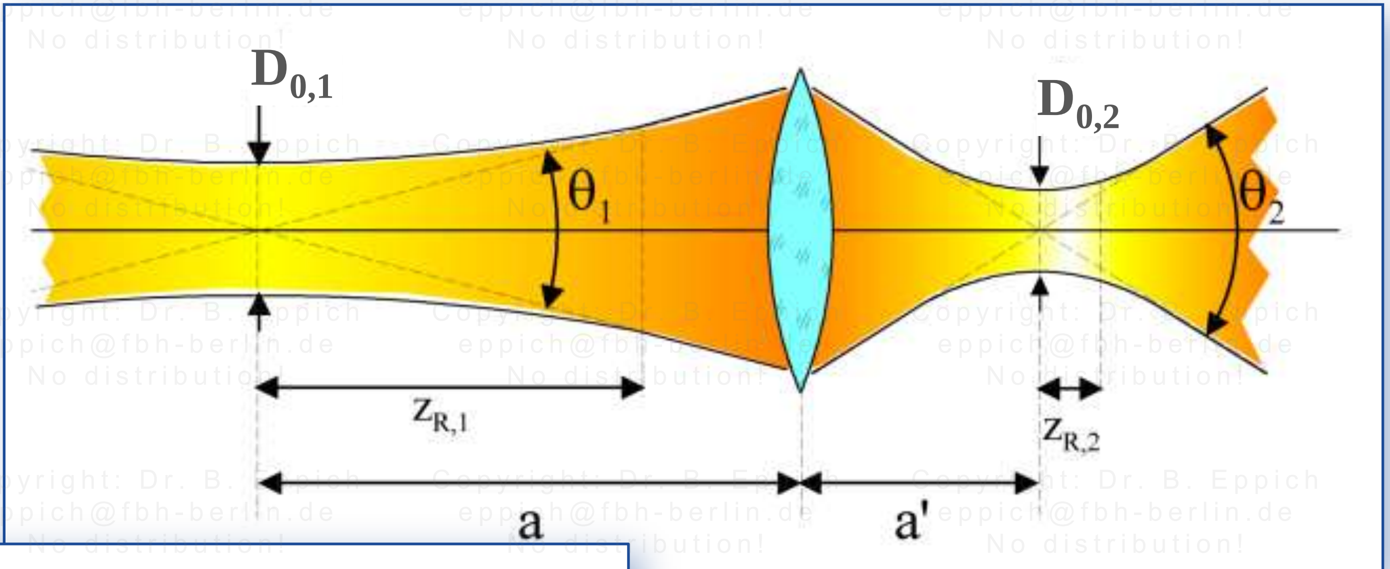
$$\frac{D_0^2}{4\, z_R} = \frac{\lambda}{\pi} \quad \Leftrightarrow \quad z_R = \frac{\pi D_0^2}{4\lambda}$$

$$\theta = \frac{4\lambda}{\pi D_0} \quad \Leftrightarrow \quad \frac{D_0\, \theta}{4} = \frac{\lambda}{\pi}$$

„Historical“ beam diameter definitions

1/e²-intensity-threshold, for Gaussian beams: re-focussing with a single lens

Valid only for Gaussian beams!



$$V = \frac{f}{\sqrt{z_{R,1}^2 + (a-f)^2}}$$

$$D_{0,2} = V \cdot D_{0,1}$$

$$z_{R,2} = V^2 \cdot z_{R,1}$$

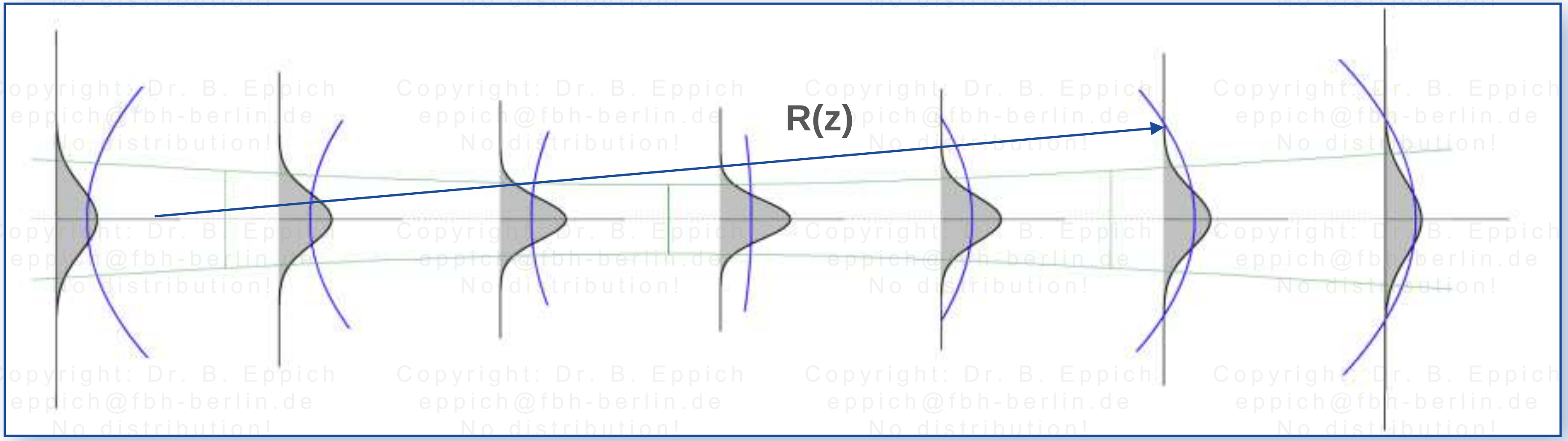
$$a' - f = V^2 \cdot (a-f)$$

$$\theta_2 = \frac{1}{V}\theta_1$$

$$\frac{D_{0,2}\cdot\theta_2}{4} = \frac{D_{0,1}\cdot\theta_1}{4} = \frac{\lambda}{\pi}$$

„Historical“ beam diameter definitions

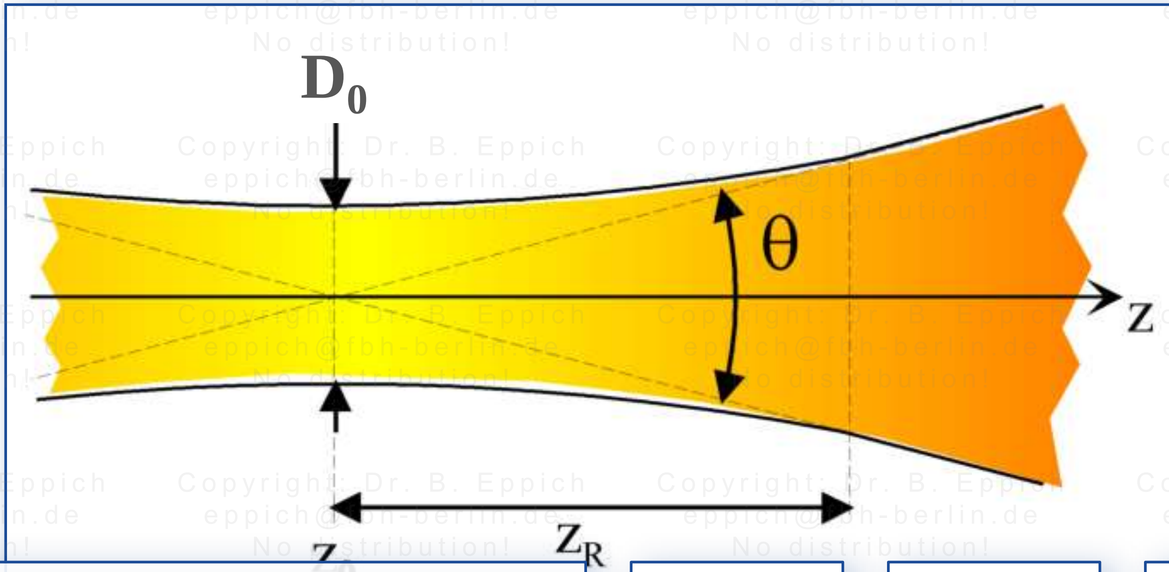
Propagation of phase curvature (Gaussian beams only!)



$$R(z) = z_R \left(\frac{z - z_0}{z_R} + \frac{z_R}{z - z_0} \right)$$

„Historical“ beam diameter definitions

$1/e^2$-intensity-threshold, for Gaussian beams: definition of the q-parameter



Valid only for Gaussian beams!

$$D(z) = D_0 \sqrt{1 + \left(\frac{z - z_0}{z_R} \right)^2} = \sqrt{D_0^2 + \theta^2 (z - z_0)^2}$$

$$\frac{D_0^2}{4 z_R} = \frac{\lambda}{\pi}$$

$$\frac{D_0 \theta}{4} = \frac{\lambda}{\pi}$$

$$R(z) = z_R \left(\frac{z}{z_R} + \frac{z_R}{z} \right)$$

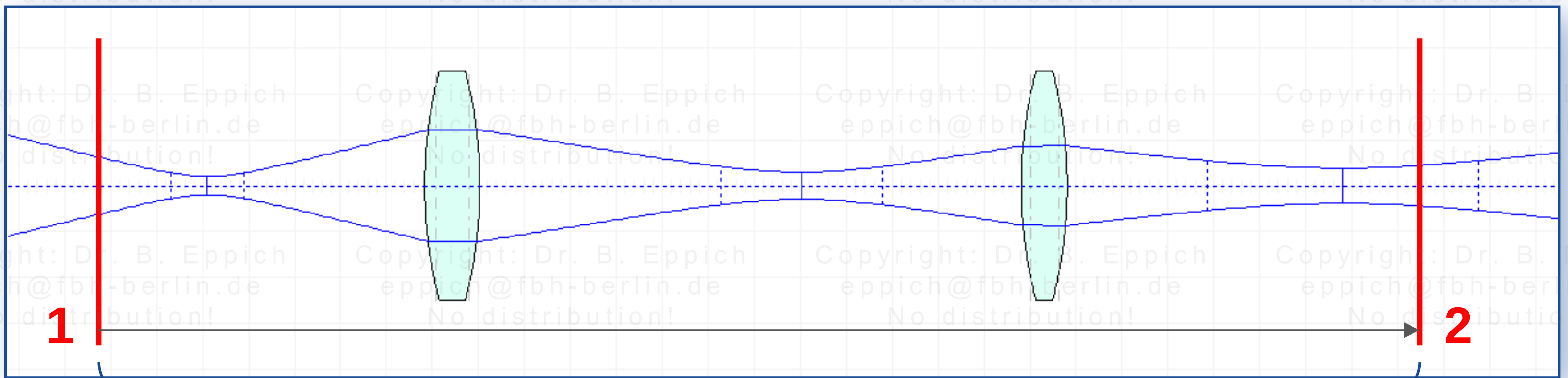
$$q = \Delta z_0 + i\, z_R$$

$$\frac{1}{q} = \frac{1}{R} - i \frac{4 \lambda}{\pi D^2}$$

$$\Delta z_0 = z - z_0$$

„Historical“ beam diameter definitions

$1/e^2$-intensity-threshold, for Gaussian beams: propagation of the q-parameter



$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

q_1 | q_2

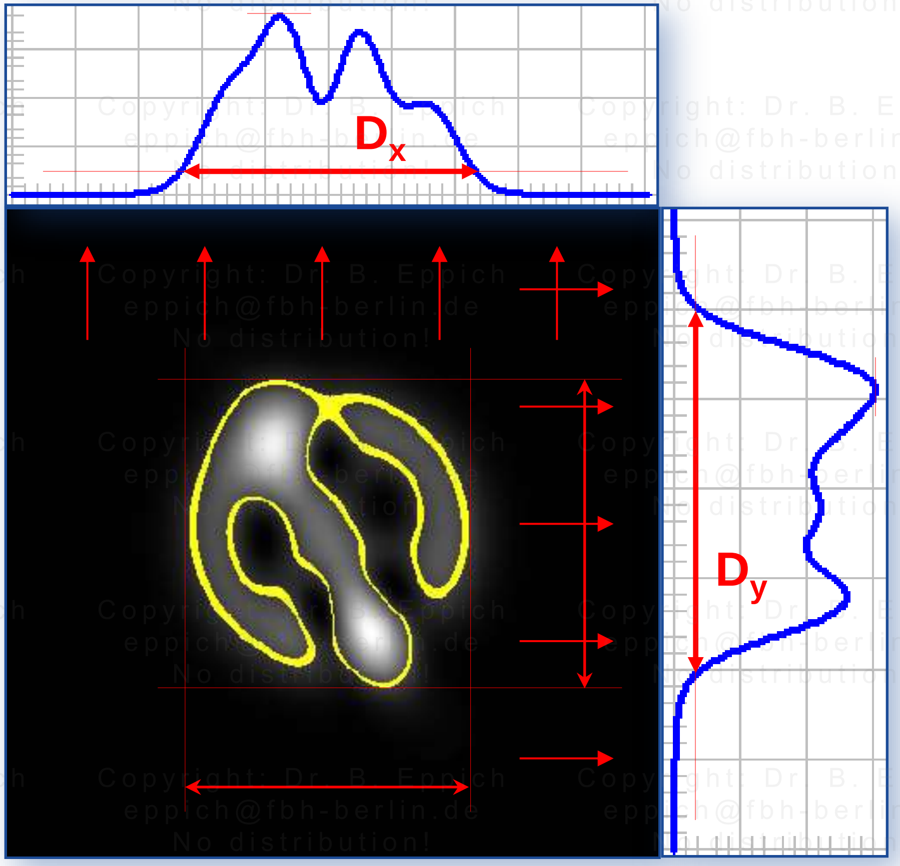
D_1, Δz_1, $z_{R,1}$, R_1

$$q_2 = \frac{A q_1 + B}{C q_1 + D}$$

D_2, Δz_2, $z_{R,2}$, R_2

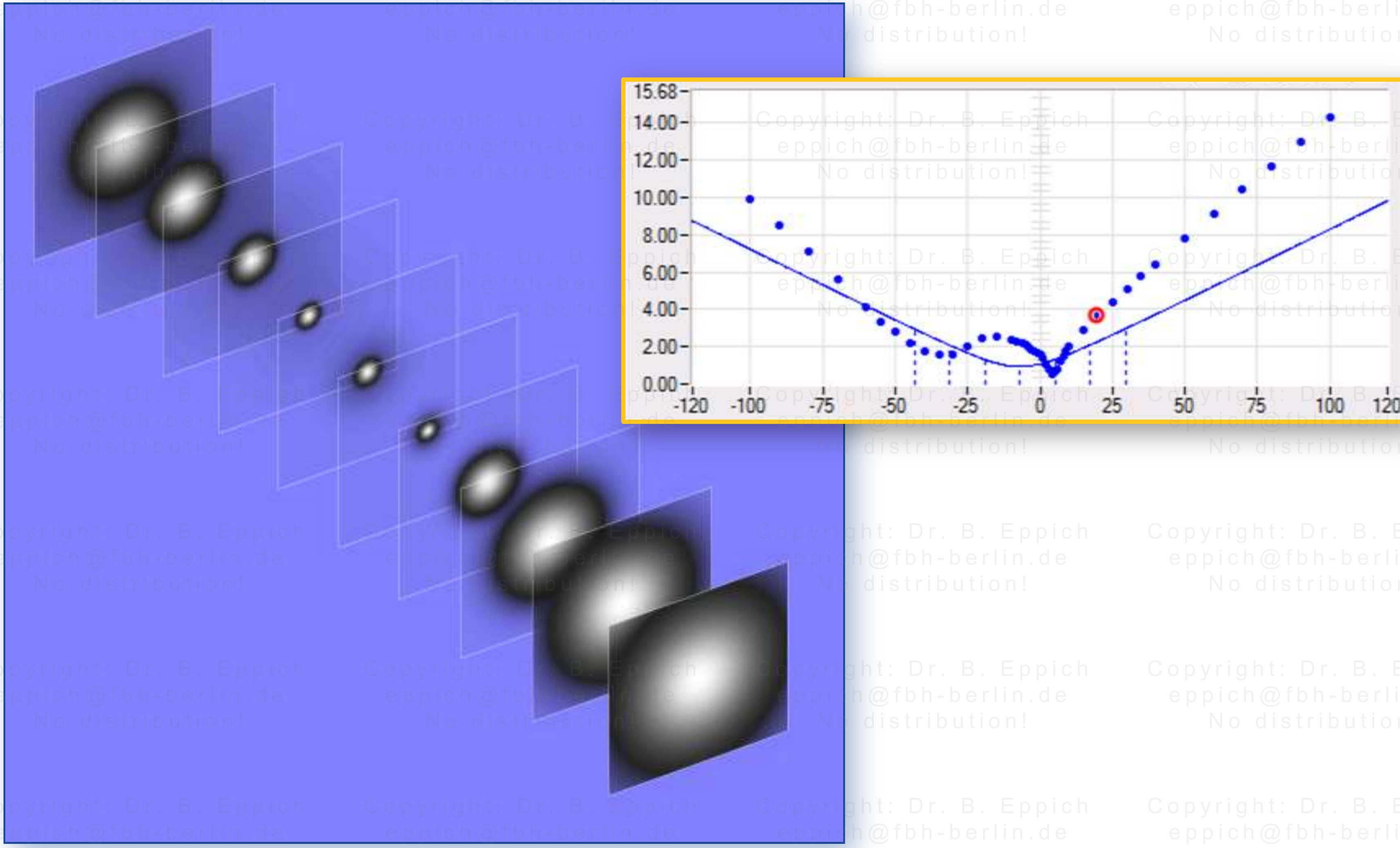
„Historical“ beam diameter definitions

Application of the threshold definition to non-Gaussian beams:



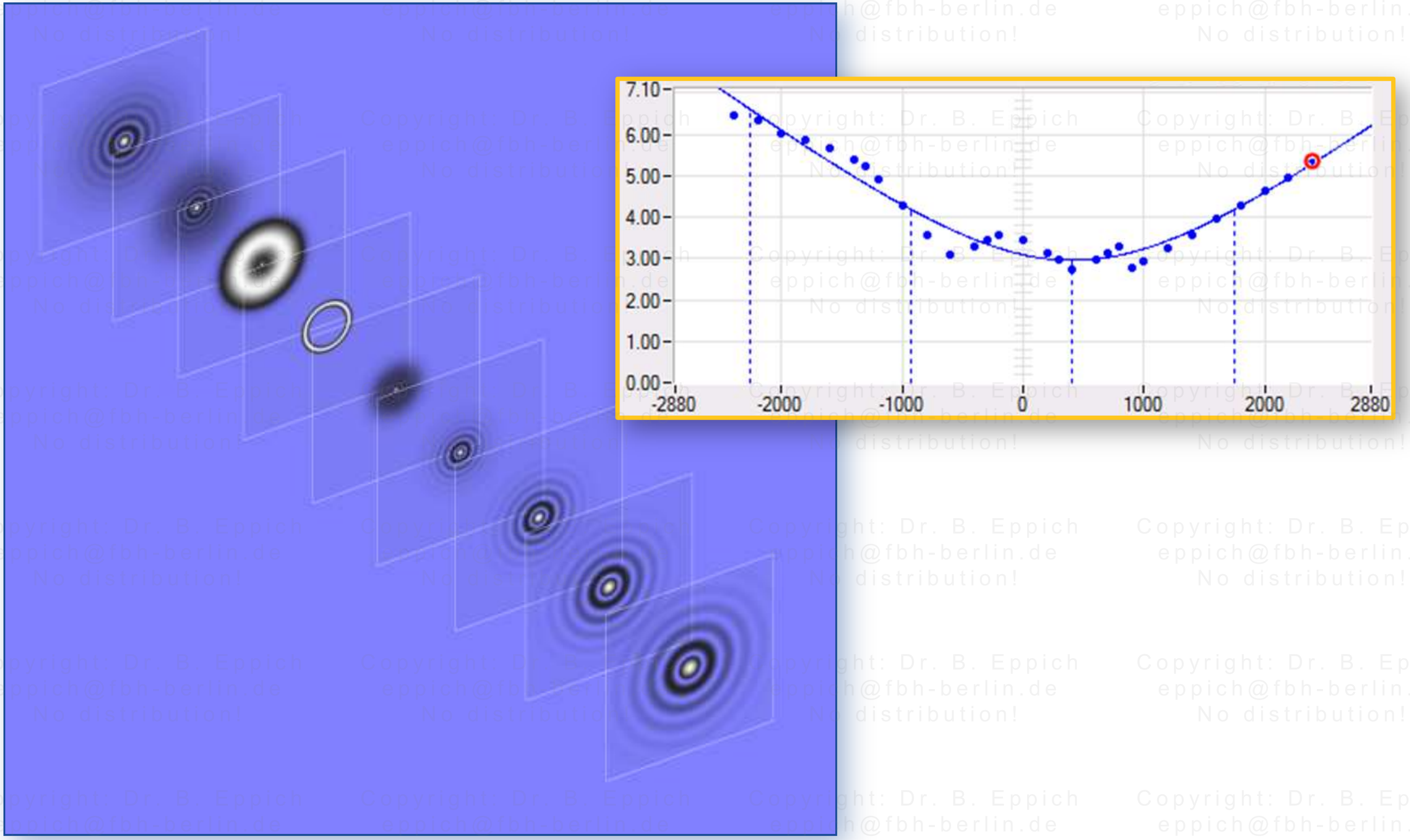
„Historical“ beam diameter definitions

Application of the threshold definition to non-Gaussian beams:



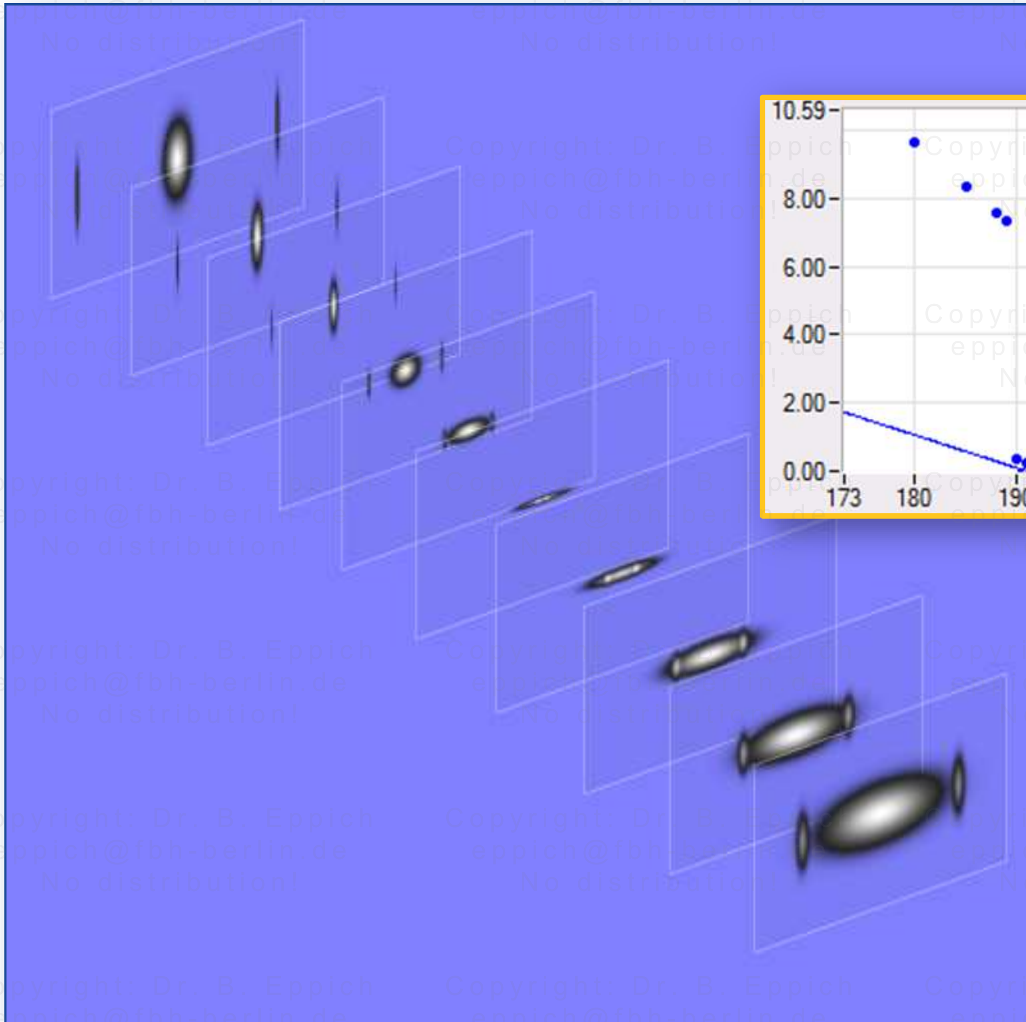
„Historical“ beam diameter definitions

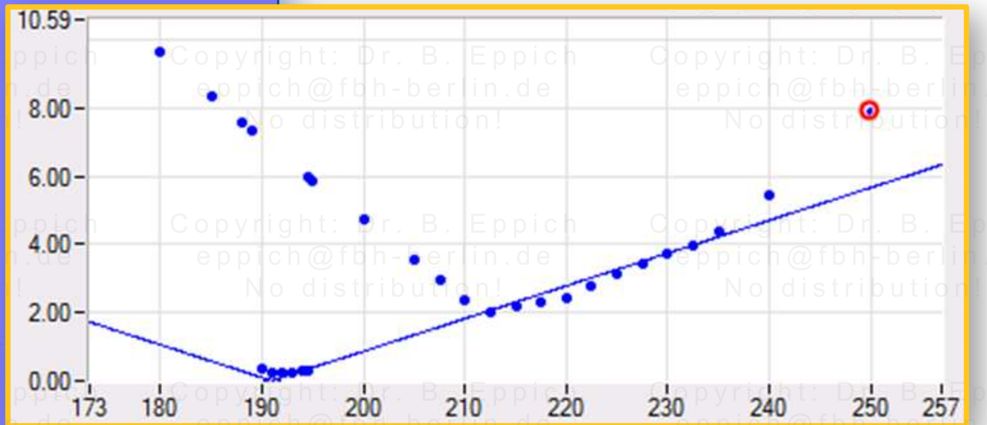
Application of the threshold definition to non-Gaussian beams:



„Historical“ beam diameter definitions

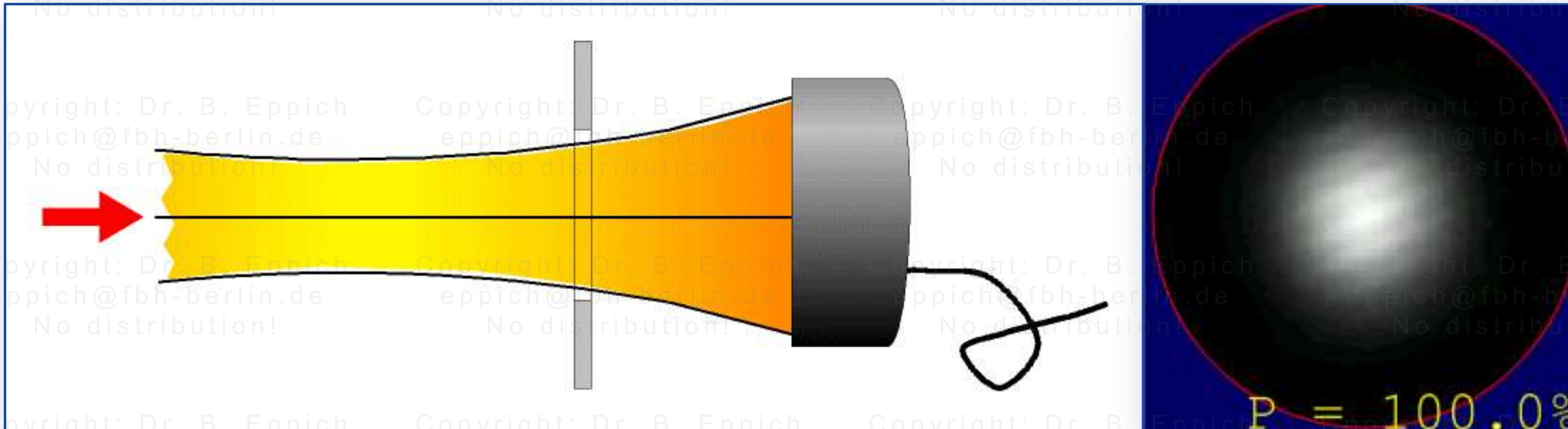
Application of the threshold definition to non-Gaussian beams:

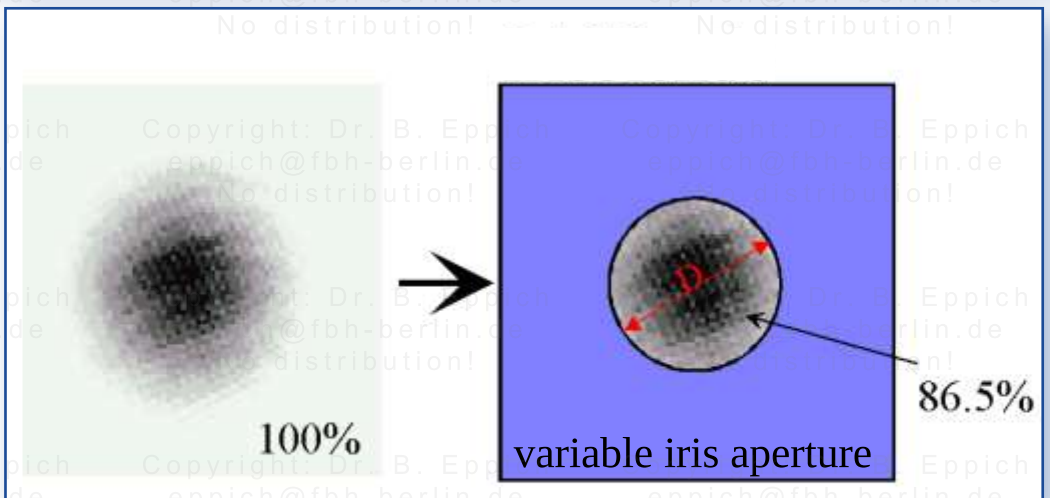




„Historical“ beam diameter definitions

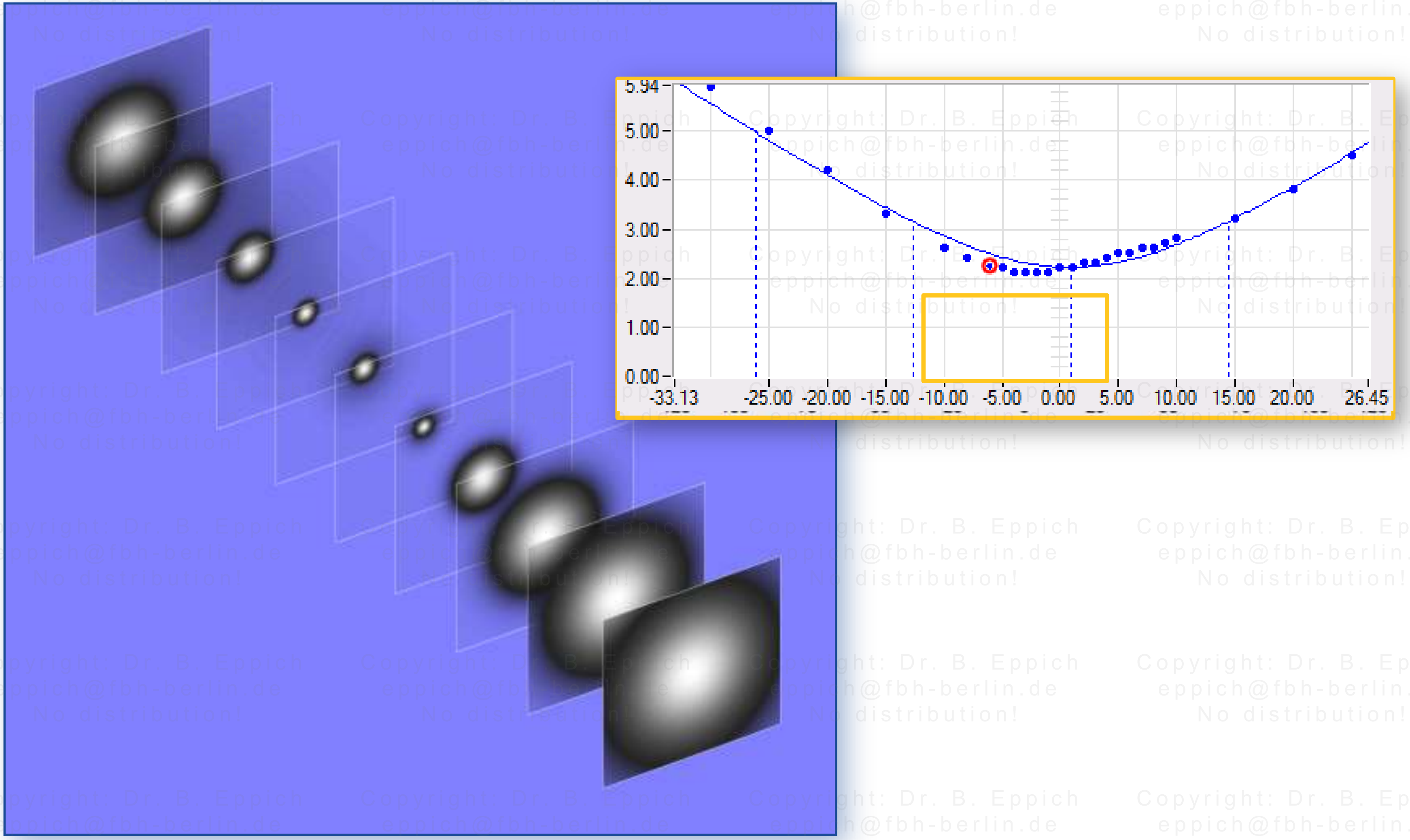
..."power content" definition for "circular" profiles





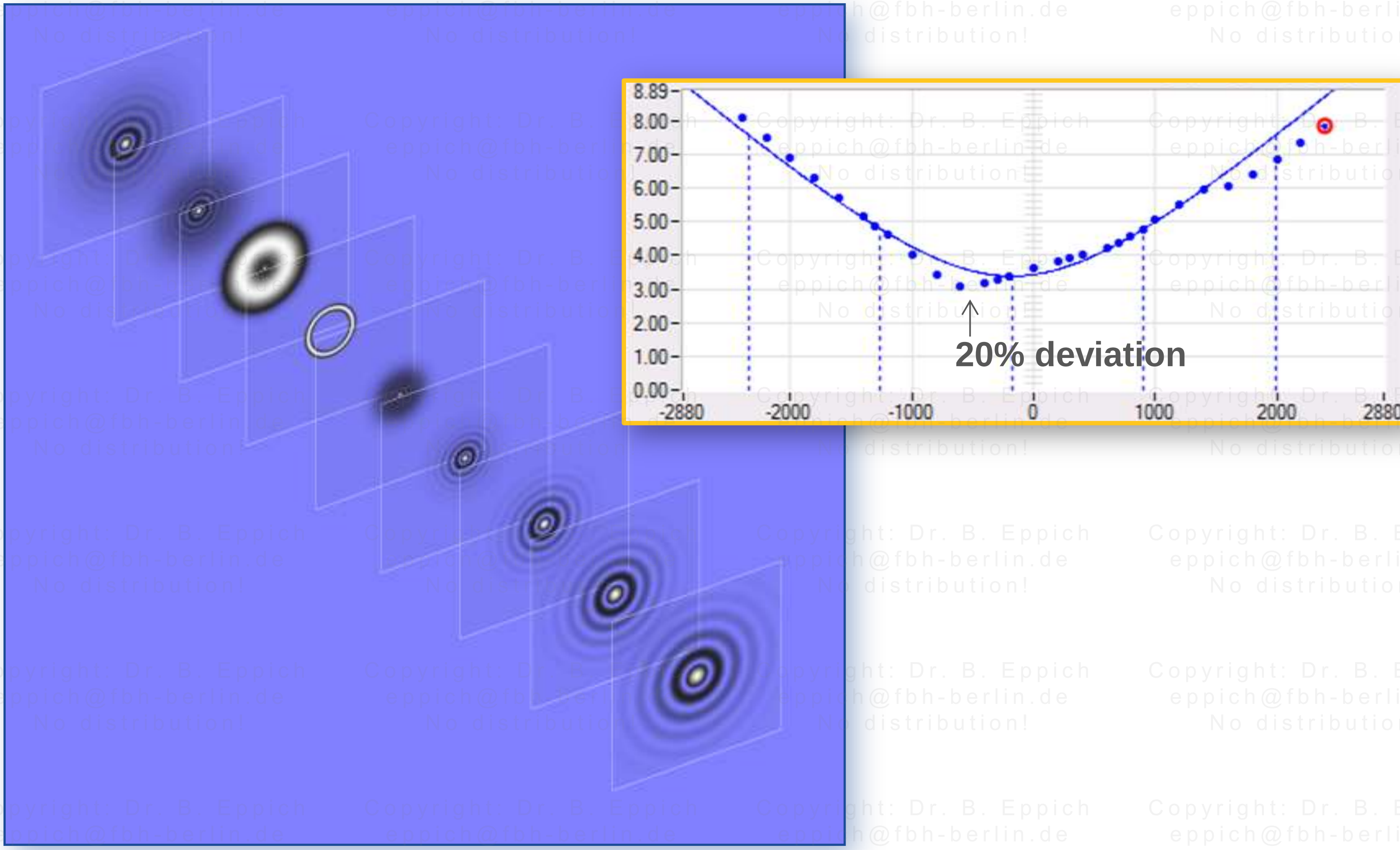
„Historical“ beam diameter definitions

Application of the power content definition to non-Gaussian beams:



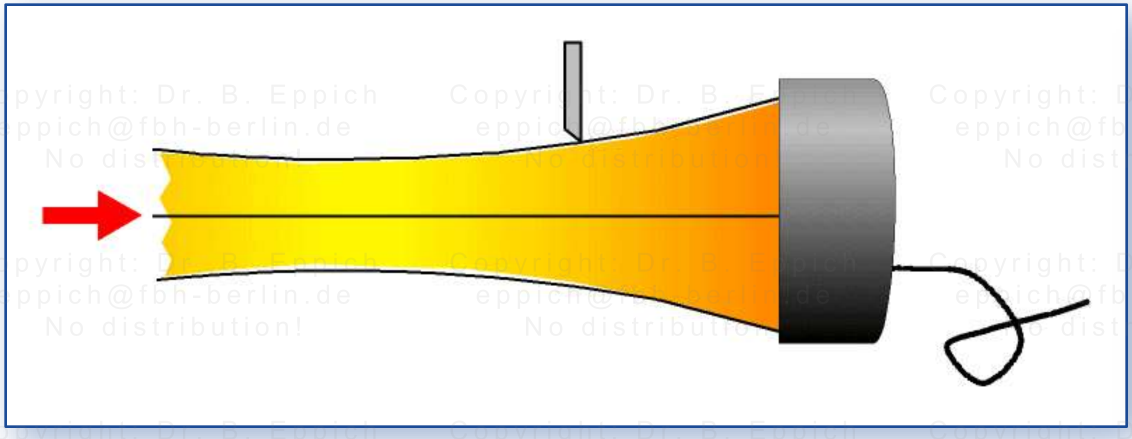
„Historical“ beam diameter definitions

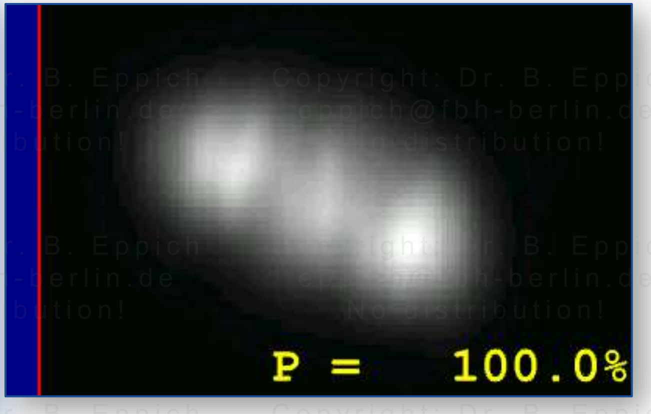
Application of the power content definition to non-Gaussian beams:

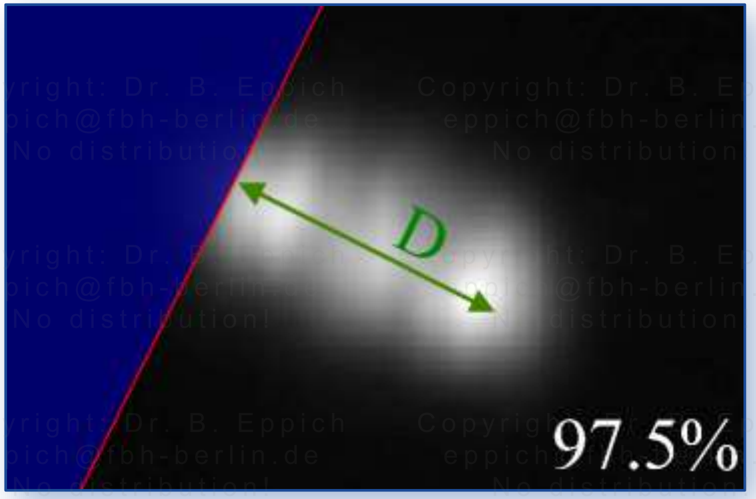


„Historical“ beam diameter definitions

...knife edge definition for „non-circular“ profiles

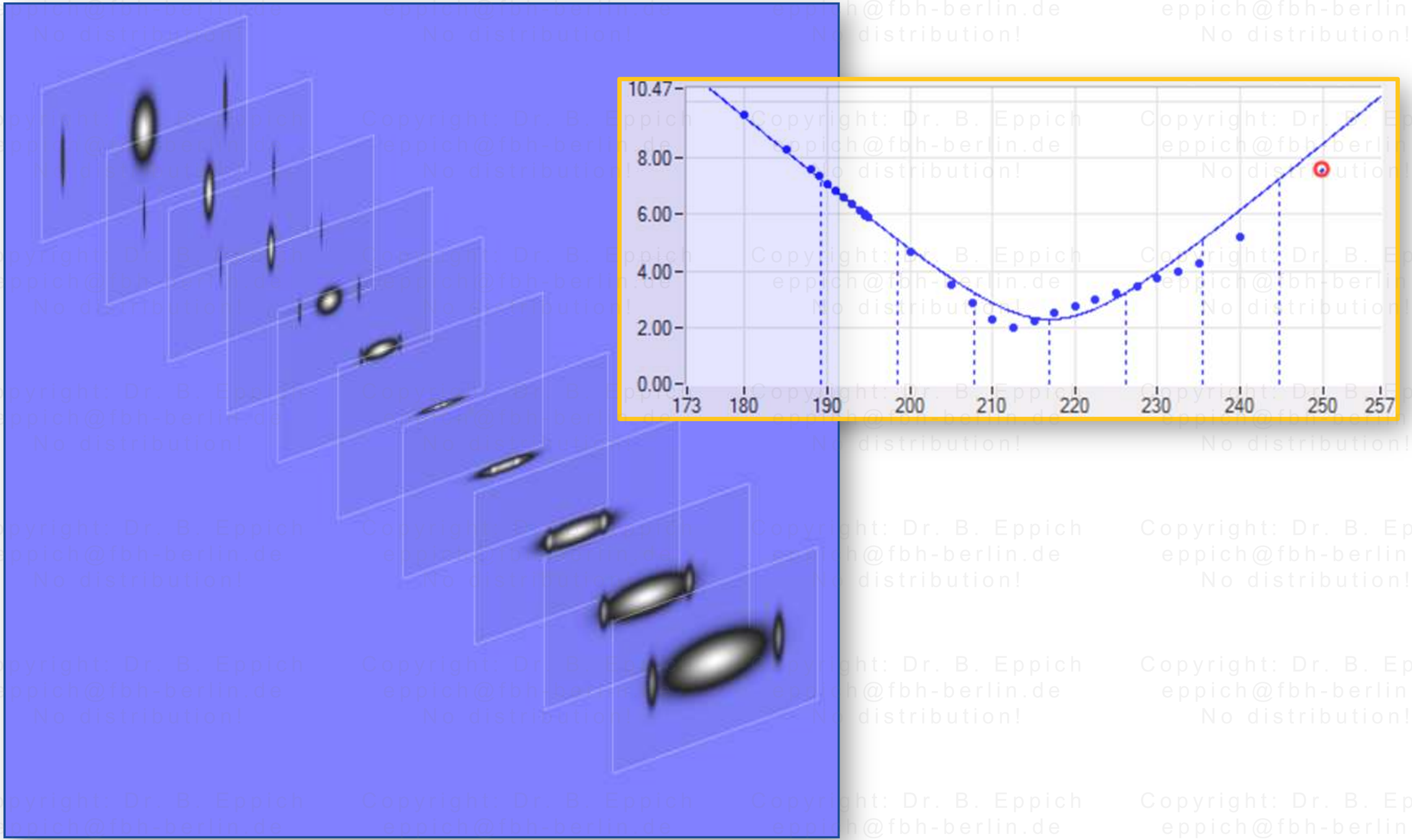




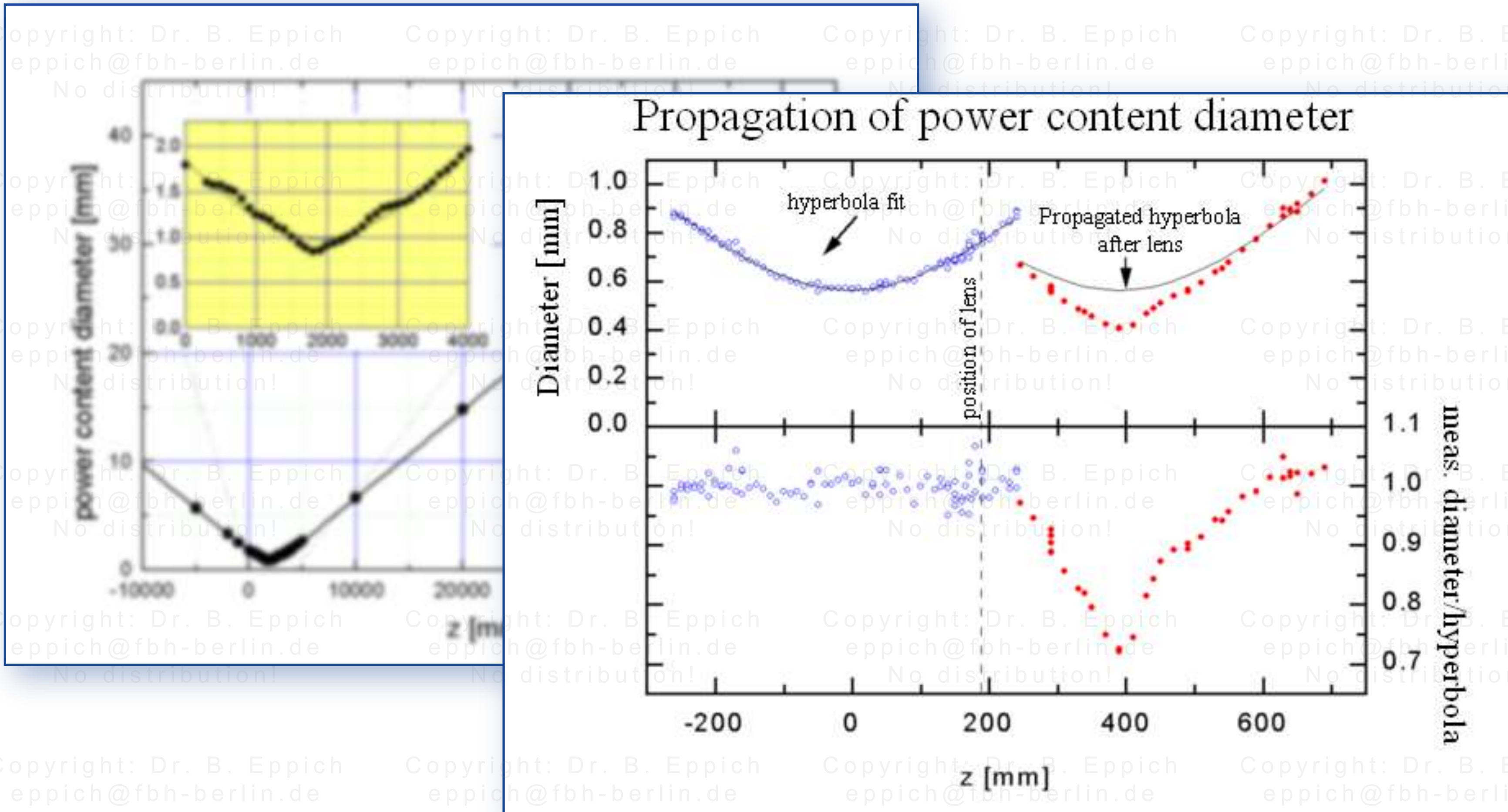


„Historical“ beam diameter definitions

Application of the knife edge definition to non-Gaussian beams:

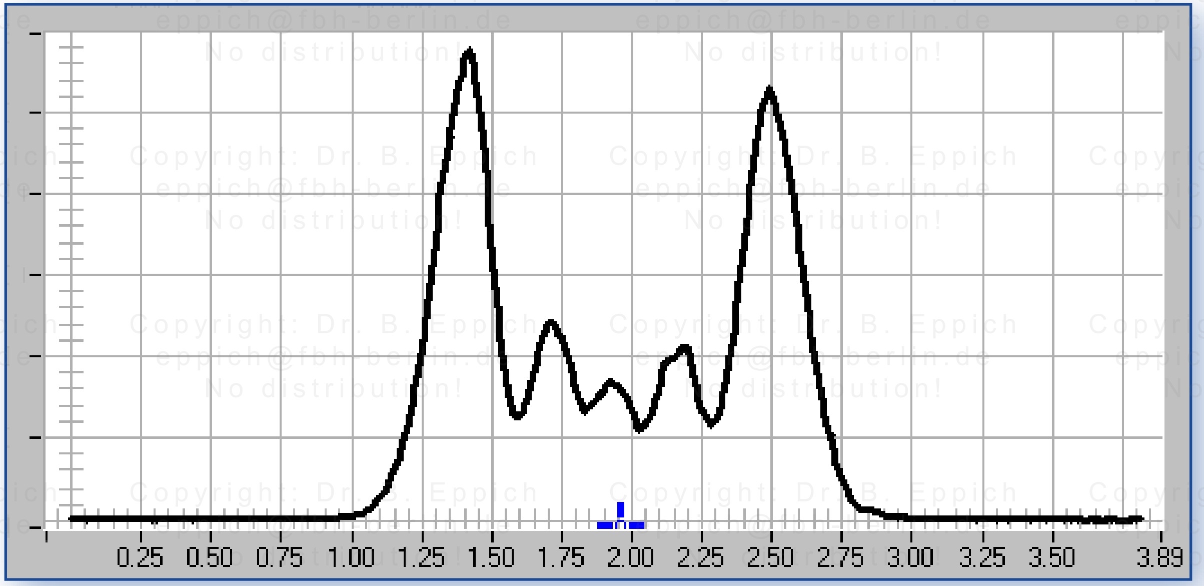


Propagation of power content beam widths



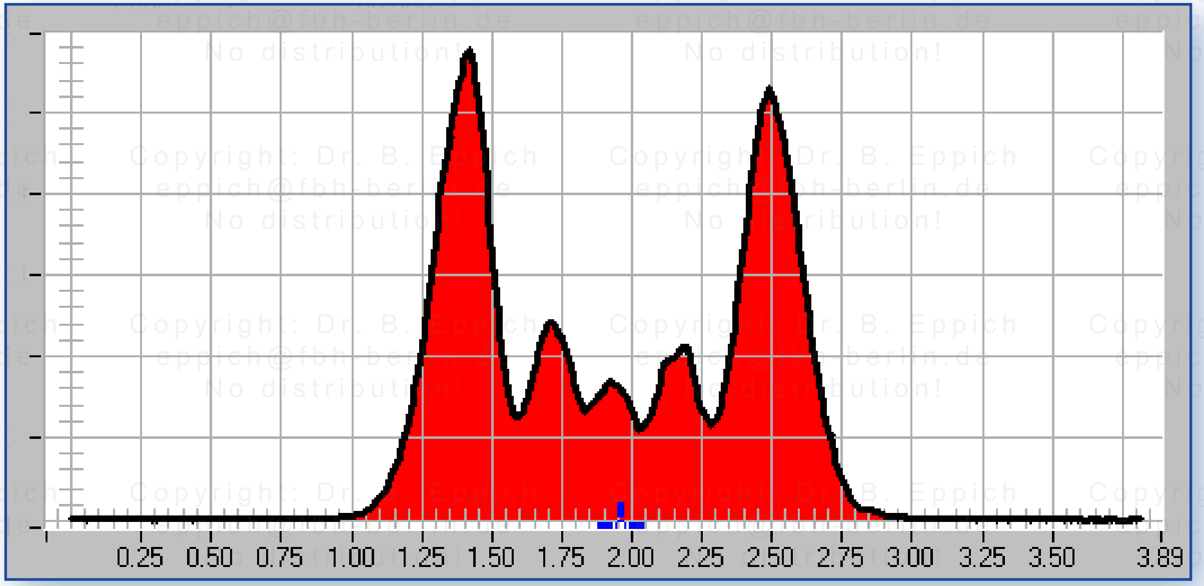
May show significant deviation from a hyperbolic propagation law!
→ Ambiguous definition of D_0, z_0, z_R ,….

Concept of intensity moments



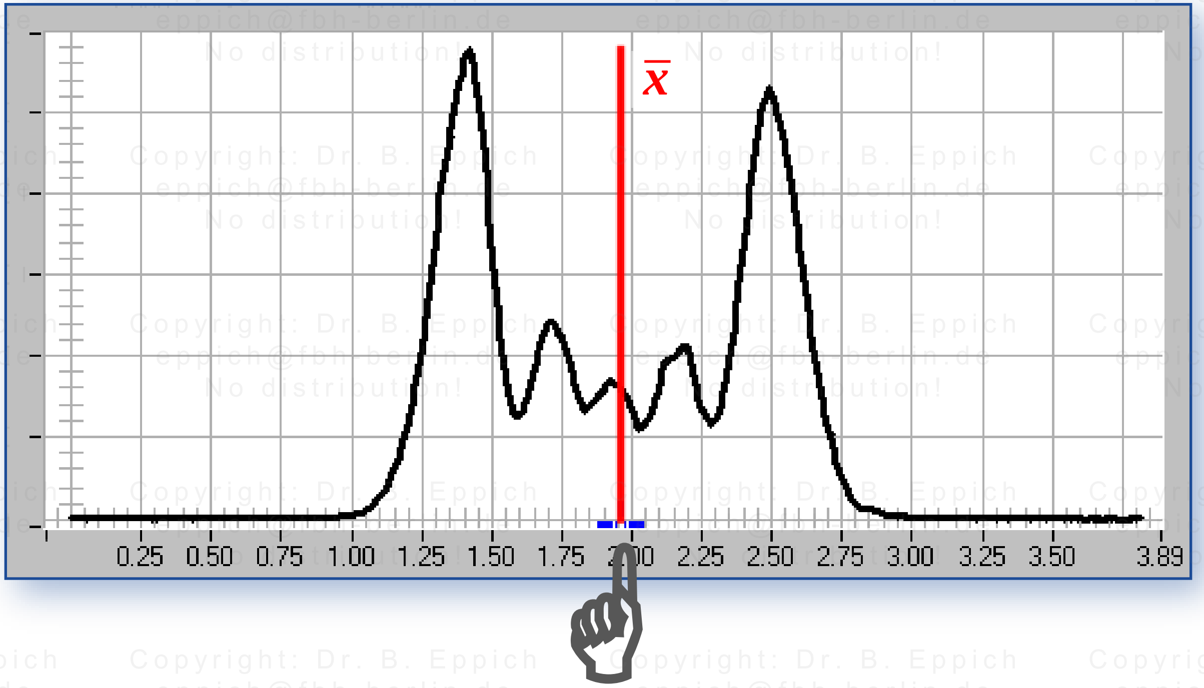
Concept of intensity moments

$$P = \int I(x)\,dx$$



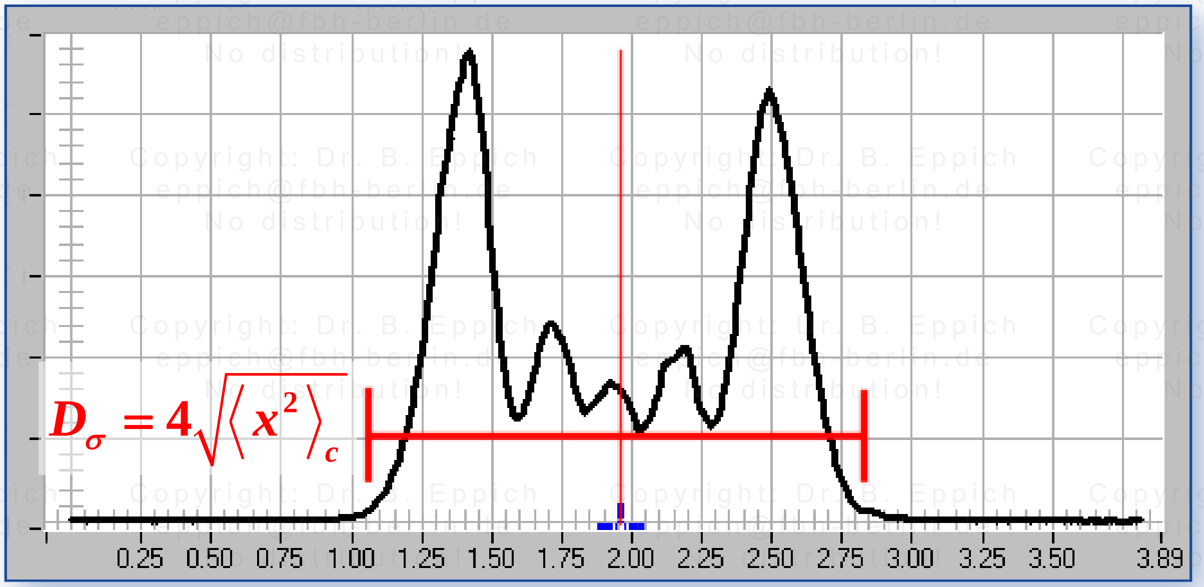
Concept of intensity moments

$$\overline{x} = \langle x \rangle = \frac{1}{P}\int I(x)\, x\, dx$$



Concept of intensity moments

$$\left\langle x^2 \right\rangle_c = \frac{1}{P}\int I(x)\left(x-\bar{x}\right)^2 dx$$



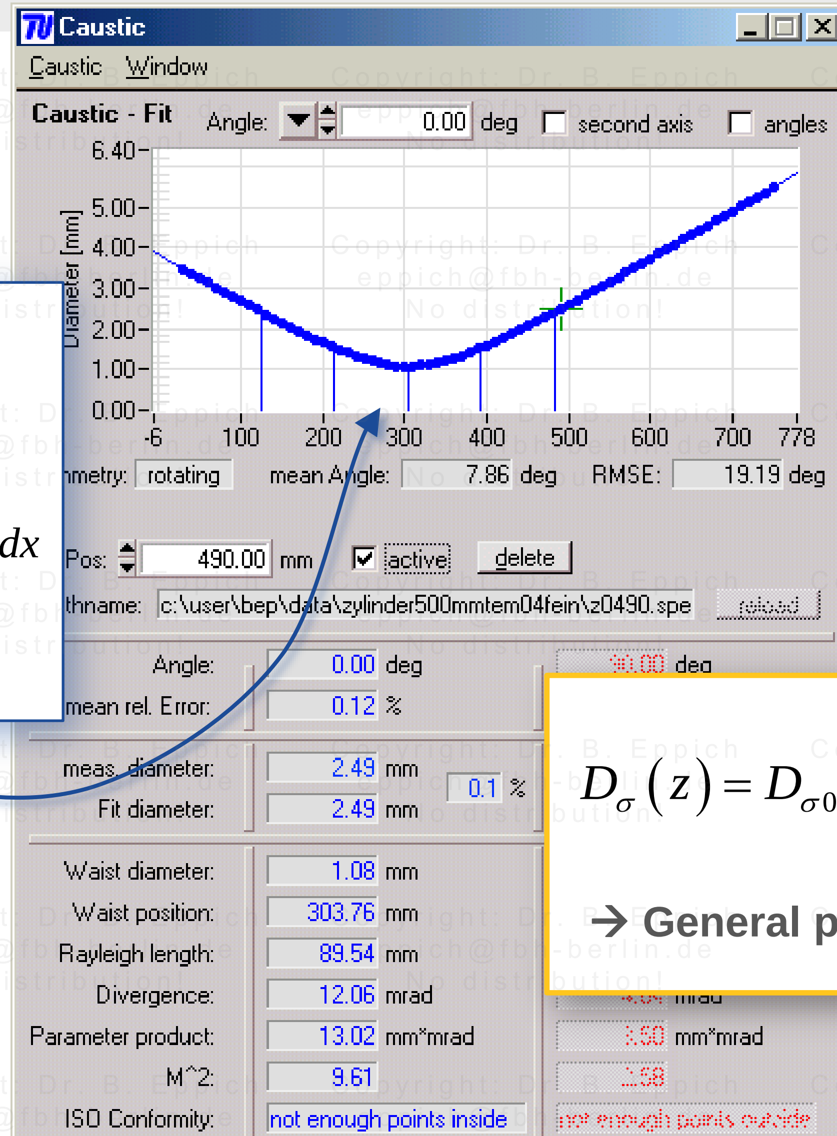
Propagation of second order moment widths

$$P = \int I(x,z)\; dx$$

$$\bar{x}(z) = \frac{1}{P}\int I(x,z)\, x\, dx$$

$$\langle x^2 \rangle_c (z) = \frac{1}{P}\int I(x,z)\left(x - \bar{x}(z)\right)^2 dx$$

$$D_\sigma(z) = 4\sqrt{\langle x^2 \rangle_c (z)}$$

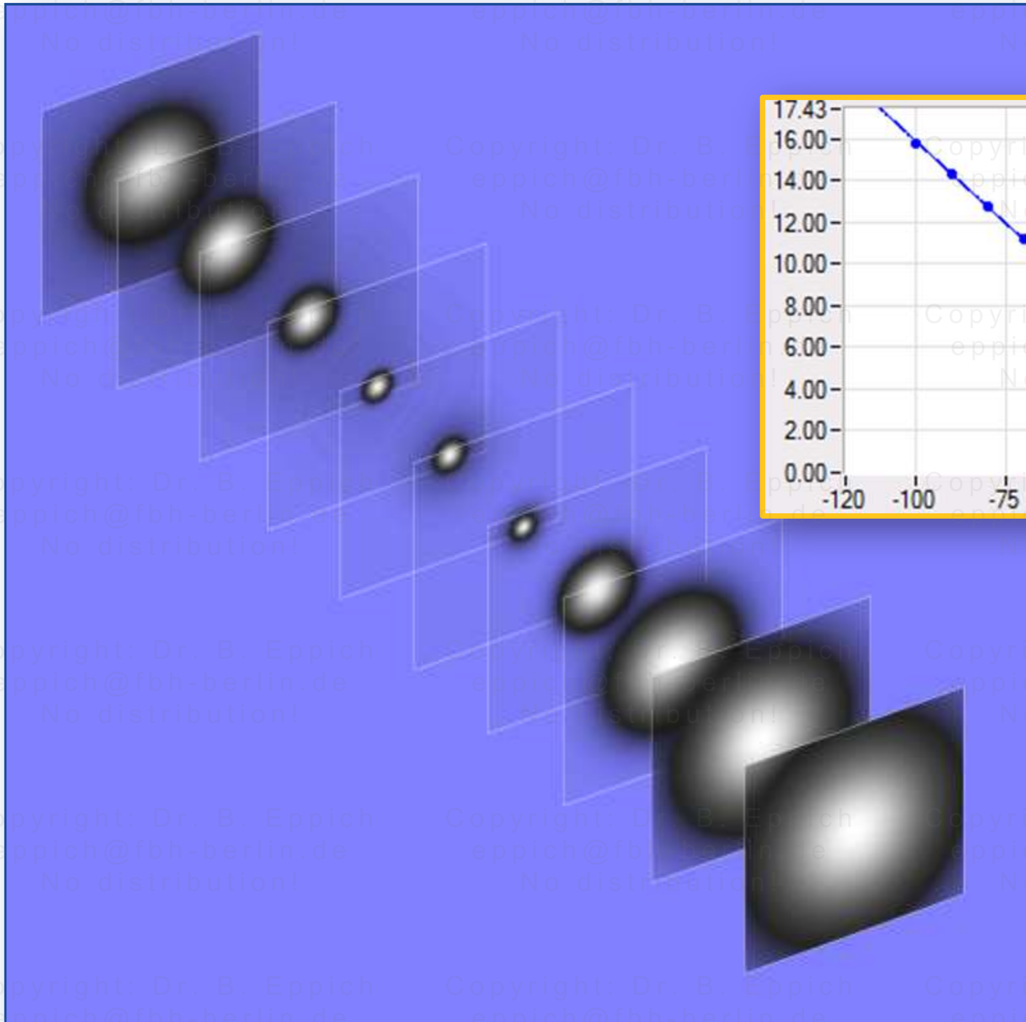


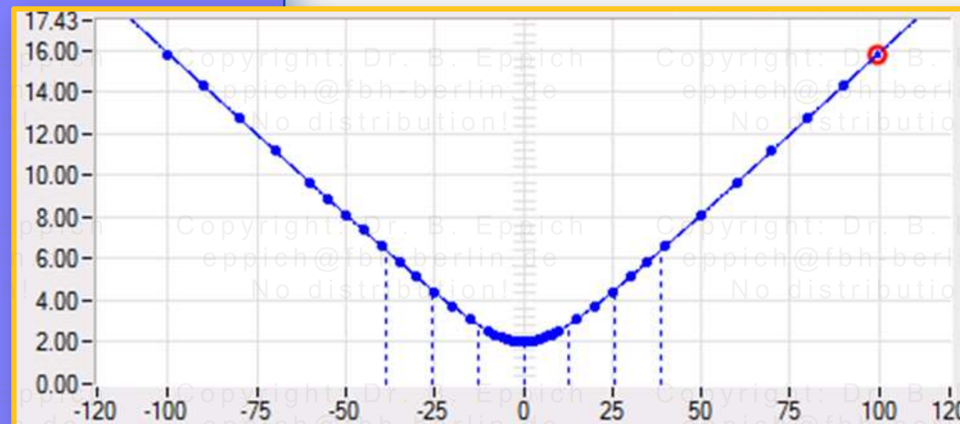
$$D_\sigma(z) = D_{\sigma 0}\sqrt{1 + \left(\frac{z - z_0}{z_R}\right)^2}$$

→ **General propagation law**

Propagation of second order moment widths

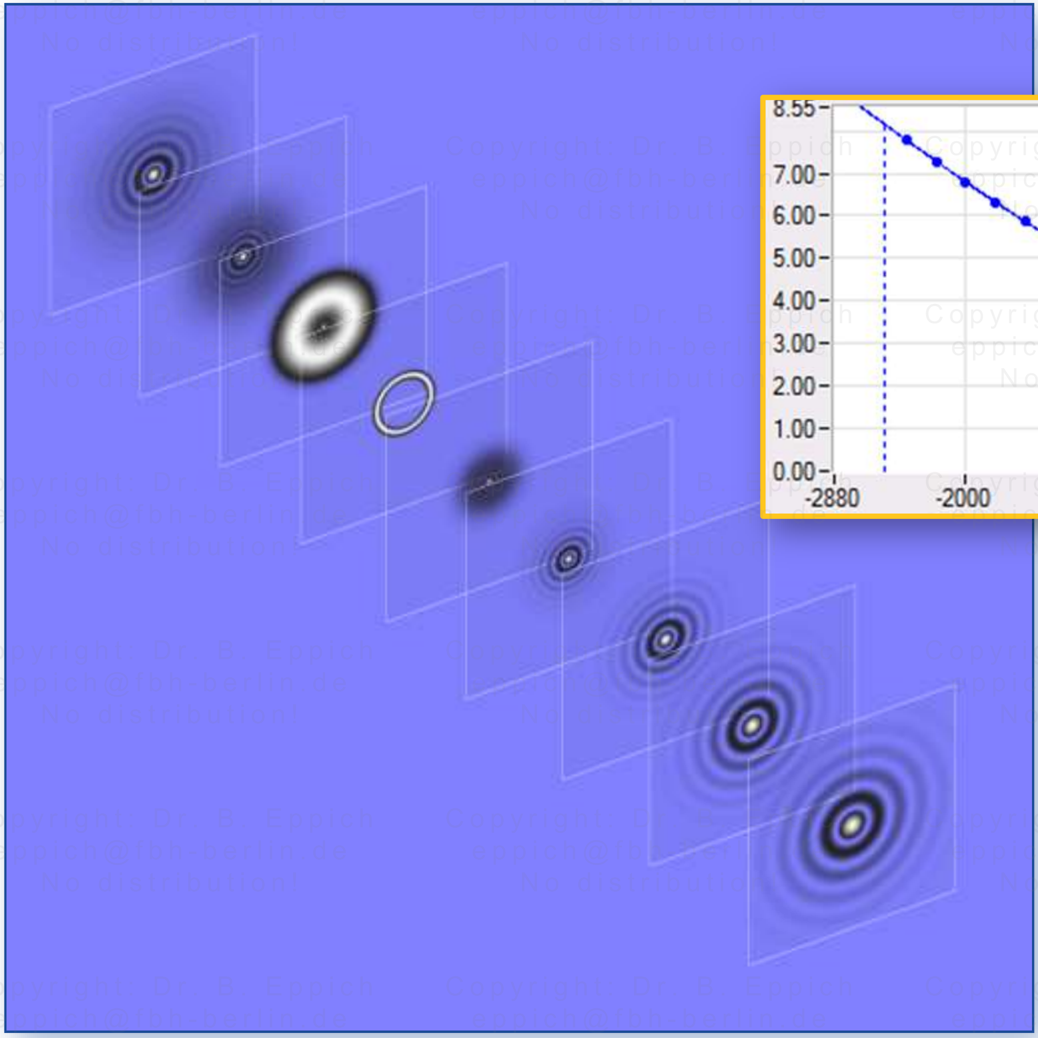
Application to non-Gaussian beams:

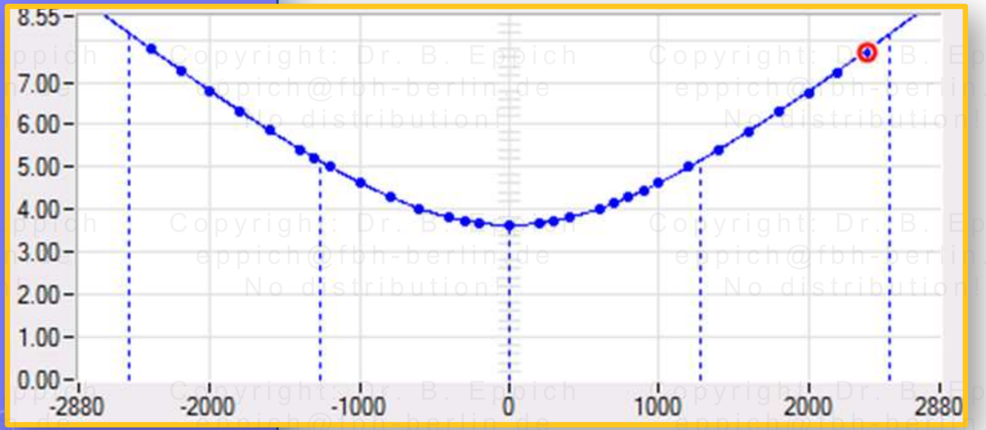




Propagation of second order moment widths

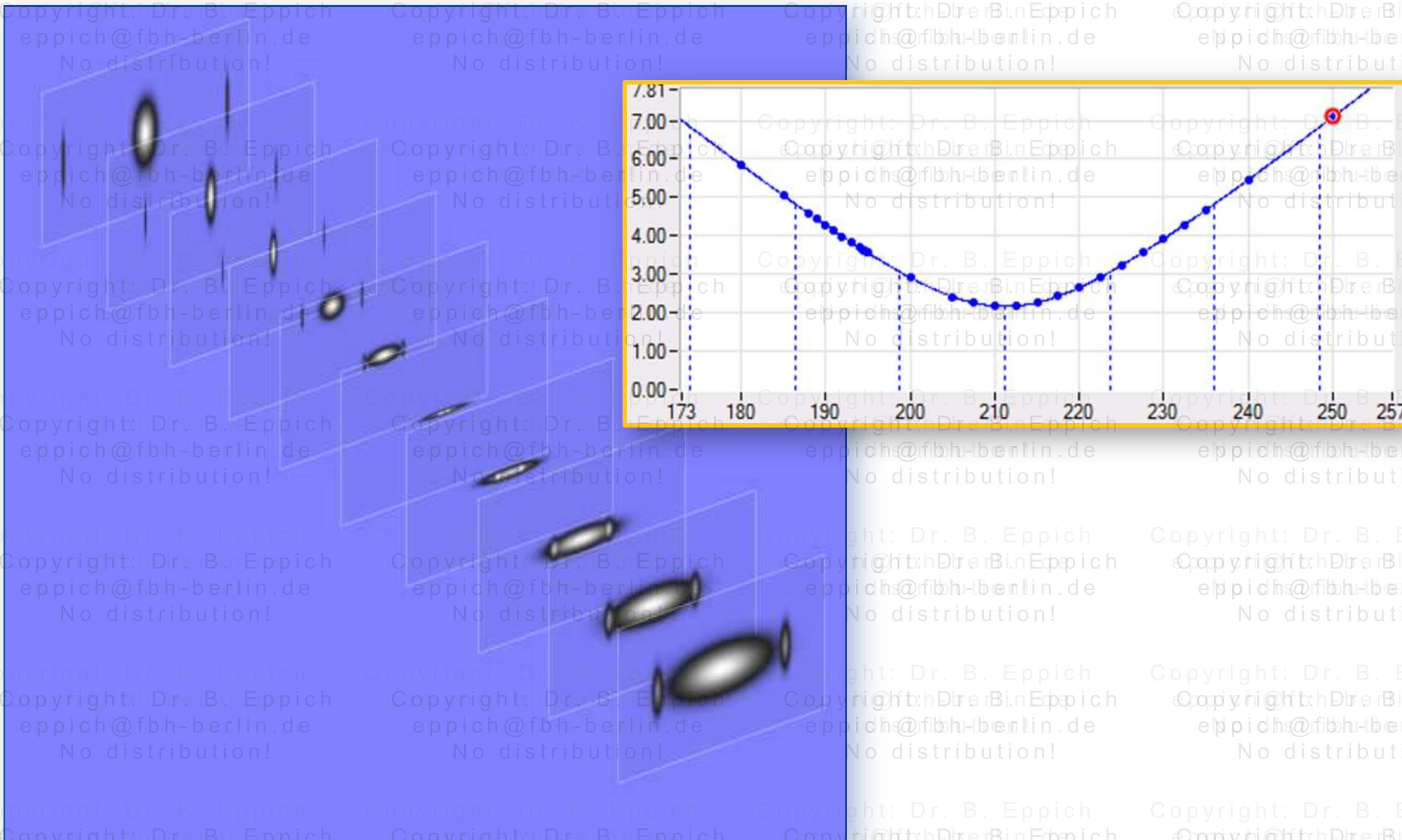
Application to non-Gaussian beams:



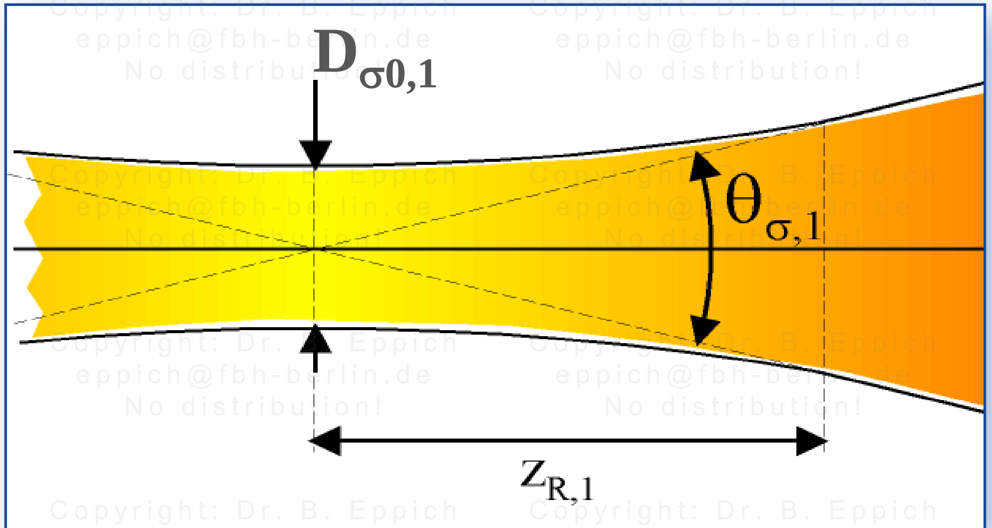


Propagation of second order moment widths

Application to non-Gaussian beams:



Beam propagation parameter

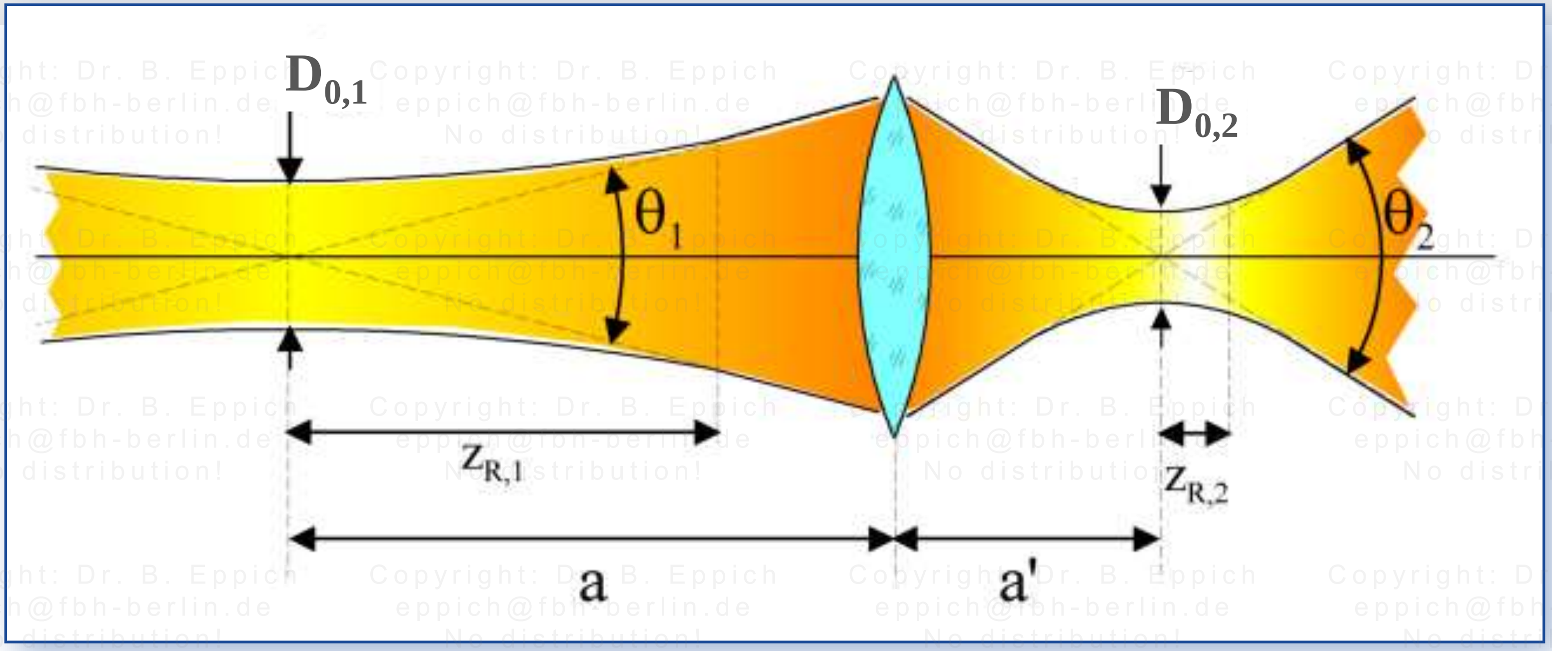


$$D_\sigma(z) = D_{\sigma,0}\sqrt{1+\left(\frac{z-z_0}{z_R}\right)^2} = \sqrt{D_{\sigma,0}^2 + \theta_\sigma^2 (z-z_0)^2}$$

$$\frac{D_{\sigma 0}\cdot\theta_\sigma}{4} \geq \frac{\lambda}{\pi}$$

$$M^2 = \frac{D_{\sigma 0}\cdot\theta}{4}\bigg/\frac{\lambda}{\pi} = \frac{D_{\sigma 0}^2}{4 z_R}\bigg/\frac{\lambda}{\pi}$$

Beam propagation parameter



$$V = \frac{f}{\sqrt{z_{R,1}^2 + (a-f)^2}}$$

$$D_{0,2} = V \cdot D_{0,1}$$

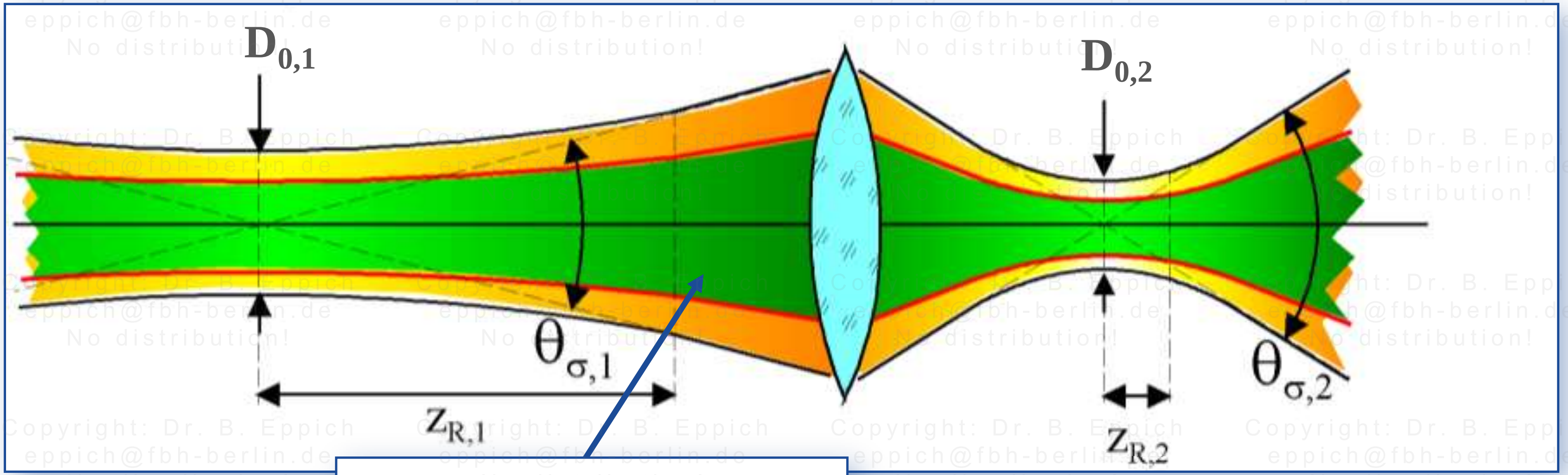
$$z_{R,2} = V^2 \cdot z_{R,1}$$

$$a' - f = V^2 \cdot (a-f)$$

$$\theta_2 = \frac{1}{V}\theta_1$$

$$M_2^2 = M_1^2$$

Embedded Gaussian beam

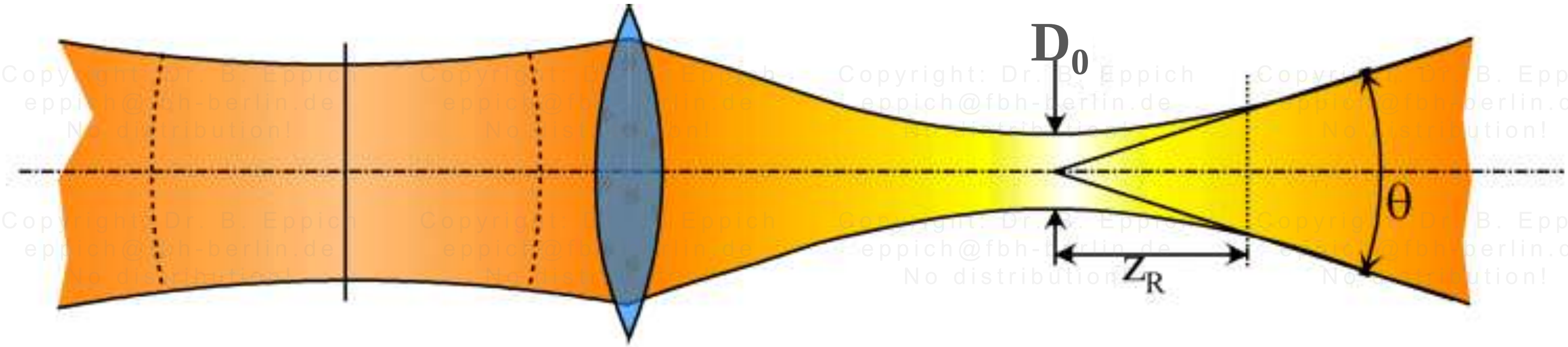


embedded Gaussian beam

$D_\sigma(z) = \sqrt{M^2}\, D_g(z)$	$z_{\sigma,0} = z_{g,0}$
$\theta_\sigma = \sqrt{M^2}\, \theta_g$	$z_{\sigma,R} = z_{g,R}$

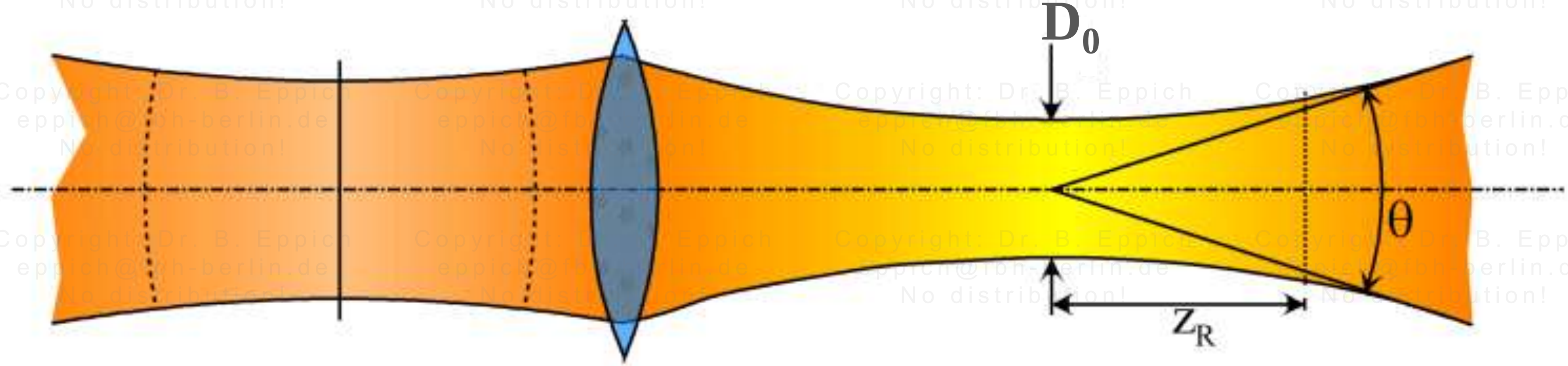
Influence of beam propagation ratio M^2

M^2:	**low**
Lens:	**week**



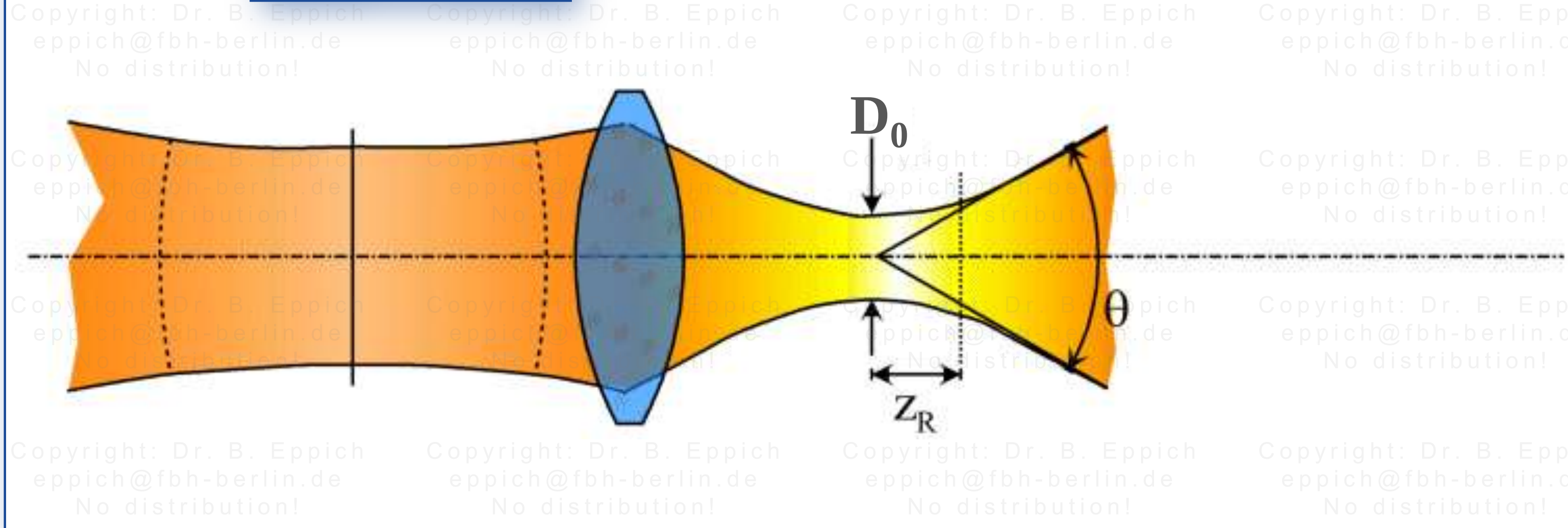
Influence of beam propagation ratio M^2

M^2:	high
Lens:	week



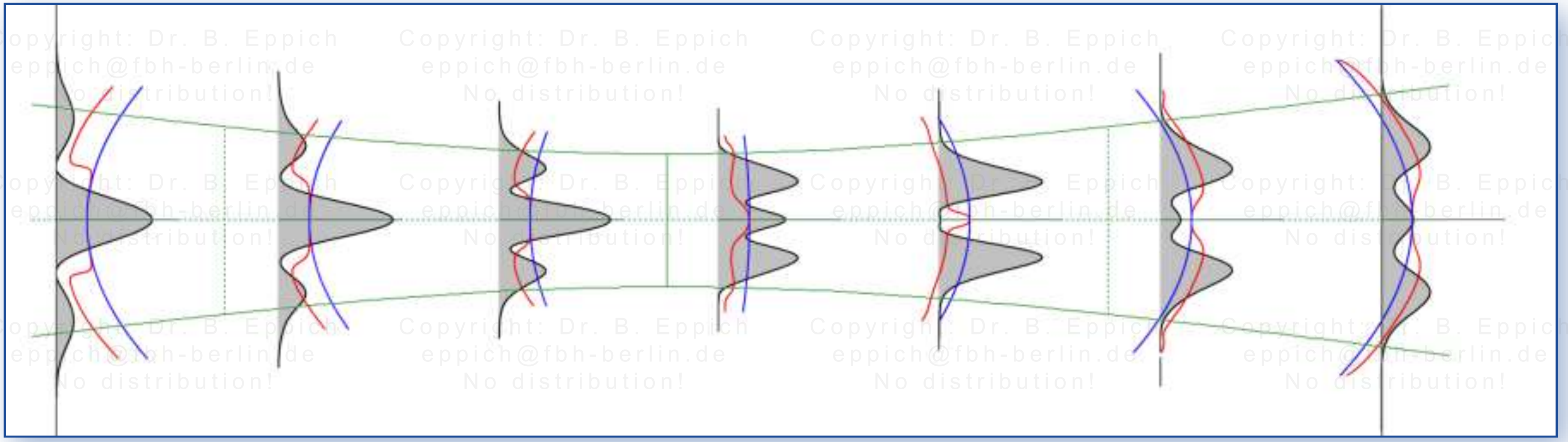
Influence of beam propagation ratio M^2

M^2:	high
Lens:	strong



→ M^2 is measure of „focusability“

Embedded Gaussian beam, phase curvature



$$R(z) = z_R\left(\frac{z-z_0}{z_R}+\frac{z_R}{z-z_0}\right)$$

Prediction of beam propagation

$$D(z) = D_0 \sqrt{1 + \left(\frac{z - z_0}{z_R}\right)^2} = \sqrt{D_0^2 + \theta^2 (z - z_0)^2}$$

Propagation through aberration-free systems given by only three parameters:

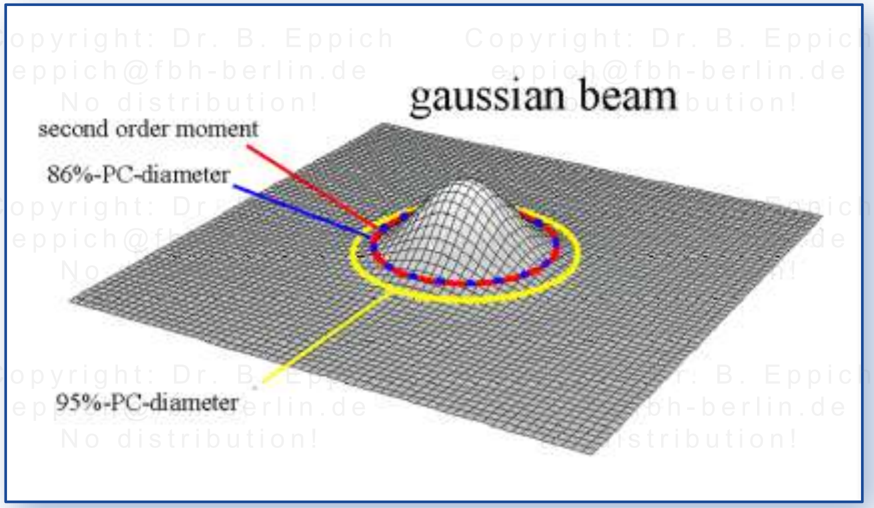
- D_0, z_0, θ
- or: D_0, z_0, z_R
- or: D_0, z_0, M^2
- or: .....

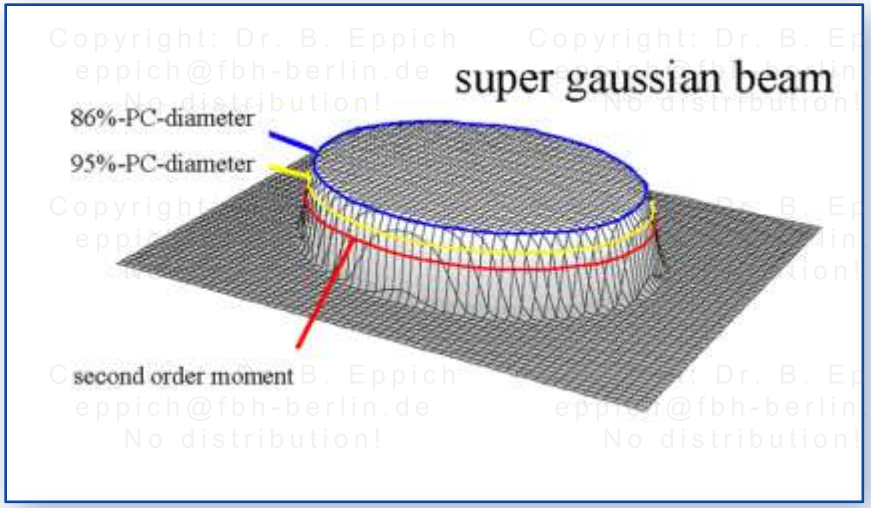
$$q = \Delta z + i\, z_R$$

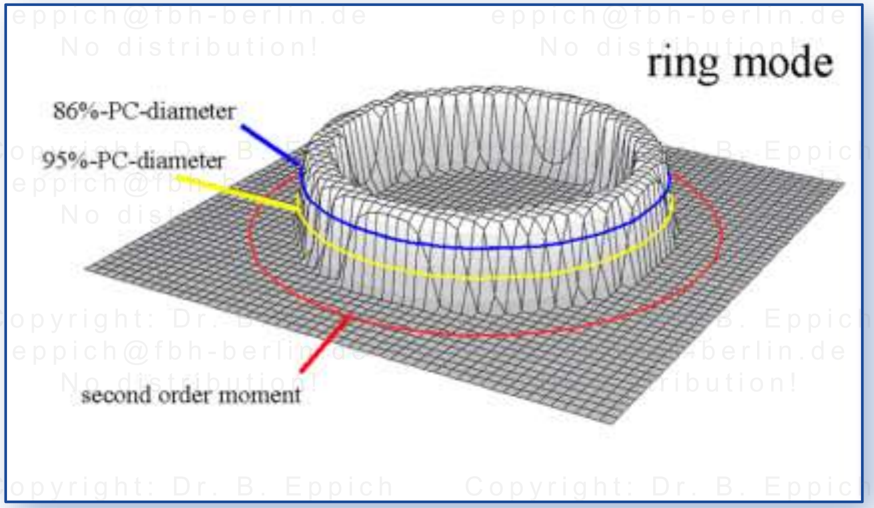
$$\frac{1}{q} = \frac{1}{R} - i \frac{4\lambda}{\pi \left(D_\sigma / \sqrt{M^2}\right)^2}$$

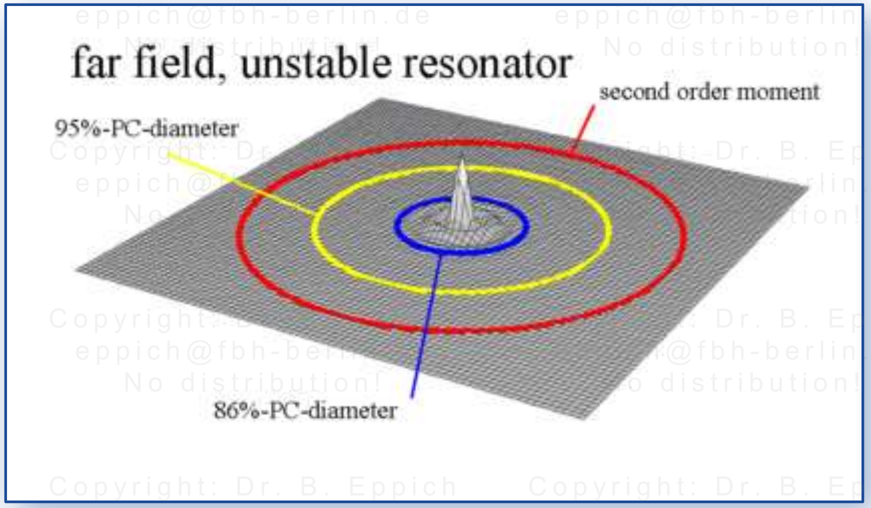
$$\rightarrow \quad q_2 = \frac{A q_1 + B}{C q_1 + D}$$

Comparison of beam diameter definitions

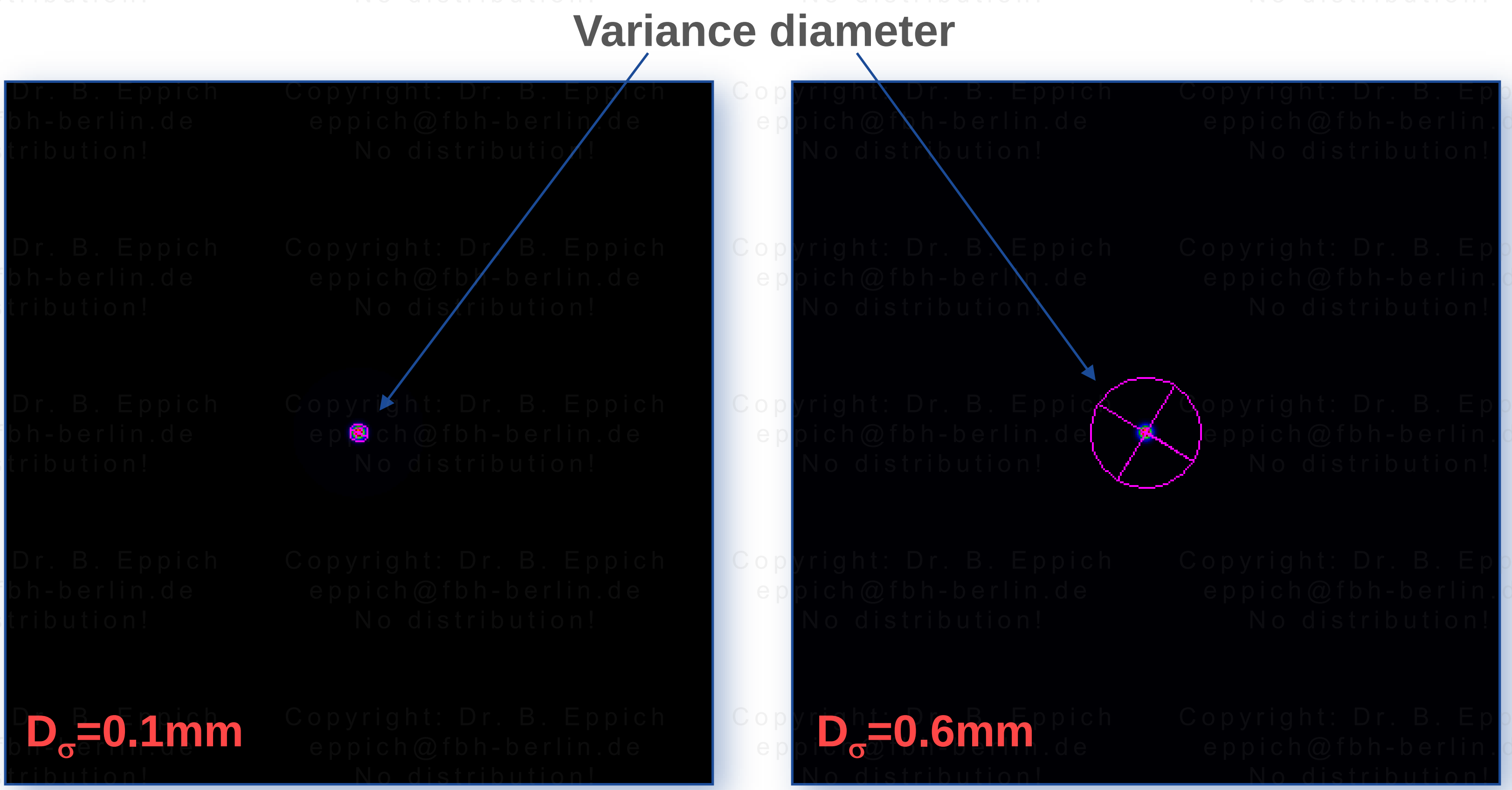




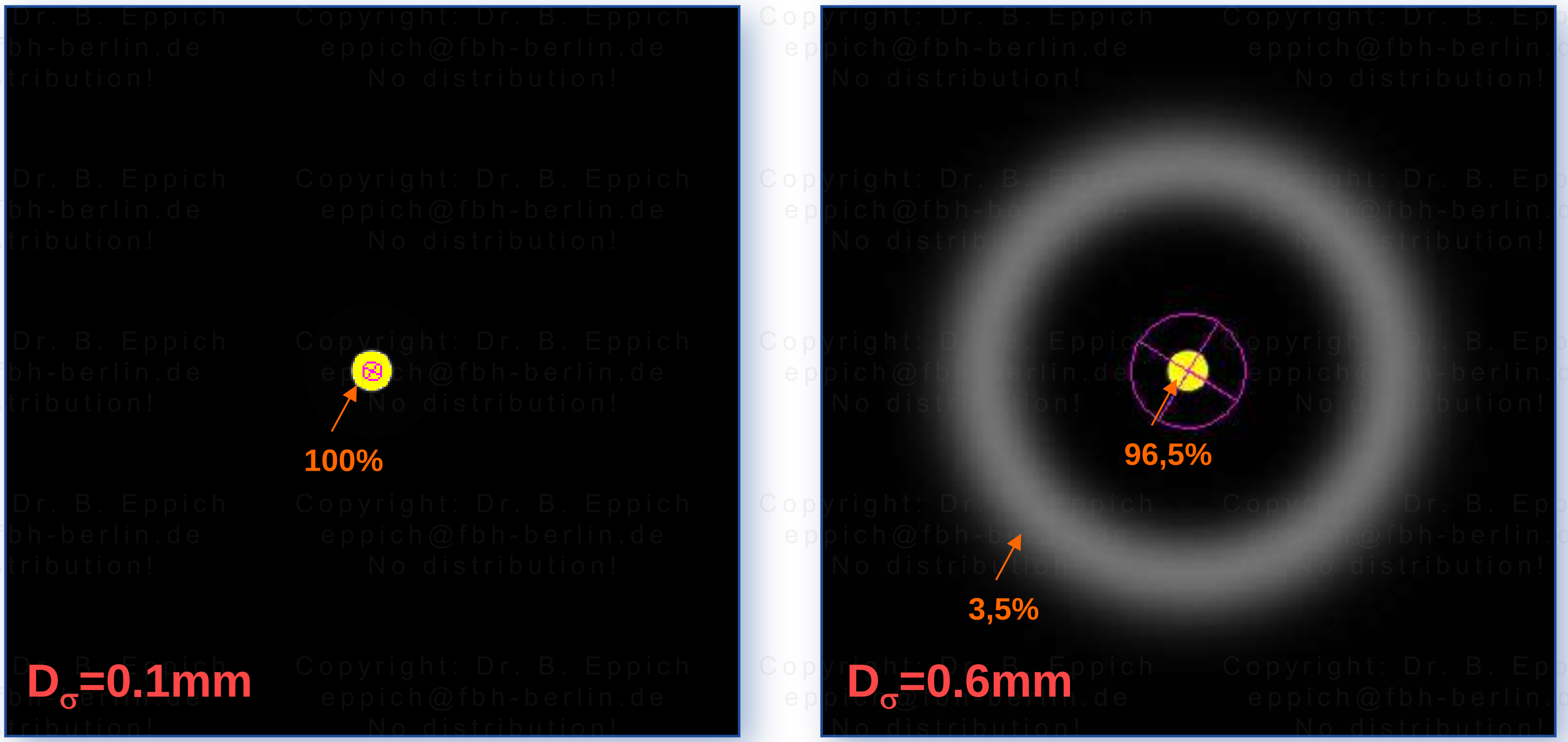




Comparison of beam diameter definitions

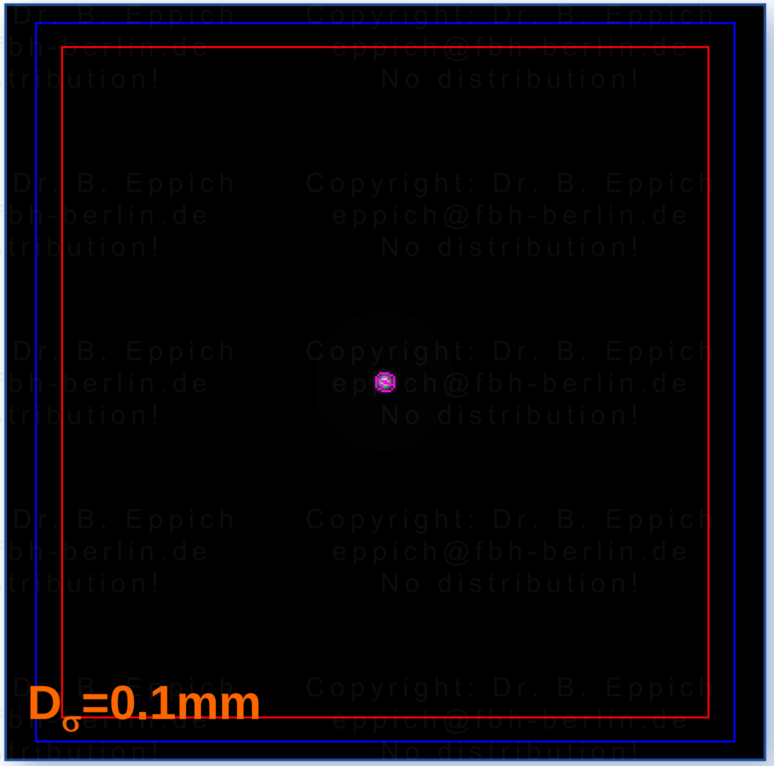


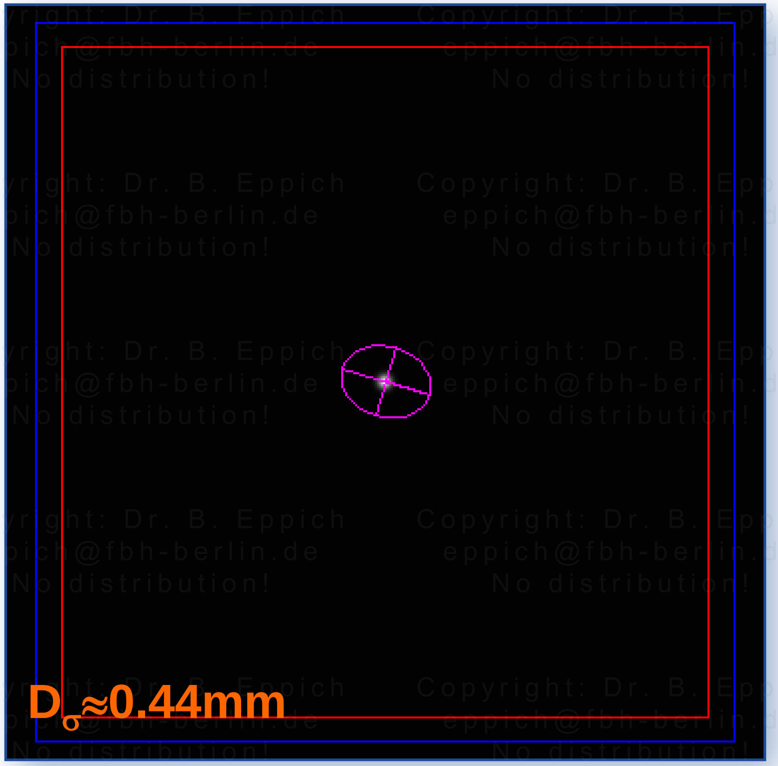
Comparison of beam diameter definitions



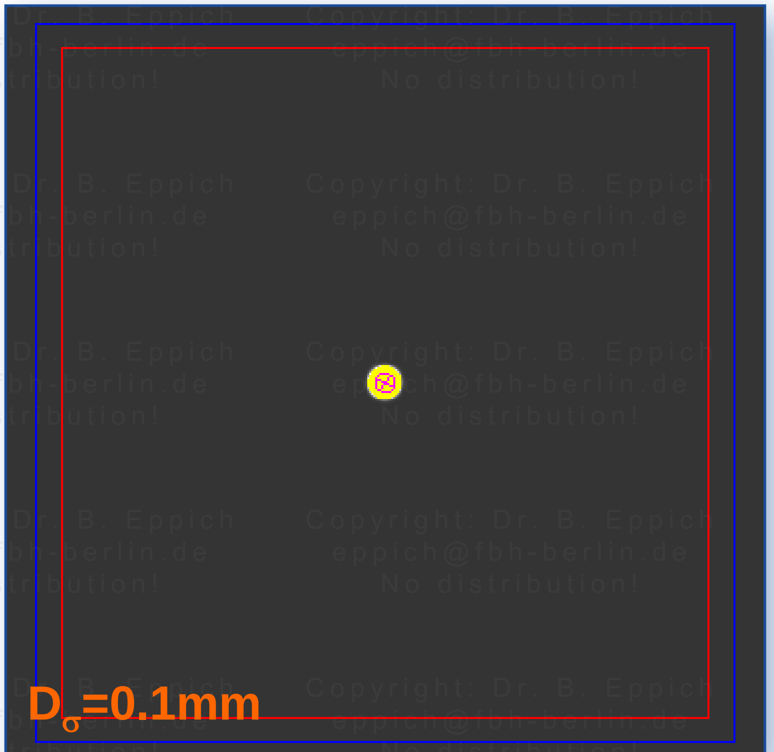
Contrast: 10000×

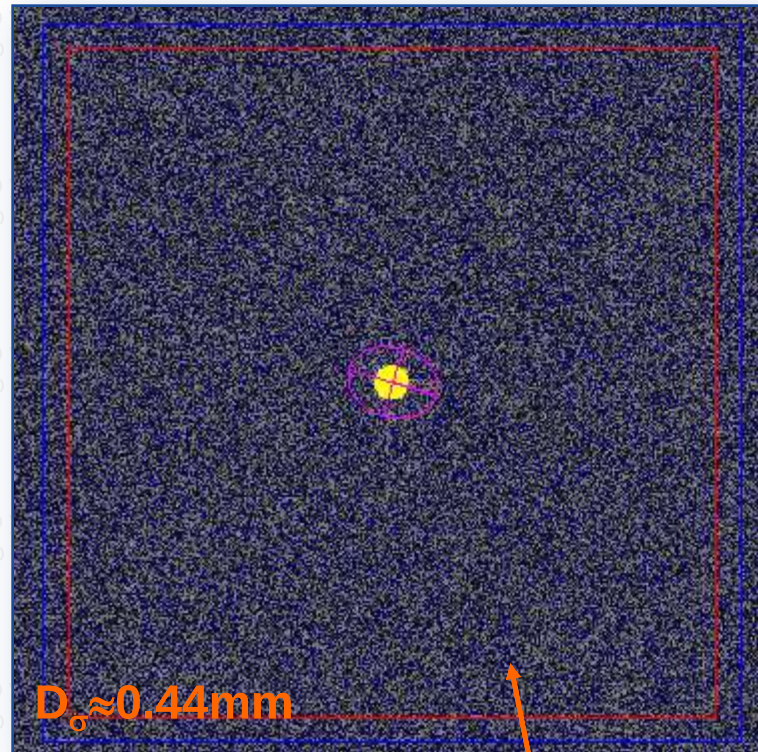
Noise sensitivity





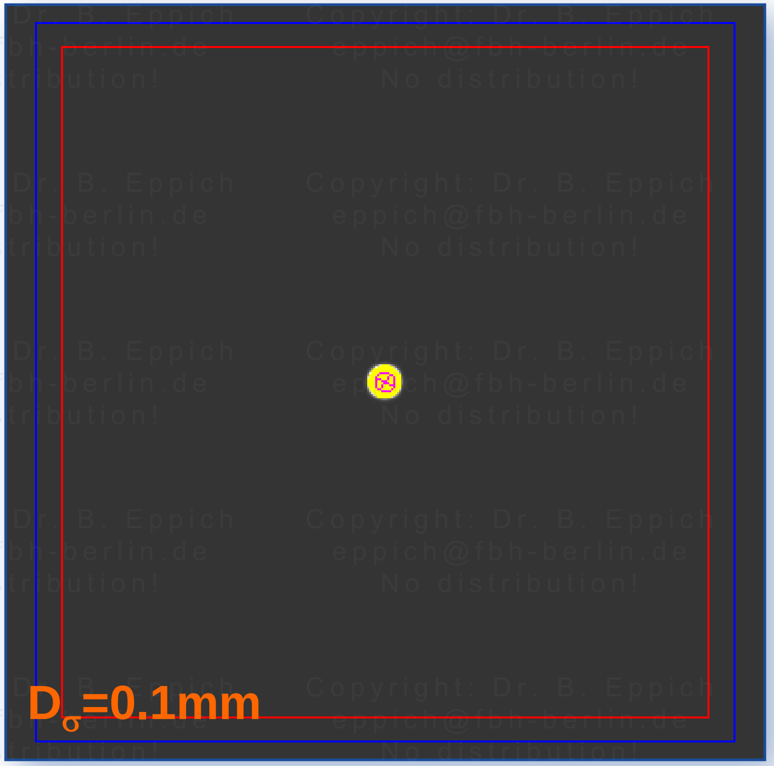
Noise sensitivity

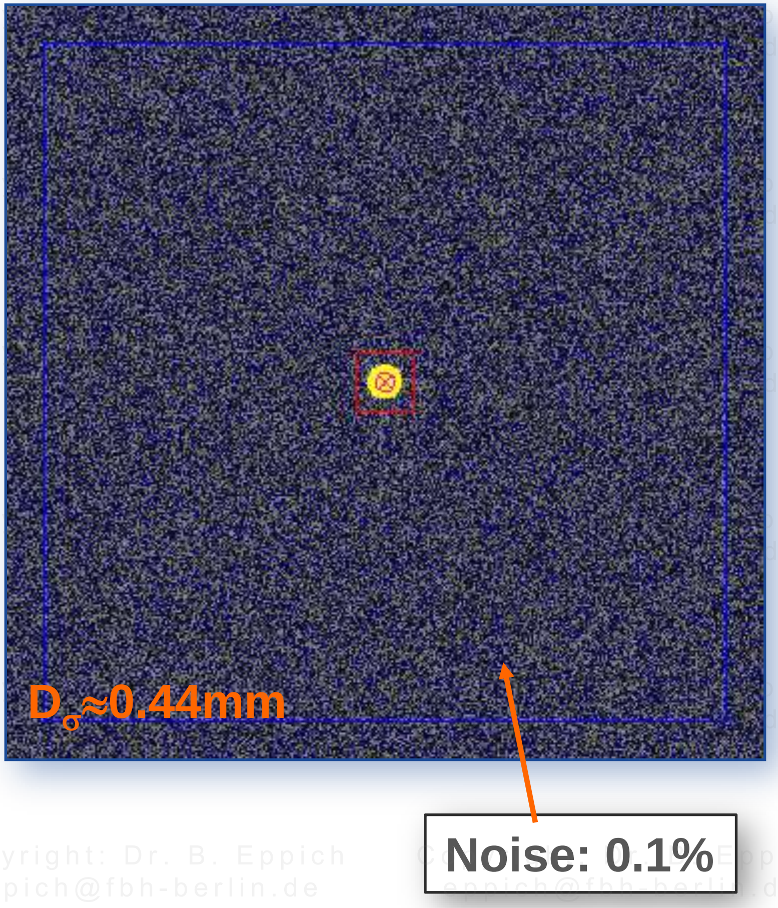




Noise: 0.1%

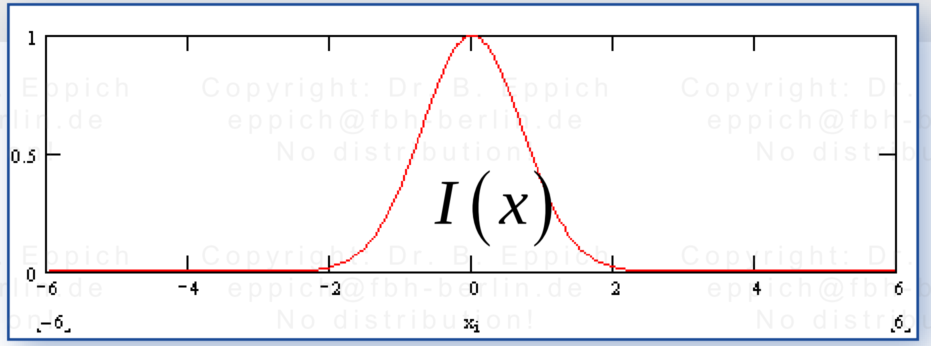
Noise sensitivity

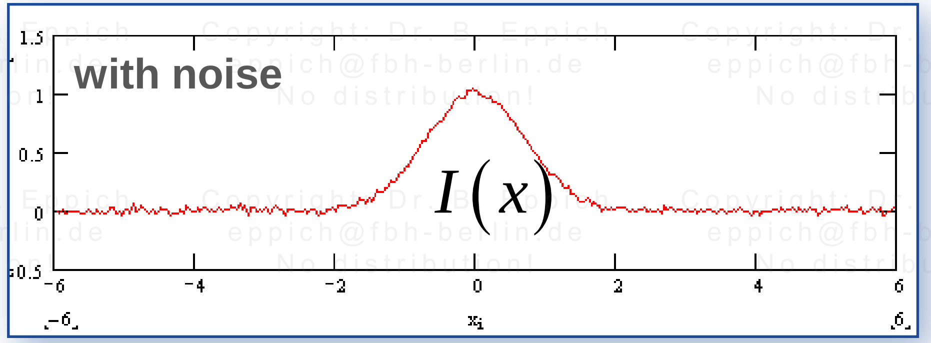


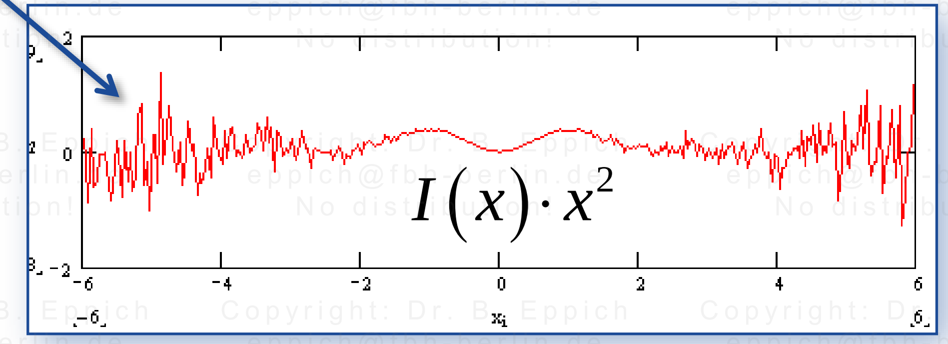


Noise sensitivity

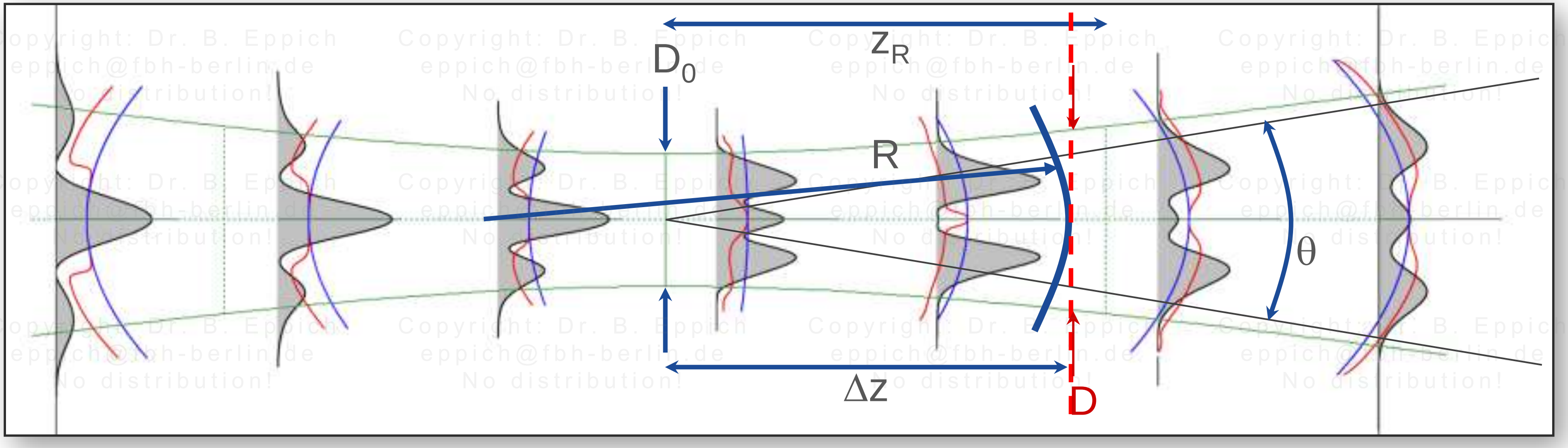
$$D = 4\sqrt{\frac{\int I(x)\,x^2\,dx}{\int I(x)\;dx}}$$







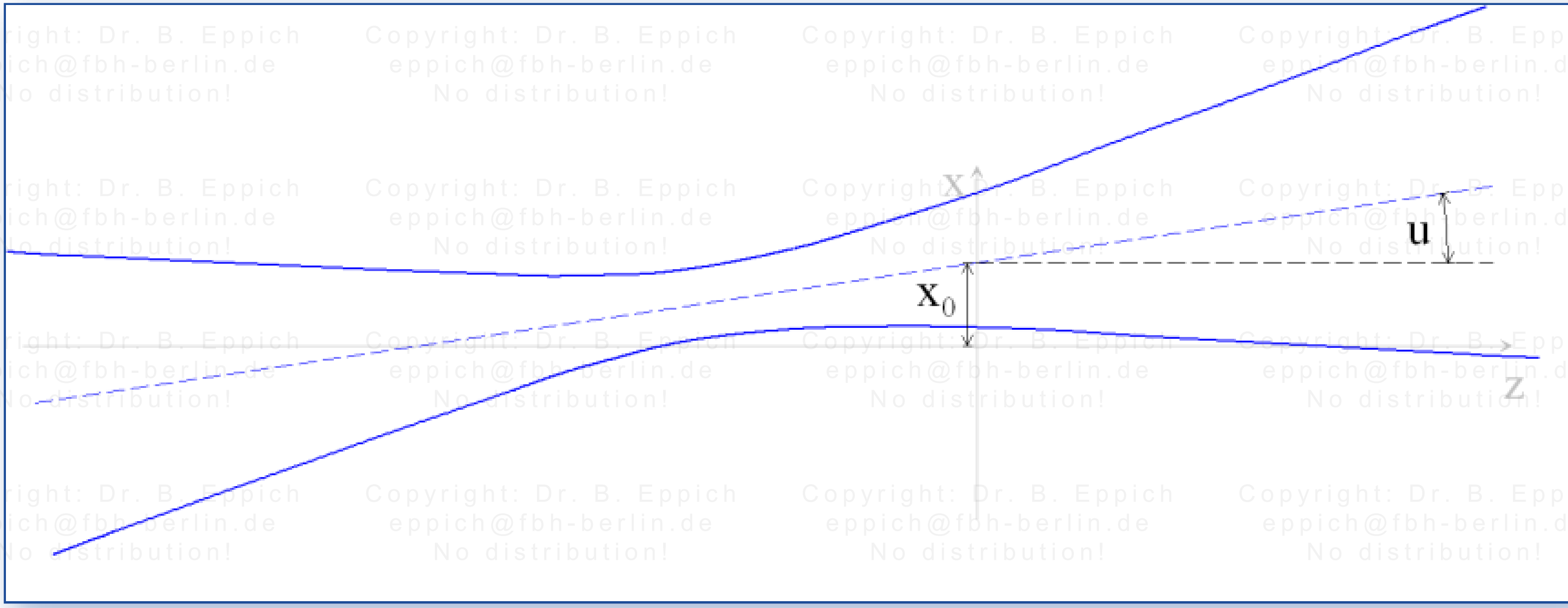
Beam propagation parameters



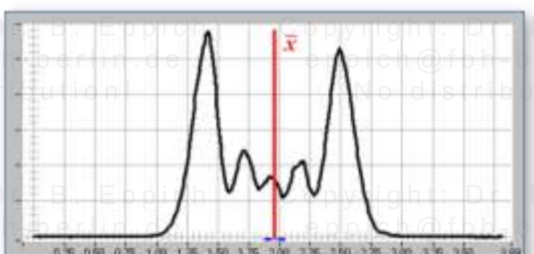
- Diameter D
- Distance to waist Δz
- Waist diameter D_0
- Radius of phase curvature R
- Divergence θ
- Rayleigh length z_R
- Beam propagation factor M^2

Only three independent parameter!

Beam position and beam direction



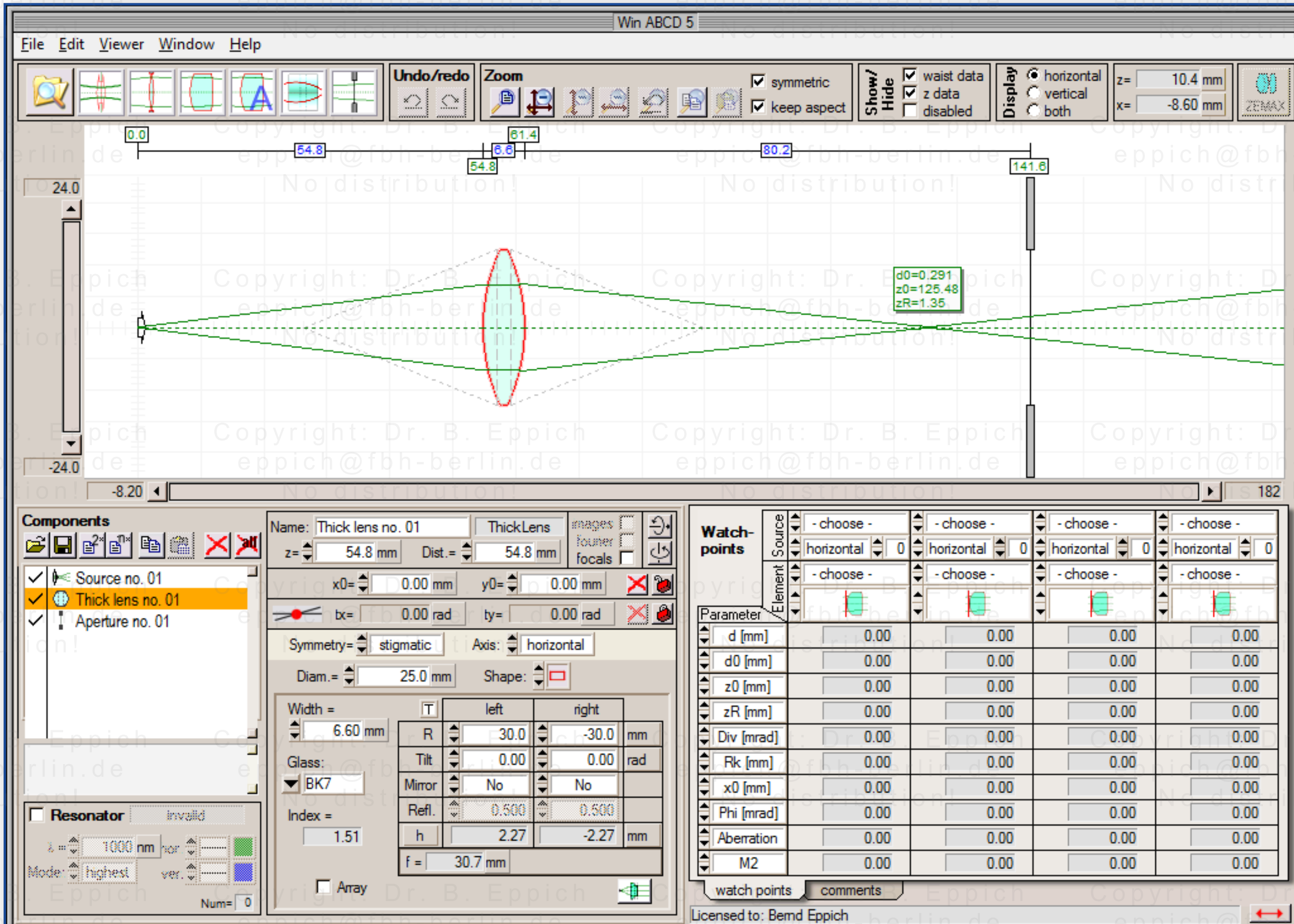
$$\bar{x} = \langle x \rangle = \frac{1}{P} \int I(x)\, x\, dx$$



$$\begin{pmatrix} x_2 \\ u_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ u_1 \end{pmatrix}$$

Benefits: Layout of optical systems

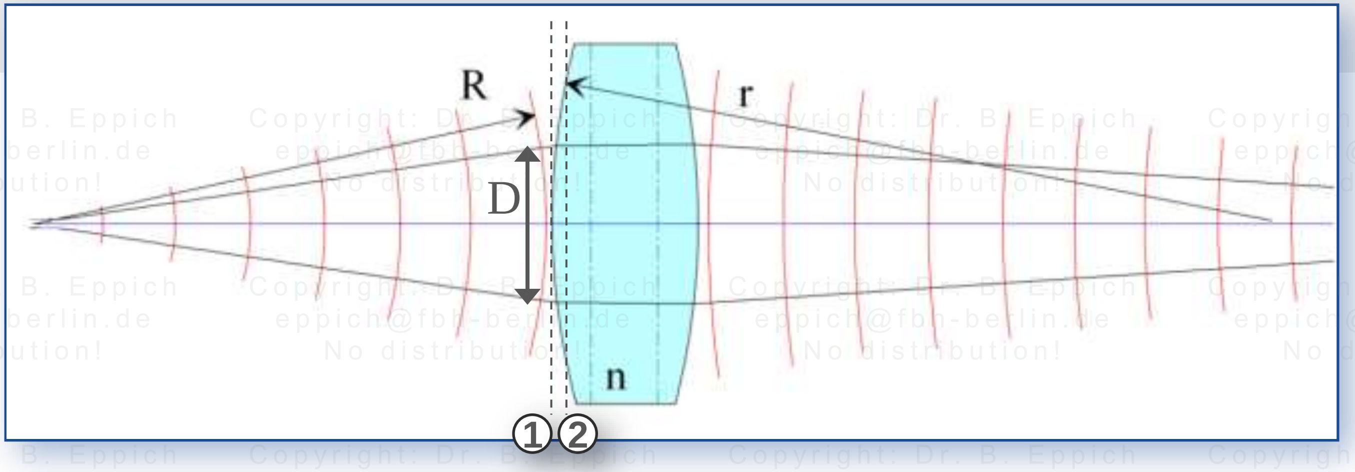
Example: *WinABCD*



(3D)

Leibniz Ferdinand-Braun-Institut

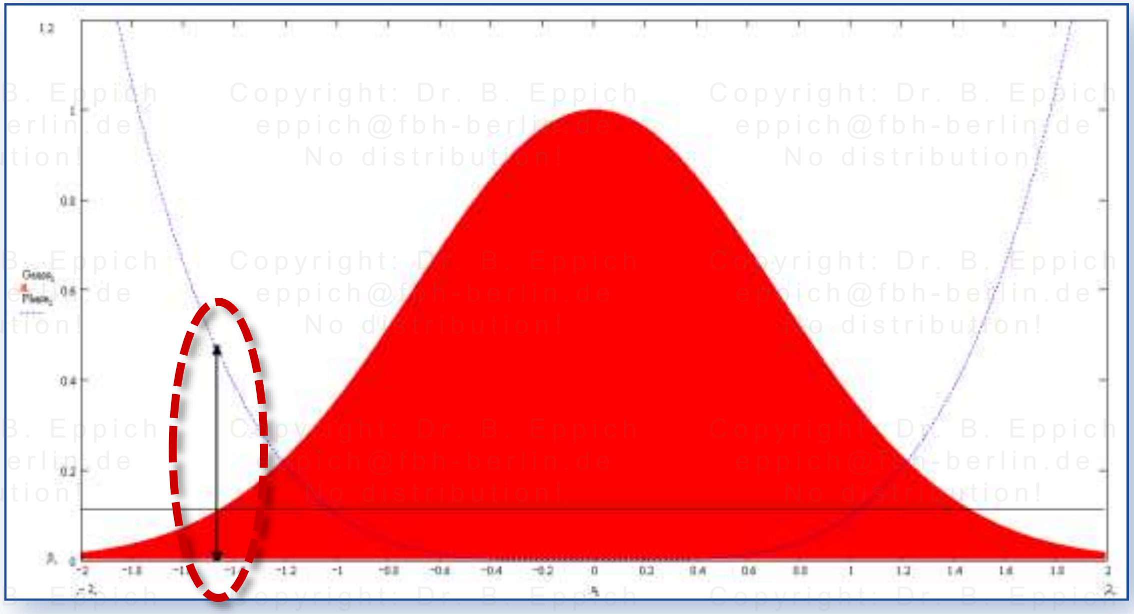
Benefits: Estimation of aberrations



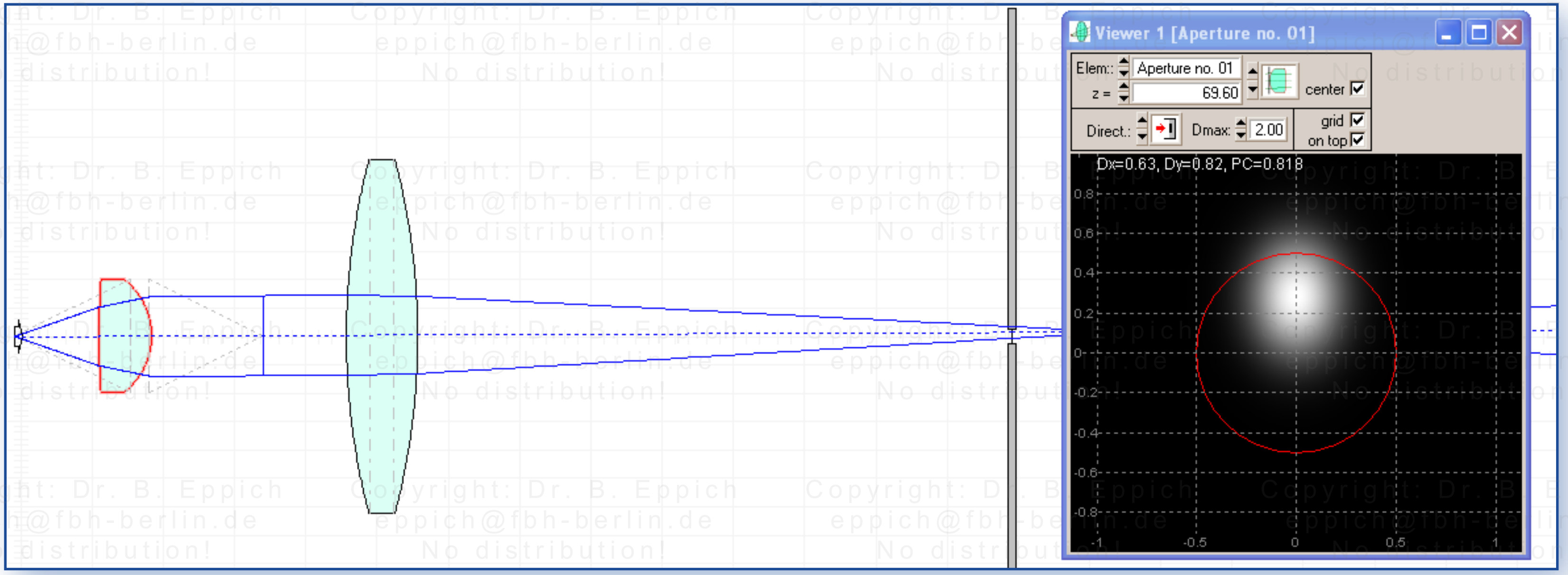
Spherical aberrations:

$$E_2(r) = E_1(r)\cdot e^{ik\left(ar^2+\mathbf{b}r^4\right)}$$

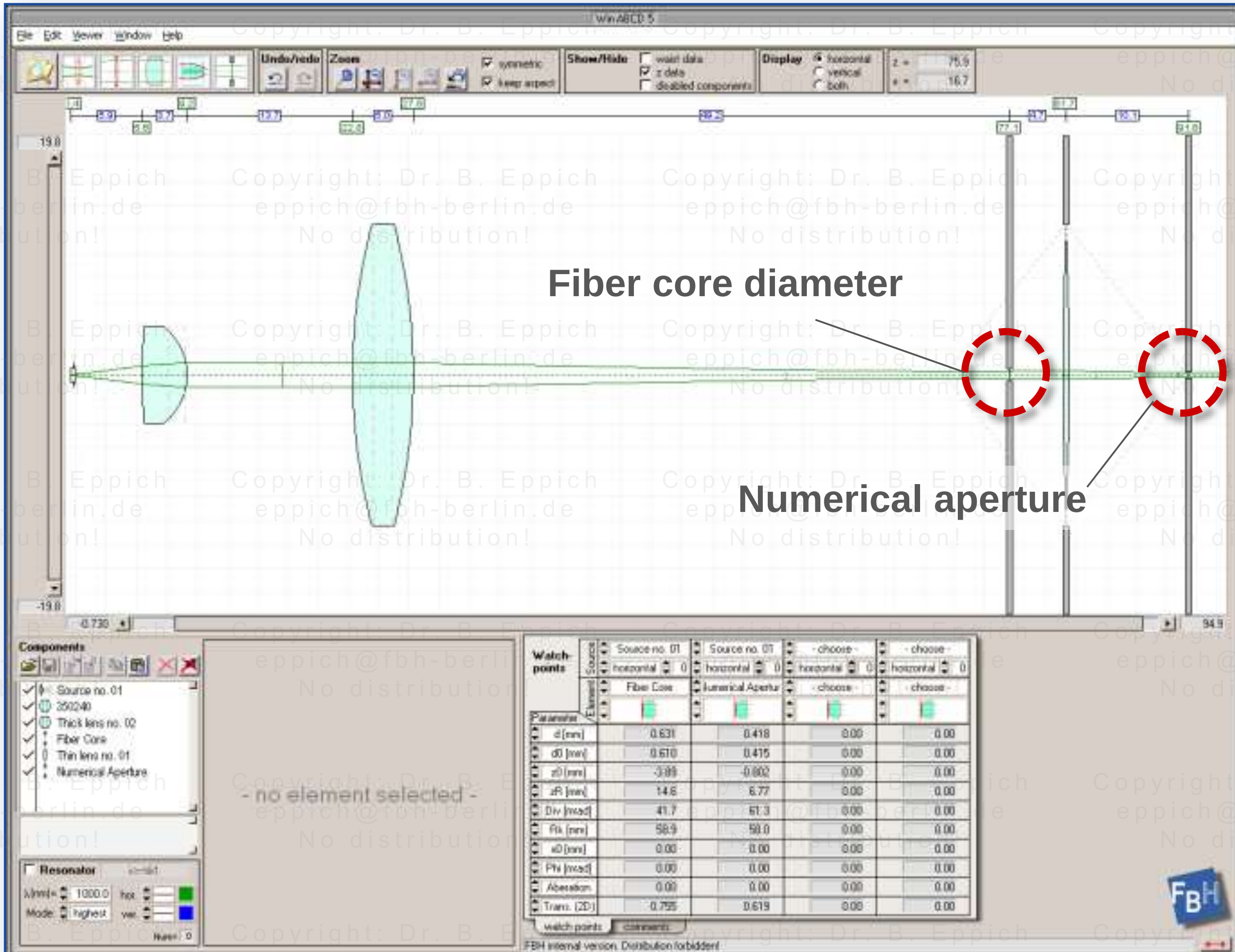
$$b = f(R, r, n)$$



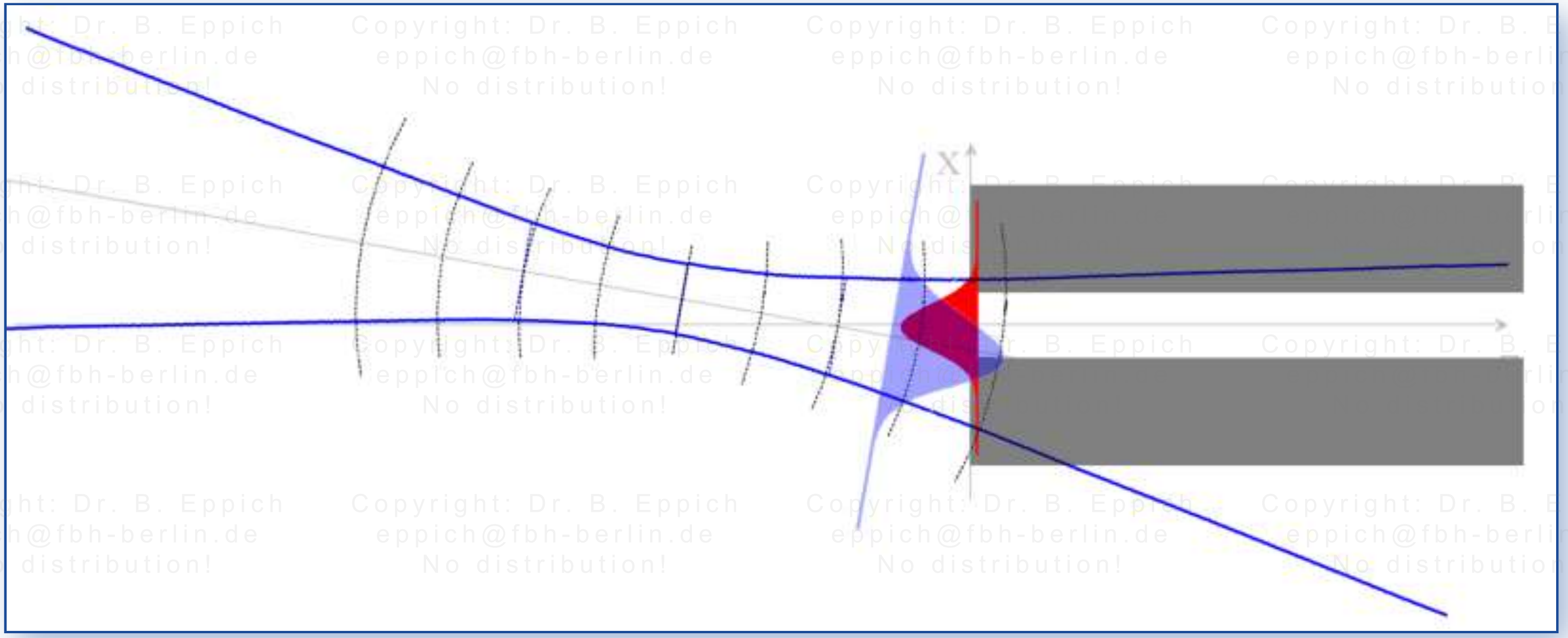
Benefits: Estimation 2D transmission



Benefits: Estimation multi-mode fiber coupling



Benefits: Monomode fiber coupling



- Valid only for coupling Gaussian beams into monomode fibers
- Coupling efficiency depends on
  - Transversal position of beam center
  - Angle of incidence
  - Radius of phase curvature (or: beam waist position)

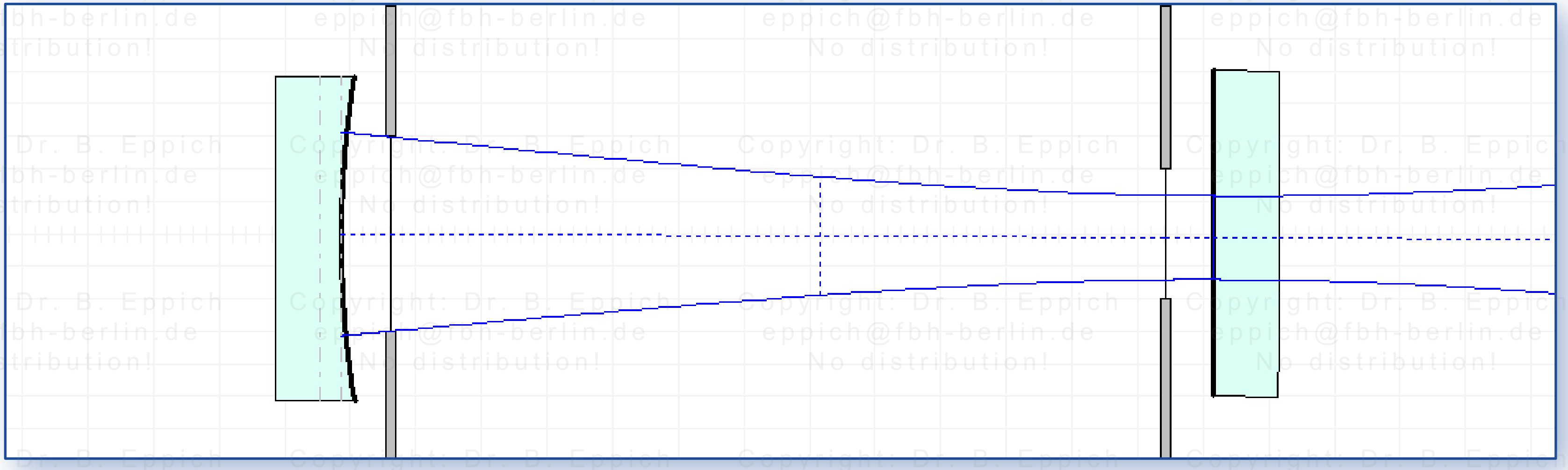
Resonators

Side note: Theory of stable resonators

- **Calculation self-consistent q-parameters → fundamental mode**

$$q = \frac{A q + B}{C q + D}$$

- **Calculation of the highest order lossless mode with same q-Parameter → M^2**



Second order beam width ellipse

$$I(x,y) \rightarrow \langle x^2 \rangle_c = \frac{1}{P} \int I(x,y)(x-\bar{x})^2 \, dx \, dy \rightarrow D_x = 4\sqrt{\langle x^2 \rangle_c}$$

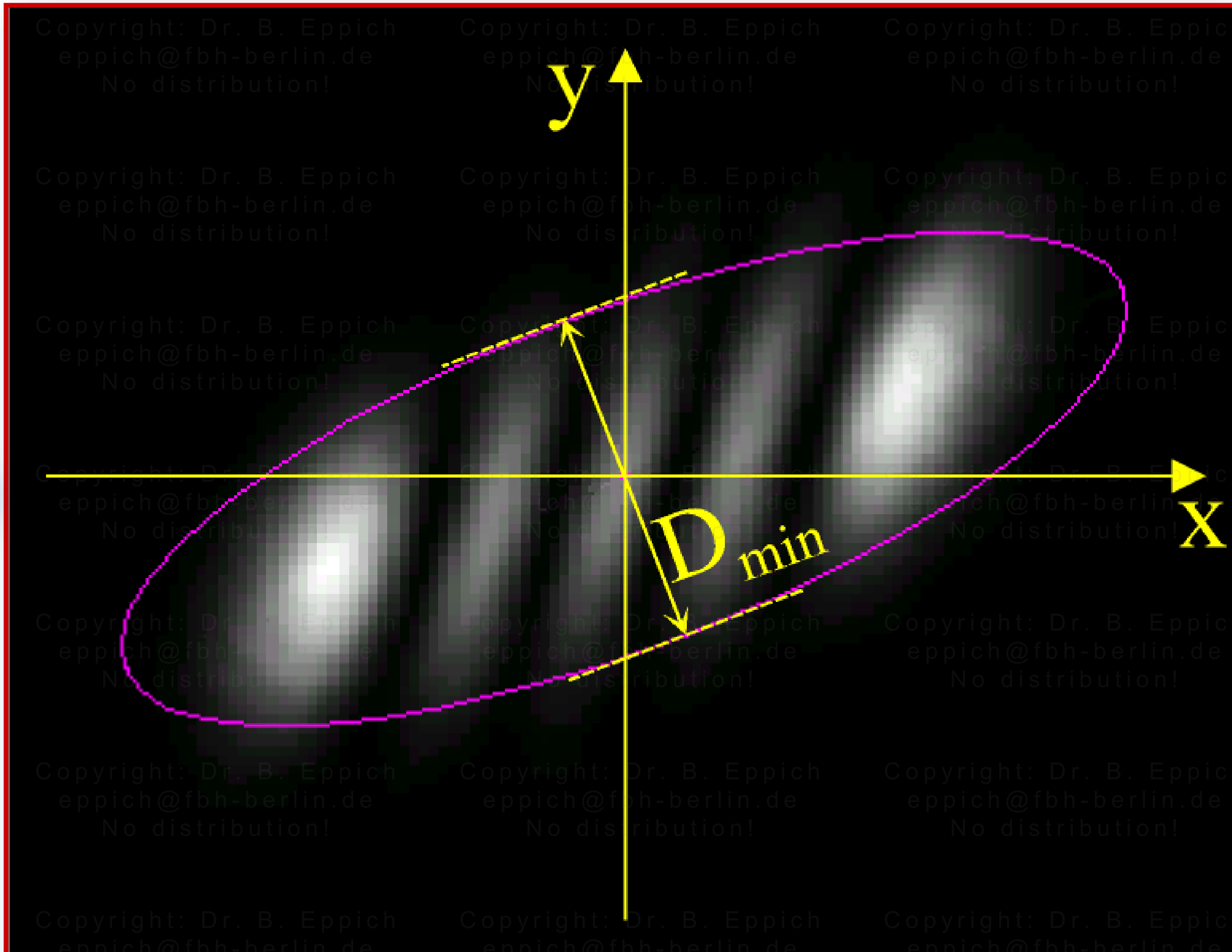
$$I(x,y) \rightarrow \langle y^2 \rangle_c = \frac{1}{P} \int I(x,y)(y-\bar{y})^2 \, dx \, dy \rightarrow D_y = 4\sqrt{\langle y^2 \rangle_c}$$

$$I(x,y) \rightarrow \langle xy \rangle_c = \frac{1}{P} \int I(x,y)(x-\bar{x})(y-\bar{y}) \, dx \, dy$$

$$D_\alpha = 4\sqrt{\cos^2\alpha \, \langle x^2 \rangle_c + \sin^2\alpha \, \langle y^2 \rangle_c + 2\sin\alpha \cos\alpha \, \langle xy \rangle_c}$$

Second order beam width ellipse

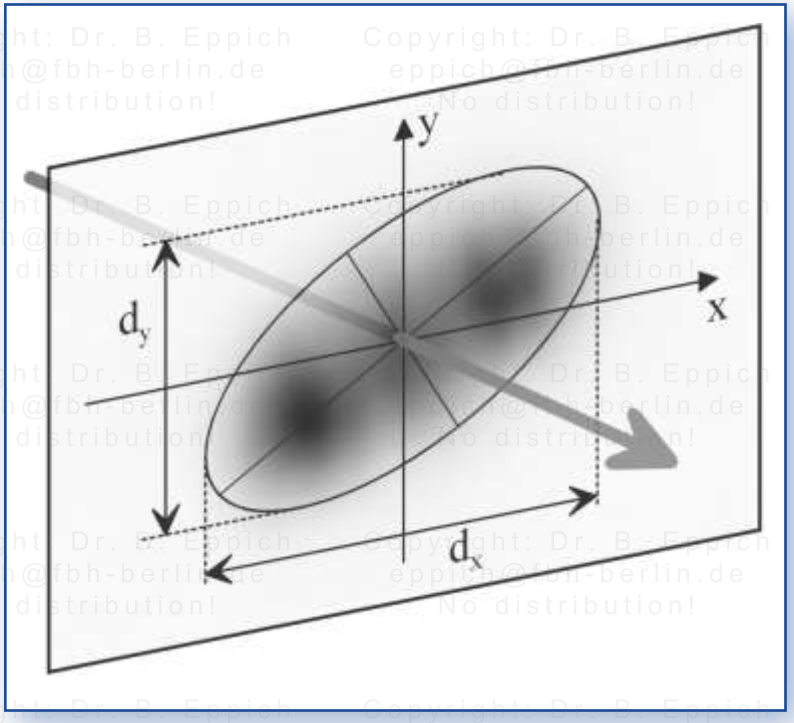
$$\langle x^2 \rangle_c , \langle xy \rangle_c , \langle y^2 \rangle_c$$



Second order beam width ellipse

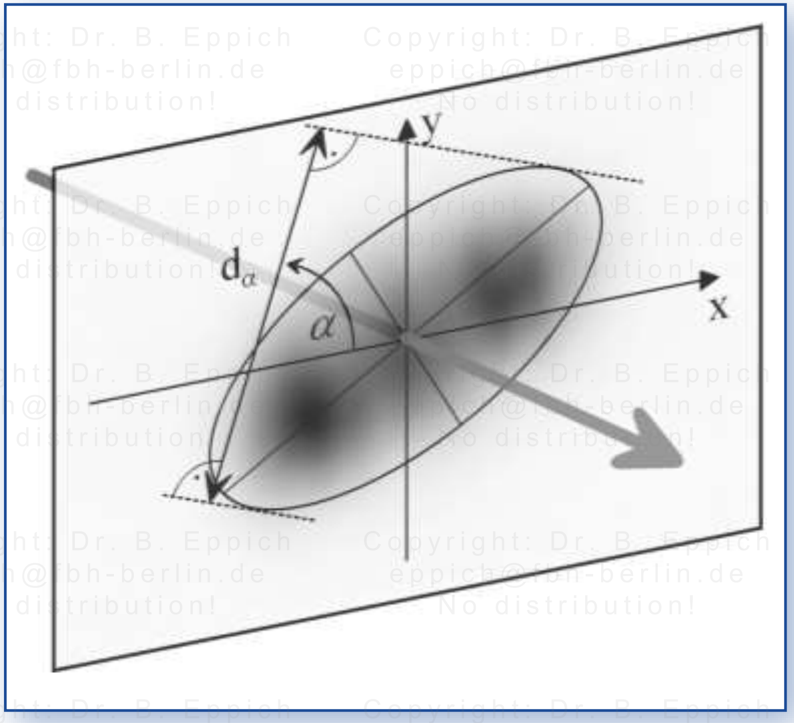
$$D_x = 4\sqrt{\langle x^2 \rangle_c}$$

$$D_y = 4\sqrt{\langle y^2 \rangle_c}$$



Second order beam width ellipse

$$D_\alpha = 4\sqrt{\cos^2\alpha \left\langle x^2 \right\rangle + \sin^2\alpha \left\langle y^2 \right\rangle + 2\sin\alpha\,\cos\alpha \left\langle xy \right\rangle}$$



Second order beam width ellipse

$$D_{\max} = 2\sqrt{2}\left\{\left(\langle x^2\rangle_c + \langle y^2\rangle_c\right) + \left[\left(\langle x^2\rangle_c - \langle y^2\rangle_c\right)^2 + 4\langle xy\rangle_c^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}$$

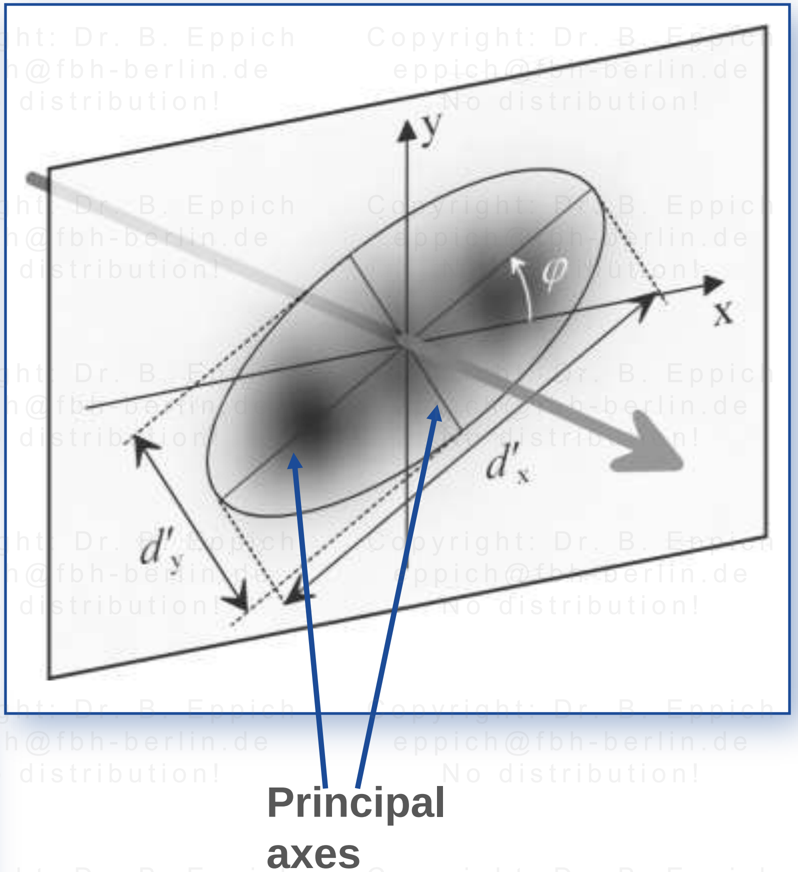
$$D_{\min} = 2\sqrt{2}\left\{\left(\langle x^2\rangle_c + \langle y^2\rangle_c\right) - \left[\left(\langle x^2\rangle_c - \langle y^2\rangle_c\right)^2 + 4\langle xy\rangle_c^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}$$

$$\varphi = \frac{1}{2}\operatorname{atan}\left(\frac{2\langle xy\rangle_c}{\langle x^2\rangle_c - \langle y^2\rangle_c}\right)$$

$$\varepsilon = \operatorname{sgn}\left(\langle x^2\rangle_c - \langle y^2\rangle_c\right)$$

$$D'_x = 2\sqrt{2}\left\{\left(\langle x^2\rangle_c + \langle y^2\rangle_c\right) + \varepsilon\left[\left(\langle x^2\rangle_c - \langle y^2\rangle_c\right)^2 + 4\langle xy\rangle_c^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}$$

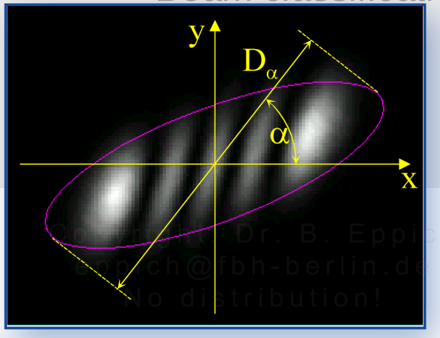
$$D'_y = 2\sqrt{2}\left\{\left(\langle x^2\rangle_c + \langle y^2\rangle_c\right) - \varepsilon\left[\left(\langle x^2\rangle_c - \langle y^2\rangle_c\right)^2 + 4\langle xy\rangle_c^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}$$



Content

„Roundness“

$$D_\alpha = 4\sqrt{\cos^2\alpha\,\langle x^2\rangle_c + \sin^2\alpha\,\langle y^2\rangle_c + 2\sin\alpha\,\cos\alpha\,\langle xy\rangle_c}$$



A beam profile is „*round*“ if

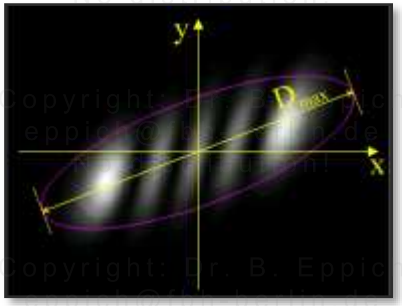
- **beam width is independent of direction →** $d(\alpha)$ = const.
- $\langle x^2\rangle = \langle y^2\rangle$ **and** $\langle xy\rangle = 0$

→ a square top hat profile is „round“:

More practical definition for roundness:

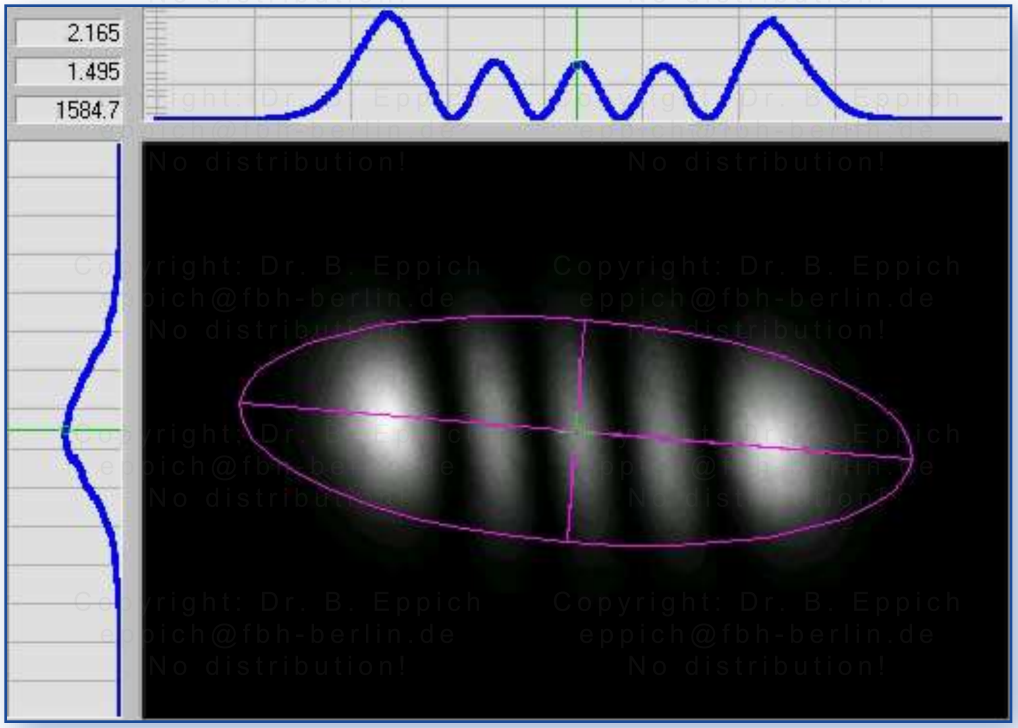
$$D_{\max} < 1.15 \cdot d_{\min}$$

If a beam profile is *elliptical*....



... it has an *orientation*.

Rotating second order beam width ellipse



„Roundness“ and orientation may changed under propagation!

„Geometrical“ beam classification

Stigmatic:
- „circular“ in any plane under free propagation
- fixed orientation behind cylindrical lens
- 3 Parameter needed (e.g. D_0, z_0, z_R)

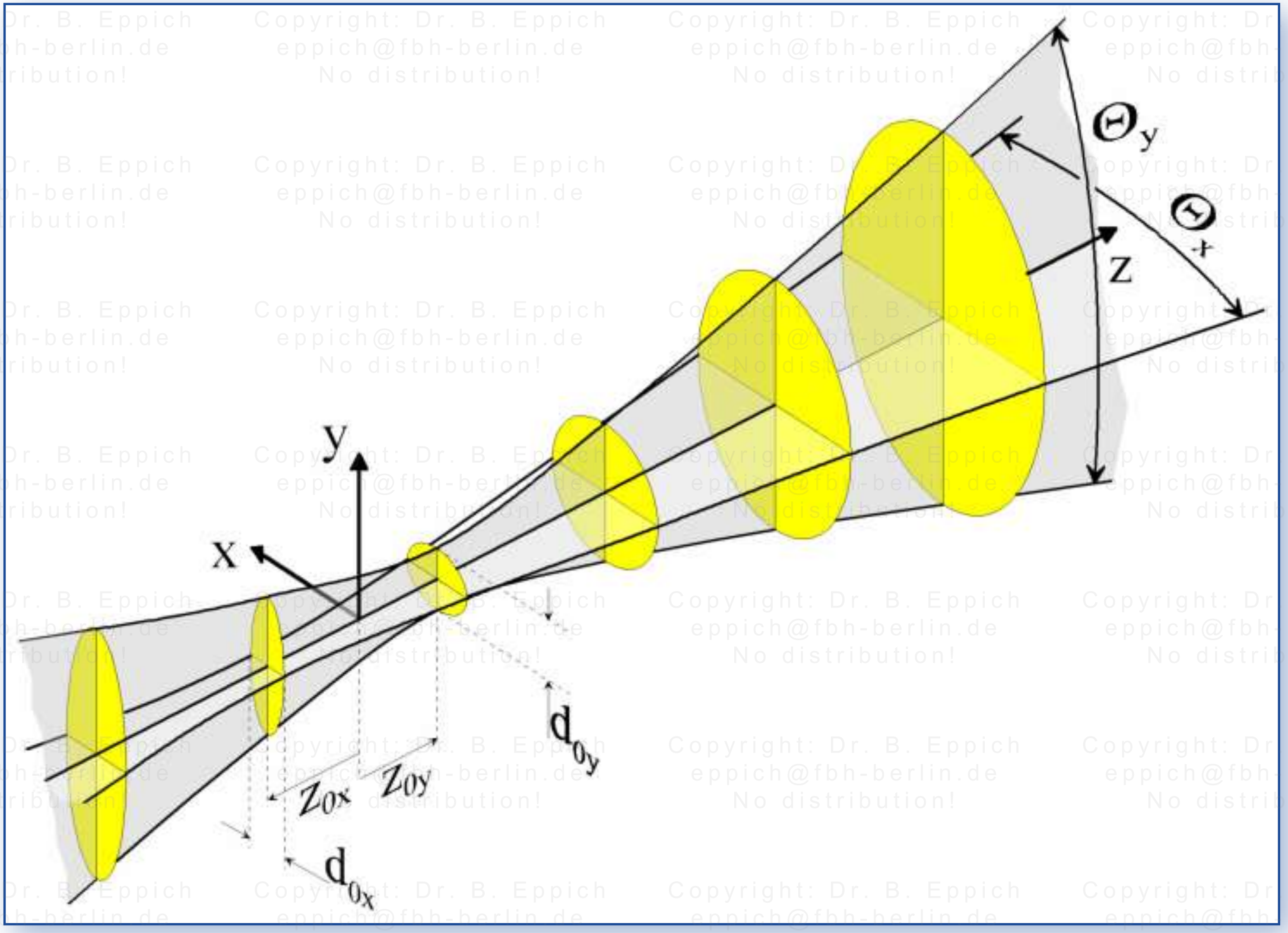
Simple astigmatic:
- non-circular, fixed orientation under free propagation
- same fixed orientation behind aligned cylindrical lens
- 6(7) Parameter needed (e.g. D_{0x}, z_{0x}, z_{Rx}, D_{0y}, z_{0y}, z_{Ry}, α)

Pseudo stigmatic:
- „circular“ in any plane under free propagation
- changing orientation behind any cylindrical lens
- 4 Parameter needed (e.g. D_0, z_0, z_R, t)

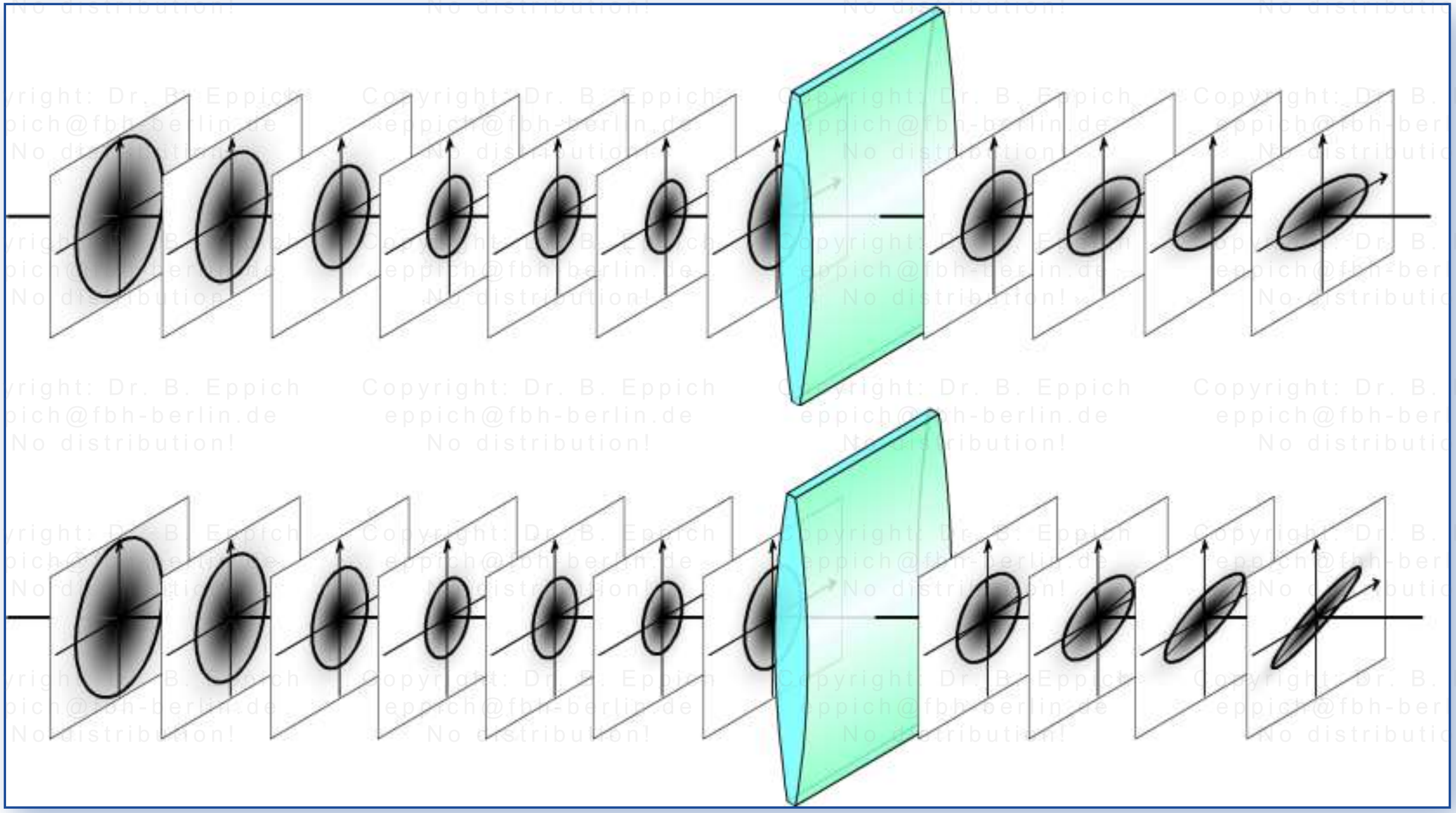
Pseudo simple astigmatic:
- non-circular, fixed orientation under free propagation
- changing orientation behind aligned cylindrical lens
- 7(8) Parameter needed (e.g. D_{0x}, z_{0x}, z_{Rx}, D_{0y}, z_{0y}, z_{Ry}, α, t)

General astigmatic:
- changing orientation under free propagation
- 10 Parameter needed (see later...)

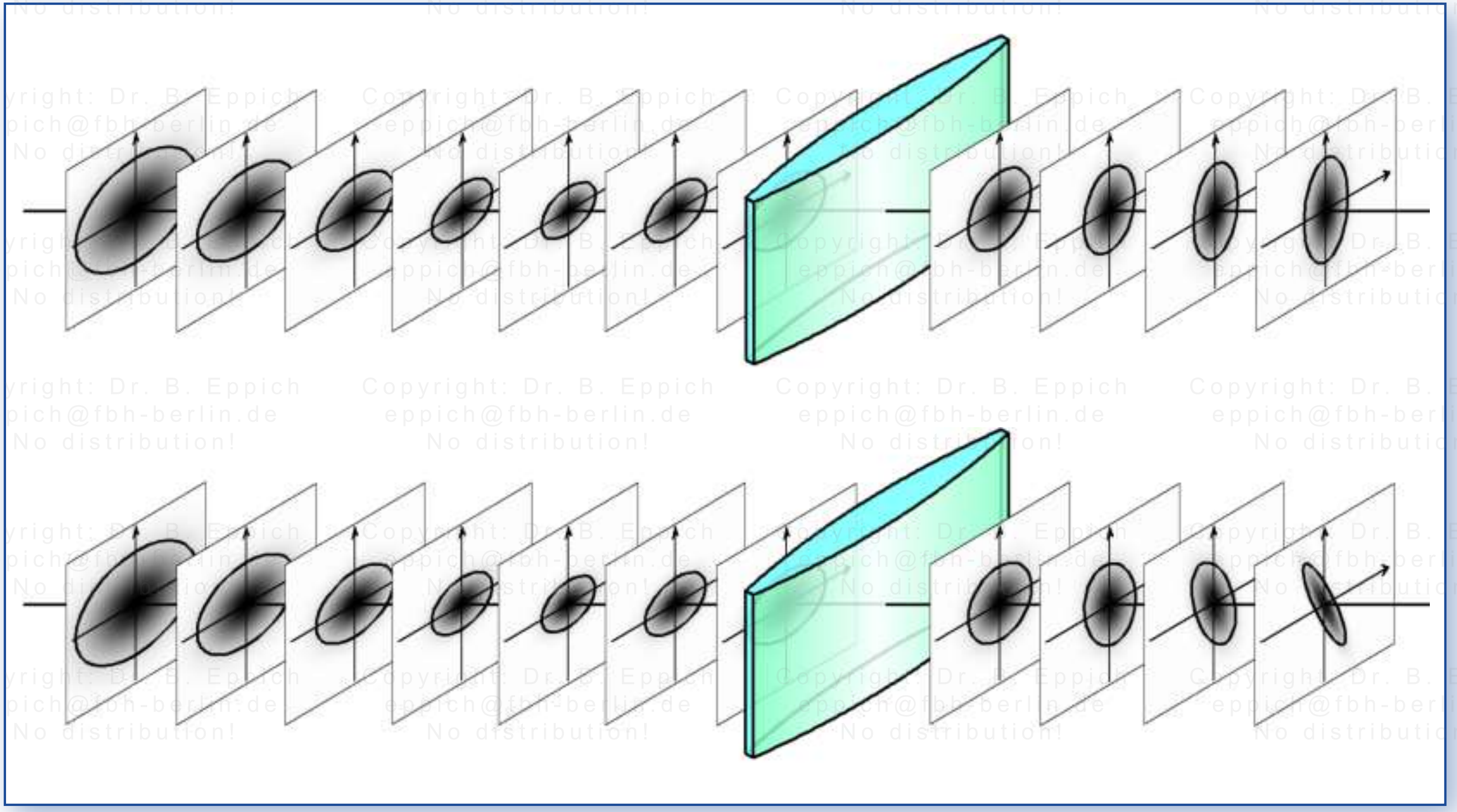
Simple astigmatism



Stigmatic versus pseudo stigmatic beams



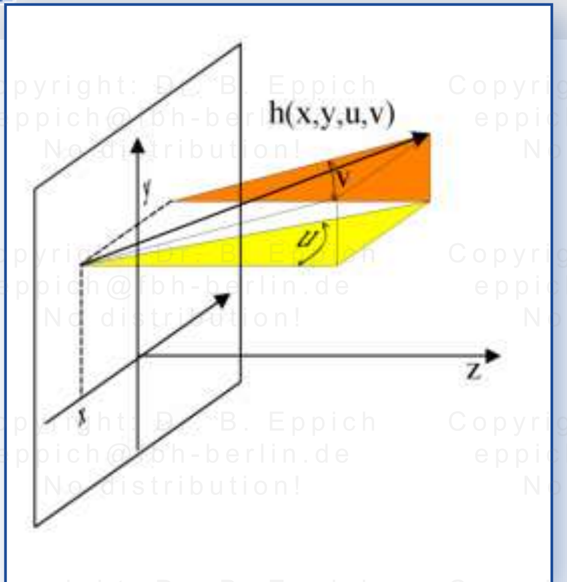
Simple-astigmatic versus pseudo-simple-astigmatic beams



Content

- Motivation
- ISO standard
- Beams and optical systems
- Beam diameter definitions
- Beam classification
- The ten second order moments
- Measurement of the ten second order moments

The four-dimensional Wigner distribution



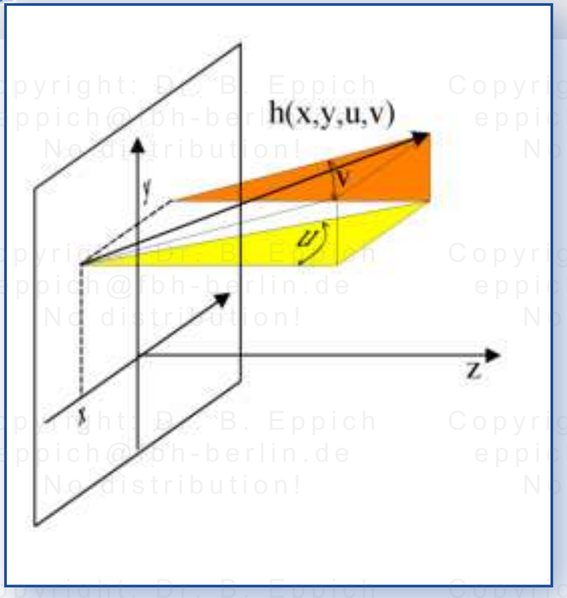
h(x,y,u,v) **gives amount of power passing the plane at point (x,y) with direction (u,v).**

$$I(x,y)=\int h(x,y,u,v)\,du\,dv$$ → **total power passing point *(x,y)***

$$I_F(u,v)=\int h(x,y,u,v)\,dx\,dy$$ → **total power passing in direction *(u,v)***

$$P=\int I(x,y)\,dx\,dy=\int I_F(u,v)\,du\,dv=\int h(x,y,u,v)\,dx\,dy\,du\,dv$$ → **total power**

The four-dimensional Wigner distribution



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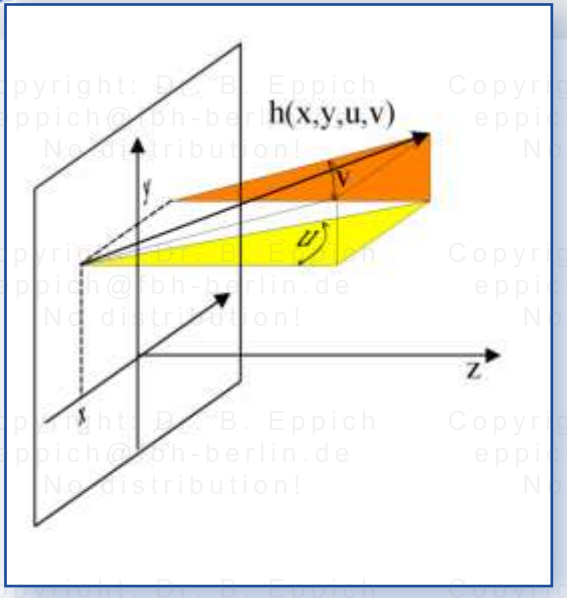
$$I(x,y)=\int h(x,y,u,v)\,du\,dv$$ → **total power passing point *(x,y)***

$$\left\langle x^n y^m \right\rangle_c = \frac{1}{P}\int I(x,y)(x-\bar{x})^n (y-\bar{y})^m\,dx\,dy = \frac{1}{P}\int h(x,y,u,v)(x-\bar{x})^n (y-\bar{y})^m\,dx\,dy\,du\,dv$$

$$n+m=2$$

$$\left\langle x^2 \right\rangle_c \qquad \left\langle x\,y \right\rangle_c \qquad \left\langle y^2 \right\rangle_c$$

The four-dimensional Wigner distribution



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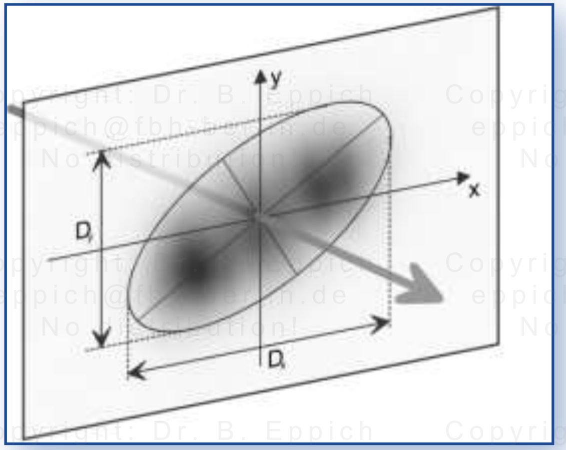
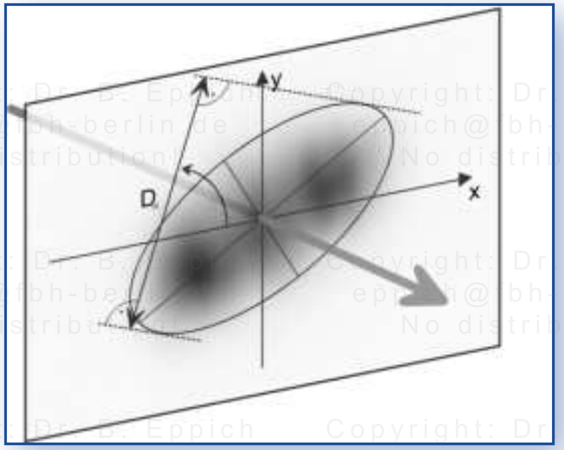
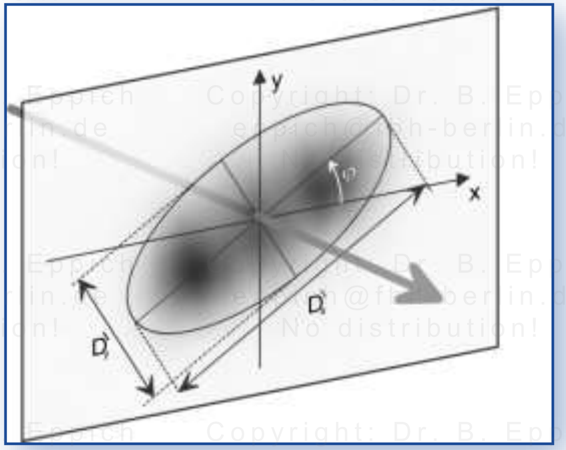
$$I(x,y)=\int h(x,y,u,v)\,du\,dv$$

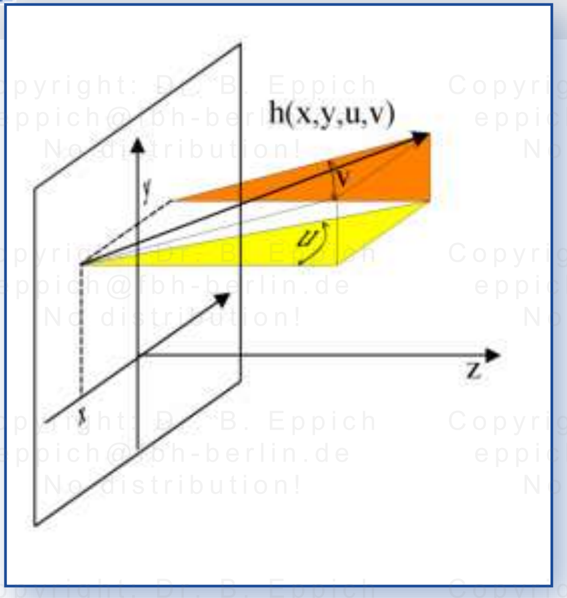
$$\langle x^2\rangle_c \qquad \langle x\,y\rangle_c \qquad \langle y^2\rangle_c$$







The four-dimensional Wigner distribution



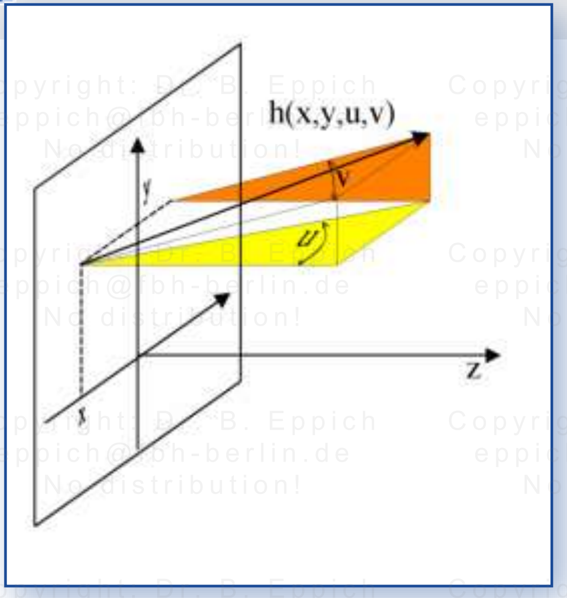
h(x,y,u,v) **gives amount of power passing the plane at point (x,y) with direction (u,v).**

$$I_F(u,v) = \int h(x,y,u,v)\,dx\,dy$$ → **total power passing in direction *(u,v)***

$$\langle u^k v^l \rangle_c = \frac{1}{P}\int I_F(u,v)(u-\bar{u})^k (v-\bar{v})^l\,du\,dv = \frac{1}{P}\int h(x,y,u,v)(u-\bar{u})^k (v-\bar{v})^l\,dx\,dy\,du\,dv \qquad k+l=2$$

$\langle u^2 \rangle_c$ $\langle u\,v \rangle_c$ $\langle v^2 \rangle_c$

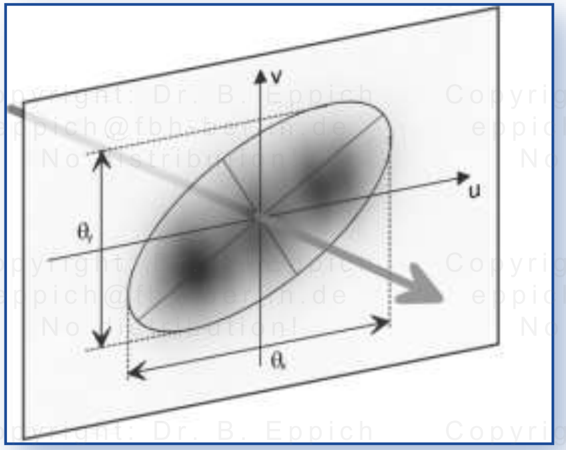
The four-dimensional Wigner distribution

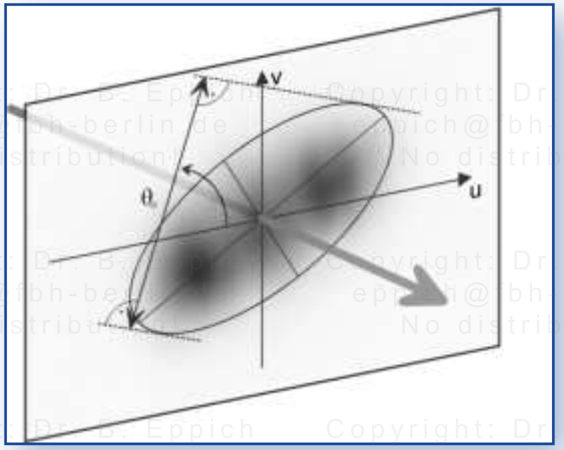


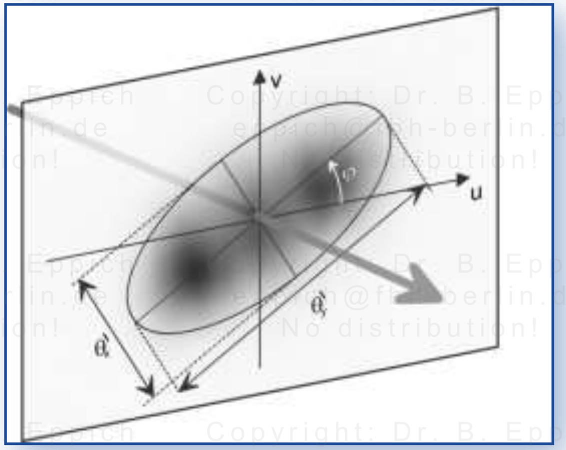
h(x,y,u,v) **gives amount of power passing the plane at point (x,y) with direction (u,v).**

$$I_F(u,v)=\int h(x,y,u,v)\,dx\,dy$$

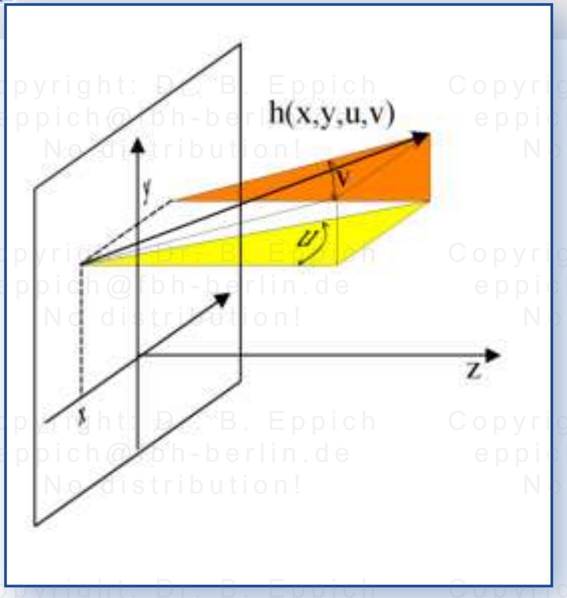
$$\langle u^2\rangle_c \qquad \langle u\,v\rangle_c \qquad \langle v^2\rangle_c$$







The four-dimensional Wigner distribution



***h(x,y,u,v)* gives amount of power passing the plane at point (x,y) with direction (u,v).**

$$\left\langle x^n y^m \right\rangle_c = \frac{1}{P}\int h(x,y,u,v)\left(x-\bar{x}\right)^n \left(y-\bar{y}\right)^m dx\,dy\,du\,dv \qquad n+m=2$$

$$\left\langle x^2 \right\rangle_c \quad \left\langle x\,y \right\rangle_c \quad \left\langle y^2 \right\rangle_c$$

near field

$$\left\langle u^k v^l \right\rangle_c = \frac{1}{P}\int h(x,y,u,v)\left(u-\bar{u}\right)^k \left(v-\bar{v}\right)^l dx\,dy\,du\,dv \qquad k+l=2$$

$$\left\langle u^2 \right\rangle_c \quad \left\langle u\,v \right\rangle_c \quad \left\langle v^2 \right\rangle_c$$

far field

$$\left\langle x^n y^m u^k v^l \right\rangle_c = \frac{1}{P}\int h(x,y,u,v)\left(x-\bar{x}\right)^n \left(y-\bar{y}\right)^m \left(u-\bar{u}\right)^k \left(v-\bar{v}\right)^l dx\,dy\,du\,dv \qquad n+m+k+l=2$$

$$\left\langle x\,u \right\rangle_c \quad \left\langle x\,v \right\rangle_c$$

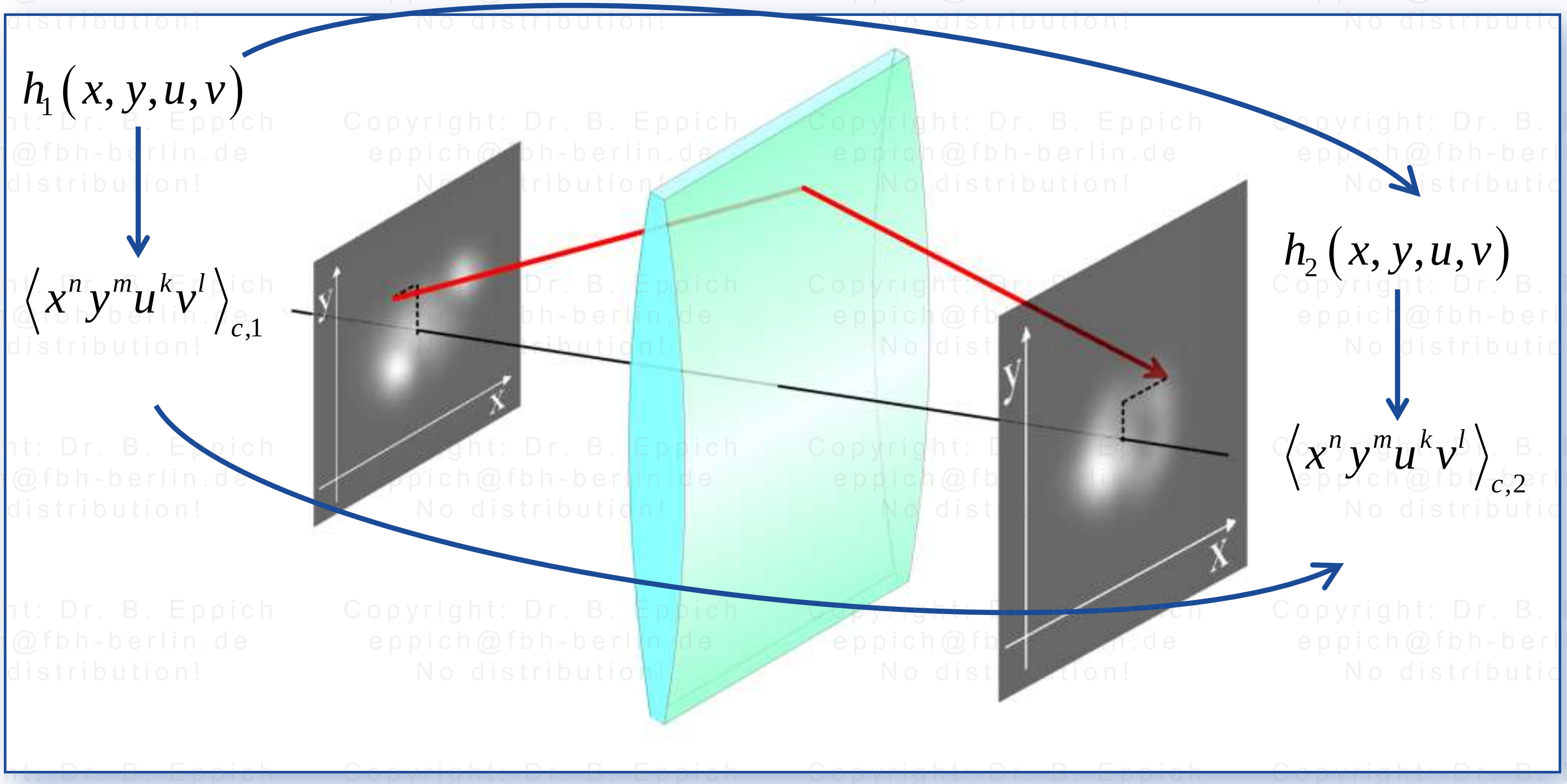
$$\left\langle y\,u \right\rangle_c \quad \left\langle y\,v \right\rangle_c$$

phase paraboloid + twist

Propagation of the second order moments

$$\left\langle x^n y^m u^k v^l \right\rangle_c = \frac{1}{P}\int h(x,y,u,v)(x-\bar{x})^n (y-\bar{y})^m (u-\bar{u})^k (v-\bar{v})^l \, dx\, dy\, du\, dv$$

$$n+m+k+l=2$$



Propagation of the second order moments

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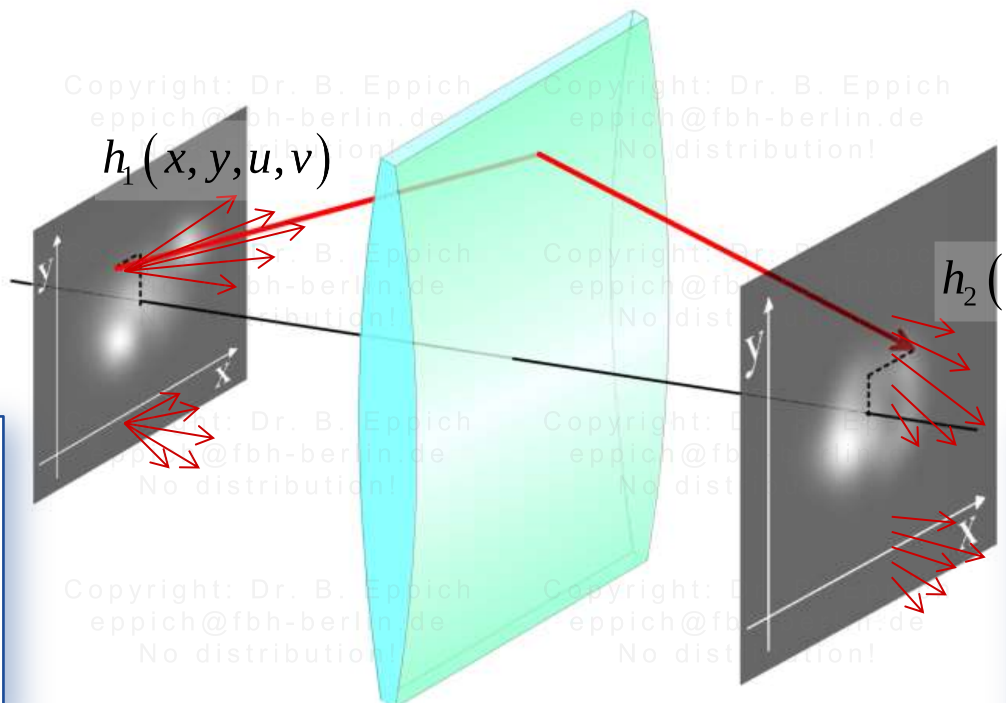
$$n+m+k+l=2$$



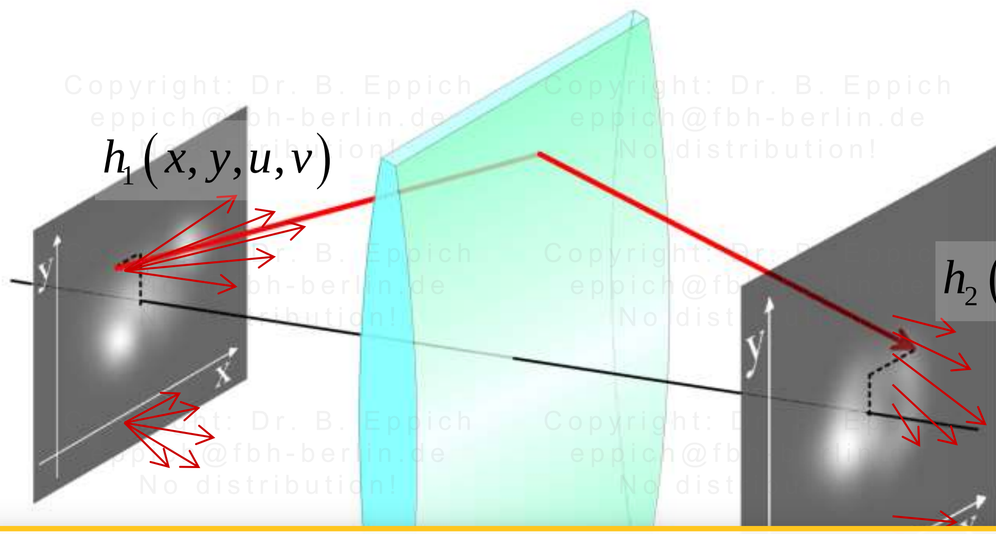
$$\begin{pmatrix} x_1 \\ y_1 \\ u_1 \\ v_1 \end{pmatrix} = \mathbf{S}^{-1} \cdot \begin{pmatrix} x_2 \\ y_2 \\ u_2 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ y_2 \\ u_2 \\ v_2 \end{pmatrix} = \mathbf{S} \cdot \begin{pmatrix} x_1 \\ y_1 \\ u_1 \\ v_1 \end{pmatrix}$$

Propagation of the second order moments

$$\left\langle x^n y^m u^k v^l \right\rangle_c = \frac{1}{P}\int h(x,y,u,v)(x-\bar{x})^n (y-\bar{y})^m (u-\bar{u})^k (v-\bar{v})^l \, dx\,dy\,du\,dv$$

$$n+m+k+l=2$$

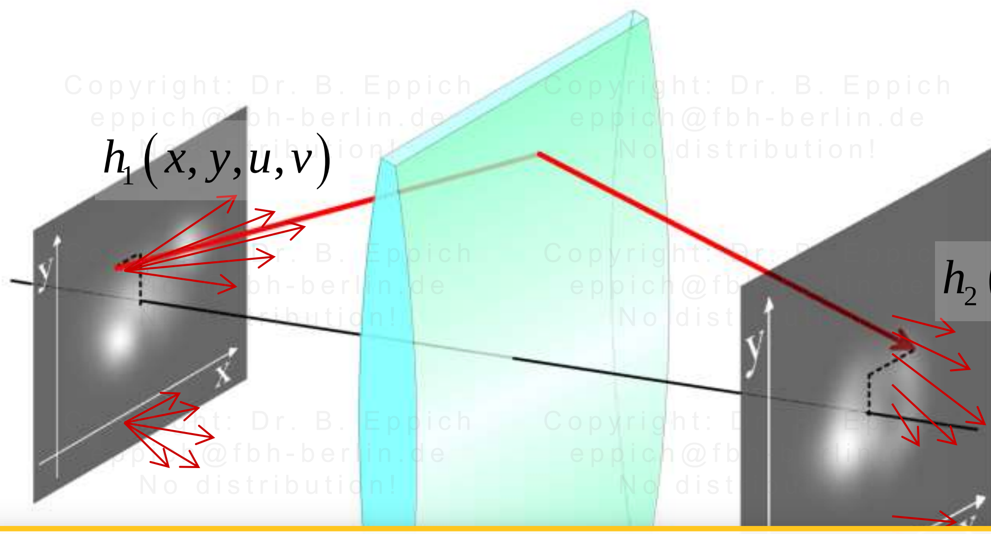


$$h_2(x_2,y_2,u_2,v_2) = h_1(x_1,y_1,u_1,v_1) \quad \textbf{with} \quad \begin{pmatrix} x_2 \\ y_2 \\ u_2 \\ v_2 \end{pmatrix} = \mathbf{S}\cdot\begin{pmatrix} x_1 \\ y_1 \\ u_1 \\ v_1 \end{pmatrix}$$

Propagation of the second order moments

$$\left\langle x^n y^m u^k v^l \right\rangle_c = \frac{1}{P}\int h(x,y,u,v)(x-\bar{x})^n (y-\bar{y})^m (u-\bar{u})^k (v-\bar{v})^l \, dx\,dy\,du\,dv$$

$$n+m+k+l=2$$



$$h_2(\vec{\xi}) = h_1(\mathbf{S}^{-1}\cdot\vec{\xi})$$ **with** $$\vec{\xi} = \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix}$$

Propagation of the second order moments

$$\left\langle x^n y^m u^k v^l \right\rangle_c = \frac{1}{P} \int h(x,y,u,v)(x-\bar{x})^n (y-\bar{y})^m (u-\bar{u})^k (v-\bar{v})^l \, dx\, dy\, du\, dv$$

$$n+m+k+l=2$$

$$h_2\left(\vec{\xi}\right) = h_1\left(\mathbf{S}^{-1}\cdot\vec{\xi}\right)$$

$$\mathbf{P} = \begin{pmatrix} \langle x^2\rangle_c & \langle xy\rangle_c & \langle xu\rangle_c & \langle xv\rangle_c \\ \langle xy\rangle_c & \langle y^2\rangle_c & \langle yu\rangle_c & \langle yv\rangle_c \\ \langle xu\rangle_c & \langle yu\rangle_c & \langle u^2\rangle_c & \langle uv\rangle_c \\ \langle xv\rangle_c & \langle yv\rangle_c & \langle uv\rangle_c & \langle v^2\rangle_c \end{pmatrix}$$

Beam matrix,
Moments matrix

$$\tilde{\mathbf{P}}_2 = \mathbf{S}\cdot\tilde{\mathbf{P}}_1\cdot\mathbf{S}^T$$

Free space propagation

Free propagation:

System matrix:

$$S = \begin{pmatrix} 1 & 0 & z & 0 \\ 0 & 1 & 0 & z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\langle x^2 \rangle_{out}(z) = \langle x^2 \rangle_{in} + 2\,z \langle xu \rangle_{in} + z^2 \langle u^2 \rangle_{in}$$

$$\langle y^2 \rangle_{out}(z) = \langle y^2 \rangle_{in} + 2\,z \langle yv \rangle_{in} + z^2 \langle v^2 \rangle_{in}$$

$$\langle xy \rangle_{out}(z) = \langle xy \rangle_{in} + z \left(\langle xv \rangle_{in} + \langle yu \rangle_{in} \right) + z^2 \langle uv \rangle_{in}$$

Free space propagation

Free propagation:

System matrix:

$$S=\begin{pmatrix}1&0&z&0\\0&1&0&z\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

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$$\langle xy\rangle_{out}(z)=\langle xy\rangle_{in}+z\left(\langle xv\rangle_{in}+\langle yu\rangle_{in}\right)+z^2\langle uv\rangle_{in}$$

$$D_x=4\sqrt{\langle x^2\rangle_c}\rightarrow D_x(z)=D_{x,0}\sqrt{1+\left(\frac{z-z_{0,x}}{z_{R,x}}\right)^2}$$

$$D_y=4\sqrt{\langle y^2\rangle_c}\rightarrow D_y(z)=D_{y,0}\sqrt{1+\left(\frac{z-z_{0,y}}{z_{R,y}}\right)^2}$$

$$\langle x^2\rangle_{c,\alpha}=\cos^2\alpha\langle x^2\rangle_c+2\cos\alpha\sin\alpha\langle xy\rangle+\sin^2\langle y^2\rangle$$

$$D_\alpha=4\sqrt{\langle x^2\rangle_{c,\alpha}}\rightarrow D_\alpha(z)=D_{\alpha,0}\sqrt{1+\left(\frac{z-z_{0,\alpha}}{z_{R,\alpha}}\right)^2}$$

Free space propagation

Directional beam properties

Parameter		horizontal	vertical	azimuthal
Diameter	D	$4\sqrt{\langle x^2\rangle_c}$	$4\sqrt{\langle y^2\rangle_c}$	$4\sqrt{\langle x^2\rangle_{c,\alpha}}$
Waist diameter	D_0	$4\sqrt{\langle x^2\rangle_c - \frac{\langle xu\rangle_c^2}{\langle u^2\rangle_c}}$	$4\sqrt{\langle y^2\rangle_c - \frac{\langle yv\rangle_c^2}{\langle v^2\rangle_c}}$	$4\sqrt{\langle x^2\rangle_{c,\alpha} - \frac{\langle xu\rangle_{c,\alpha}^2}{\langle u^2\rangle_{c,\alpha}}}$
Waist position	z_0	$-\frac{\langle xu\rangle_c}{\langle u^2\rangle_c}$	$-\frac{\langle yv\rangle_c}{\langle v^2\rangle_c}$	$-\frac{\langle xu\rangle_{c,\alpha}}{\langle u^2\rangle_{c,\alpha}}$
Rayleigh length	z_R	$\sqrt{\frac{\langle x^2\rangle_c}{\langle u^2\rangle_c} - \frac{\langle xu\rangle_c^2}{\langle u^2\rangle_c^2}}$	$\sqrt{\frac{\langle y^2\rangle_c}{\langle v^2\rangle_c} - \frac{\langle yv\rangle_c^2}{\langle v^2\rangle_c^2}}$	$\sqrt{\frac{\langle x^2\rangle_{c,\alpha}}{\langle u^2\rangle_{c,\alpha}} - \frac{\langle xu\rangle_{c,\alpha}^2}{\langle u^2\rangle_{c,\alpha}^2}}$
Divergence	θ	$4\sqrt{\langle u^2\rangle_c}$	$4\sqrt{\langle v^2\rangle_c}$	$4\sqrt{\langle u^2\rangle_{c,\alpha}}$
Radius of phase curvature	R	$\frac{\langle x^2\rangle_c}{\langle xu\rangle_c}$	$\frac{\langle y^2\rangle_c}{\langle yv\rangle_c}$	$\frac{\langle x^2\rangle_{c,\alpha}}{\langle xu\rangle_{c,\alpha}}$
Beam propagation factor	M^2	$\frac{4\pi}{\lambda}\sqrt{\langle x^2\rangle_c\langle u^2\rangle_c - \langle xu\rangle_c^2}$	$\frac{4\pi}{\lambda}\sqrt{\langle y^2\rangle_c\langle v^2\rangle_c - \langle yv\rangle_c^2}$	$\frac{4\pi}{\lambda}\sqrt{\langle x^2\rangle_{c,\alpha}\langle u^2\rangle_{c,\alpha} - \langle xu\rangle_{c,\alpha}^2}$

Propagation through separable ABCD-Systems (2D)

$$\mathbf{P}_{in} = \begin{pmatrix} \langle x^2\rangle_c & \langle xy\rangle_c & \langle xu\rangle_c & \langle xv\rangle_c \\ \langle xy\rangle_c & \langle y^2\rangle_c & \langle yu\rangle_c & \langle yv\rangle_c \\ \langle xu\rangle_c & \langle yu\rangle_c & \langle u^2\rangle_c & \langle uv\rangle_c \\ \langle xv\rangle_c & \langle yv\rangle_c & \langle uv\rangle_c & \langle v^2\rangle_c \end{pmatrix}_{in}$$

$$\mathbf{S}_{sep} = \begin{pmatrix} A_{xx} & 0 & B_{xx} & 0 \\ 0 & A_{yy} & 0 & B_{yy} \\ C_{xx} & 0 & D_{xx} & 0 \\ 0 & C_{yy} & 0 & D_{yy} \end{pmatrix}$$

(separable system)

$$\mathbf{P}_{out} = \mathbf{S}\cdot\mathbf{P}_{in}\cdot\mathbf{S}^T$$

$$\langle x^2\rangle_{c,out} = A_{xx}^2\langle x^2\rangle_{c,in} + 2A_{xx}B_{xx}\langle xu\rangle_{c,in} + B_{xx}^2\langle u^2\rangle_{c,in}$$

$$\langle xu\rangle_{c,out} = A_{xx}C_{xx}\langle x^2\rangle_{c,in} + \left(A_{xx}D_{xx} - B_{xx}C_{xx}\right)\langle xu\rangle_{c,in} + B_{xx}D_{xx}\langle u^2\rangle_{c,in}$$

$$\langle u^2\rangle_{c,out} = C_{xx}^2\langle x^2\rangle_{c,in} + 2C_{xx}D_{xx}\langle xu\rangle_{c,in} + D_{xx}^2\langle u^2\rangle_{c,in}$$

Propagation through separable ABCD-Systems (2D)

$$\mathbf{P}_{in} = \begin{pmatrix} \langle x^2 \rangle_c & \langle x\,y \rangle_c & \langle x u \rangle_c & \langle x\,v \rangle_c \\ \langle x\,y \rangle_c & \langle y^2 \rangle_c & \langle y\,u \rangle_c & \langle y\,v \rangle_c \\ \langle x u \rangle_c & \langle y\,u \rangle_c & \langle u^2 \rangle_c & \langle u\,v \rangle_c \\ \langle x\,v \rangle_c & \langle y\,v \rangle_c & \langle u\,v \rangle_c & \langle v^2 \rangle_c \end{pmatrix}_{in}$$

$$\mathbf{S}_{sep} = \begin{pmatrix} A_{xx} & 0 & B_{xx} & 0 \\ 0 & A_{yy} & 0 & B_{yy} \\ C_{xx} & 0 & D_{xx} & 0 \\ 0 & C_{yy} & 0 & D_{yy} \end{pmatrix}$$

(separable system)

$$\mathbf{P}_{out} = \mathbf{S} \cdot \mathbf{P}_{in} \cdot \mathbf{S}^T$$

$$\langle y^2 \rangle_{c,out} = A_{yy}^2 \langle y^2 \rangle_{c,in} + 2 A_{yy} B_{yy} \langle y\,v \rangle_{c,in} + B_{yy}^2 \langle v^2 \rangle_{c,in}$$

$$\langle y\,v \rangle_{c,out} = A_{yy} C_{yy} \langle y^2 \rangle_{c,in} + \left(A_{yy} D_{yy} - B_{yy} C_{yy} \right) \langle y\,v \rangle_{c,in} + B_{yy} D_{yy} \langle v^2 \rangle_{c,in}$$

$$\langle v^2 \rangle_{c,out} = C_{yy}^2 \langle y^2 \rangle_{c,in} + 2 C_{yy} D_{yy} \langle y\,v \rangle_{c,in} + D_{xx}^2 \langle v^2 \rangle_{c,in}$$

Propagation through separable ABCD-Systems (2D)

$$\mathbf{P}_{in} = \begin{pmatrix} \langle x^2\rangle_c & \langle xy\rangle_c & \langle xu\rangle_c & \langle xv\rangle_c \\ \langle xy\rangle_c & \langle y^2\rangle_c & \langle yu\rangle_c & \langle yv\rangle_c \\ \langle xu\rangle_c & \langle yu\rangle_c & \langle u^2\rangle_c & \langle uv\rangle_c \\ \langle xv\rangle_c & \langle yv\rangle_c & \langle uv\rangle_c & \langle v^2\rangle_c \end{pmatrix}_{in}$$

$$\mathbf{S}_{sep} = \begin{pmatrix} A_{xx} & 0 & B_{xx} & 0 \\ 0 & A_{yy} & 0 & B_{yy} \\ C_{xx} & 0 & D_{xx} & 0 \\ 0 & C_{yy} & 0 & D_{yy} \end{pmatrix}$$

(separable system)

$$\mathbf{P}_{out} = \mathbf{S}\cdot\mathbf{P}_{in}\cdot\mathbf{S}^T$$

$$\langle x^2\rangle_{c,\alpha,out} = A^2\langle x^2\rangle_{c,\alpha,in} + 2AB\langle xu\rangle_{c,\alpha,in} + B^2\langle u^2\rangle_{c,\alpha,in}$$

$$\langle xu\rangle_{c,\alpha,out} = AC\langle x^2\rangle_{c,\alpha,in} + (AD-BC)\langle xu\rangle_{c,\alpha,in} + BD\langle u^2\rangle_{c,\alpha,in}$$

$$\langle u^2\rangle_{c,\alpha,out} = C^2\langle x^2\rangle_{c,\alpha,in} + 2CD\langle xu\rangle_{c,\alpha,in} + D^2\langle u^2\rangle_{c,\alpha,in}$$

Beam classification

$$D_{max} = 2\sqrt{2}\left\{\left(\langle x^2\rangle_c + \langle y^2\rangle_c\right) + \left[\left(\langle x^2\rangle_c - \langle y^2\rangle_c\right)^2 + 4\langle xy\rangle_c^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}$$

$$D_{min} = 2\sqrt{2}\left\{\left(\langle x^2\rangle_c + \langle y^2\rangle_c\right) - \left[\left(\langle x^2\rangle_c - \langle y^2\rangle_c\right)^2 + 4\langle xy\rangle_c^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}$$

$$\varphi = \frac{1}{2}\operatorname{atan}\left(\frac{2\langle xy\rangle_c}{\langle x^2\rangle_c - \langle y^2\rangle_c}\right)$$

$$\varepsilon = \operatorname{sgn}\left(\langle x^2\rangle_c - \langle y^2\rangle_c\right)$$

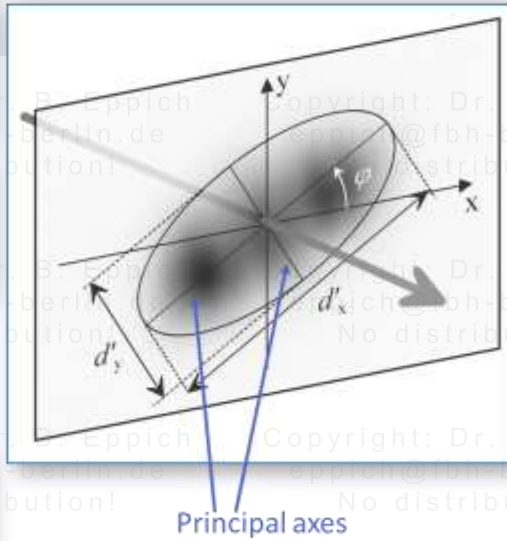
$$D'_x = 2\sqrt{2}\left\{\left(\langle x^2\rangle_c + \langle y^2\rangle_c\right) + \varepsilon\left[\left(\langle x^2\rangle_c - \langle y^2\rangle_c\right)^2 + 4\langle xy\rangle_c^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}$$

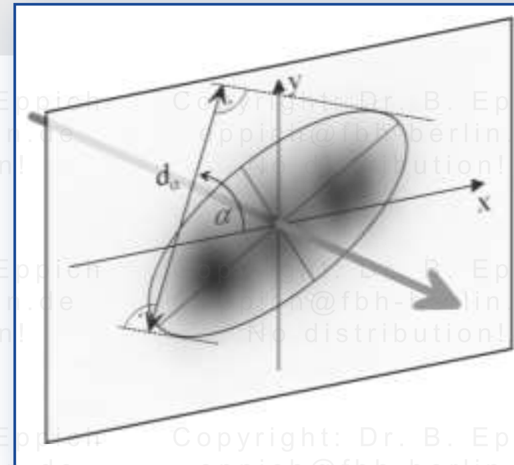
$$D'_y = 2\sqrt{2}\left\{\left(\langle x^2\rangle_c + \langle y^2\rangle_c\right) - \varepsilon\left[\left(\langle x^2\rangle_c - \langle y^2\rangle_c\right)^2 + 4\langle xy\rangle_c^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}$$

$$D_x = 4\sqrt{\langle x^2\rangle_c}$$

$$D_y = 4\sqrt{\langle y^2\rangle_c}$$

$$D_\alpha = 4\sqrt{\cos^2\alpha\,\langle x^2\rangle + \sin^2\alpha\,\langle y^2\rangle + 2\sin\alpha\,\cos\alpha\,\langle xy\rangle}$$





Round beam profile:

$$D_x = D_y = D'_x = D'_y = D_{min} = D_{max} \Leftrightarrow \begin{array}{c}\langle x^2\rangle_c = \langle y^2\rangle_c \\ \langle x\,y\rangle_c = 0\end{array}$$

Aligned elliptic beam profile:

$$\varphi = 0 \Leftrightarrow \langle x\,y\rangle_c = 0$$

Beam classification

Free space propagation:

$$\langle x^2\rangle_c(z)=\langle x^2\rangle_c+2z\langle xu\rangle_c+z^2\langle u^2\rangle_c$$

$$\langle xy\rangle_c(z)=\langle xy\rangle_c+z\left(\langle xv\rangle_c+\langle yu\rangle_c\right)+z^2\langle uv\rangle_c$$

$$\langle y^2\rangle_c(z)=\langle y^2\rangle_c+2z\langle yv\rangle_c+z^2\langle v^2\rangle_c$$

Necessary condition for $\langle x\,y\rangle_c(z)=0$

$$\langle xv\rangle_c+\langle yu\rangle_c=0$$

$$\langle uv\rangle_c=0$$

Necessary condition for $\langle x^2\rangle_c(z)=\langle y^2\rangle_c(z)$

$$\langle xu\rangle_c=\langle yv\rangle_c$$

$$\langle u^2\rangle_c=\langle v^2\rangle_c$$

(pseudo) stigmatic beam:

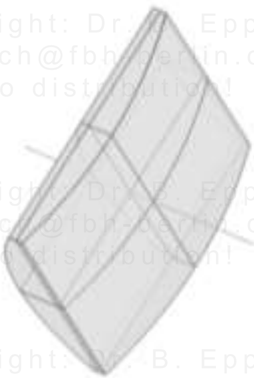
$$\mathbf{P}_{(P)ST}=\begin{pmatrix}\langle x^2\rangle_c & 0 & \langle xu\rangle_c & \langle xv\rangle_c\\ 0 & \langle x^2\rangle_c & -\langle xv\rangle_c & \langle xu\rangle_c\\ \langle xu\rangle_c & -\langle xv\rangle_c & \langle u^2\rangle_c & 0\\ \langle xv\rangle_c & \langle xu\rangle_c & 0 & \langle u^2\rangle_c\end{pmatrix}$$

(pseudo) aligned simple astigmatic beam:

$$\mathbf{P}_{(P)ASA}=\begin{pmatrix}\langle x^2\rangle_c & 0 & \langle xu\rangle_c & \langle xv\rangle_c\\ 0 & \langle y^2\rangle_c & -\langle xv\rangle_c & \langle yv\rangle_c\\ \langle xu\rangle_c & -\langle xv\rangle_c & \langle u^2\rangle_c & 0\\ \langle xv\rangle_c & \langle yv\rangle_c & 0 & \langle v^2\rangle_c\end{pmatrix}$$

"pseudo" $\Leftrightarrow \langle x\,v\rangle_c \neq 0$

Phase paraboloid



$$\mathbf{S}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{\cos^2\alpha}{f_x}-\frac{\sin^2\alpha}{f_y} & \cos\alpha\sin\alpha\left(\frac{1}{f_x}-\frac{1}{f_y}\right) & 1 & 0 \\ \cos\alpha\sin\alpha\left(\frac{1}{f_x}-\frac{1}{f_y}\right) & -\frac{\sin^2\alpha}{f_x}-\frac{\cos^2\alpha}{f_y} & 0 & 1 \end{pmatrix}$$

Find f_x, f_y, and α to…. minimize $tr\left(\mathbf{U}_{out}\right)=\left\langle u^2\right\rangle_c+\left\langle v^2\right\rangle_c$

$$a=\frac{\langle y^2\rangle\langle xu\rangle\left(\langle x^2\rangle+\langle y^2\rangle\right)-\langle xy\rangle^2\left(\langle xu\rangle-\langle yv\rangle\right)-\langle xy\rangle\langle y^2\rangle\left(\langle xv\rangle+\langle yu\rangle\right)}{\left(\langle x^2\rangle+\langle y^2\rangle\right)\left(\langle x^2\rangle\langle y^2\rangle-\langle xy\rangle^2\right)}$$

$$b=\frac{\langle x^2\rangle\langle y^2\rangle\left(\langle xv\rangle+\langle yu\rangle\right)-\langle xy\rangle\left(\langle x^2\rangle\langle yv\rangle+\langle y^2\rangle\langle xu\rangle\right)}{\left(\langle x^2\rangle+\langle y^2\rangle\right)\left(\langle x^2\rangle\langle y^2\rangle-\langle xy\rangle^2\right)}$$

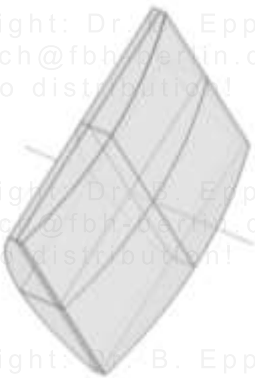
$$c=\frac{\langle x^2\rangle\langle yv\rangle\left(\langle x^2\rangle+\langle y^2\rangle\right)+\langle xy\rangle^2\left(\langle xu\rangle-\langle yv\rangle\right)-\langle xy\rangle\langle x^2\rangle\left(\langle xv\rangle+\langle yu\rangle\right)}{\left(\langle x^2\rangle+\langle y^2\rangle\right)\left(\langle x^2\rangle\langle y^2\rangle-\langle xy\rangle^2\right)}$$

$$R'_x=\frac{2}{(a+c)+\mu\sqrt{(a-c)^2+4b^2}}$$

$$R'_y=\frac{2}{(a+c)-\mu\sqrt{(a-c)^2+4b^2}}$$

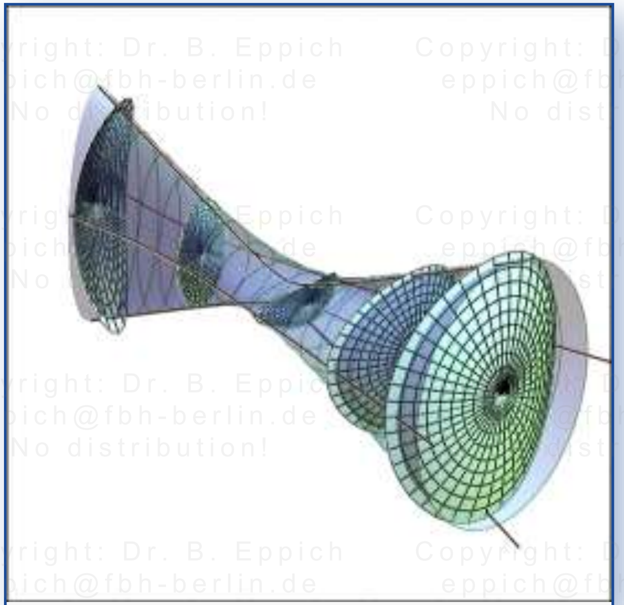
$$\varphi_P=\frac{1}{2}\operatorname{atan}\left(\frac{2b}{a-c}\right)\qquad \mu=\operatorname{sgn}(a-c)$$

Phase paraboloid



$$\mathbf{S}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{\cos^2\alpha}{f_x}-\frac{\sin^2\alpha}{f_y} & \cos\alpha\sin\alpha\left(\frac{1}{f_x}-\frac{1}{f_y}\right) & 1 & 0 \\ \cos\alpha\sin\alpha\left(\frac{1}{f_x}-\frac{1}{f_y}\right) & -\frac{\sin^2\alpha}{f_x}-\frac{\cos^2\alpha}{f_y} & 0 & 1 \end{pmatrix}$$

Find f_x, f_y, and α to…. minimize $tr\left(\mathbf{U}_{out}\right)=\left\langle u^2\right\rangle_c+\left\langle v^2\right\rangle_c$

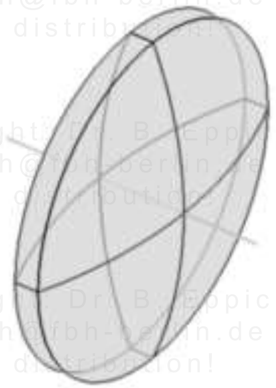


$$R'_x=\frac{2}{(a+c)+\mu\sqrt{(a-c)^2+4b^2}}$$

$$R'_y=\frac{2}{(a+c)-\mu\sqrt{(a-c)^2+4b^2}}$$

$$\varphi_P=\frac{1}{2}\operatorname{atan}\left(\frac{2b}{a-c}\right)$$

Phase paraboloid



$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1/f & 0 & 1 & 0 \\ 0 & -1/f & 0 & 1 \end{pmatrix}$$

Find *f* to…. minimize $tr\left(\mathbf{U}_{out}\right) = \langle u^2 \rangle_c + \langle v^2 \rangle_c$

$$R = f = \frac{\langle x^2 \rangle_c + \langle y^2 \rangle_c}{\langle xu \rangle_c + \langle yv \rangle_c}$$

(best fitting spherical phase front)

The twist parameter

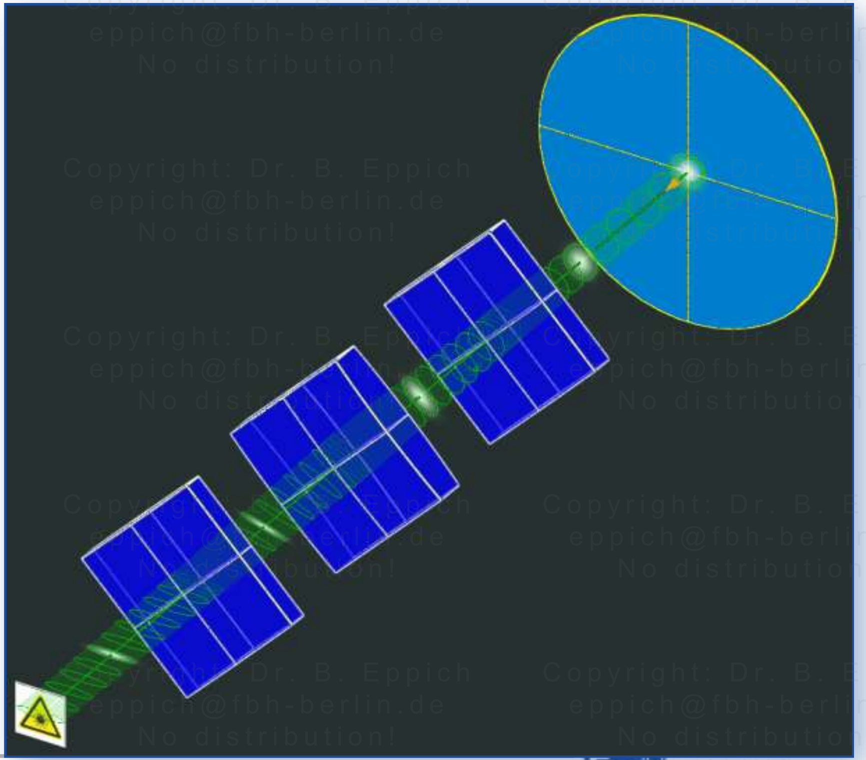
$$\mathbf{P}_{in} = \begin{pmatrix} \langle x^2\rangle_c & \langle xy\rangle_c & \langle xu\rangle_c & \langle xv\rangle_c \\ \langle xy\rangle_c & \langle y^2\rangle_c & \langle yu\rangle_c & \langle yv\rangle_c \\ \langle xu\rangle_c & \langle yu\rangle_c & \langle u^2\rangle_c & \langle uv\rangle_c \\ \langle xv\rangle_c & \langle yv\rangle_c & \langle uv\rangle_c & \langle v^2\rangle_c \end{pmatrix}_{in}$$

$$\begin{aligned} \langle x^2\rangle_{out}(z) &= \langle x^2\rangle_{in} + 2z\langle xu\rangle_{in} + z^2\langle u^2\rangle_{in} \\ \langle xy\rangle_{out}(z) &= \langle xy\rangle_{in} + z\left(\langle xv\rangle_{in} + \langle yu\rangle_{in}\right) + z^2\langle uv\rangle_{in} \\ \langle y^2\rangle_{out}(z) &= \langle y^2\rangle_{in} + 2z\langle yv\rangle_{in} + z^2\langle v^2\rangle_{in} \end{aligned}$$

(free space propagation)

Twist parameter $t = \langle xv\rangle - \langle yu\rangle$

Orbital angular momentum of light



Invariants in general (non-symmetric) systems

From...

$$\mathbf{S}\cdot\mathbf{J}\cdot\mathbf{S}^{T}=\mathbf{J}\quad,\quad \mathbf{J}=\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{pmatrix}$$

(symplecticity)

...the following two invariants can be derived:

The **effective beam propagation** ratio

$$M_{eff}^{2}=\frac{4\pi}{\lambda}\left(\det\left(\mathbf{P}\right)\right)^{\frac{1}{4}}\geq 1$$

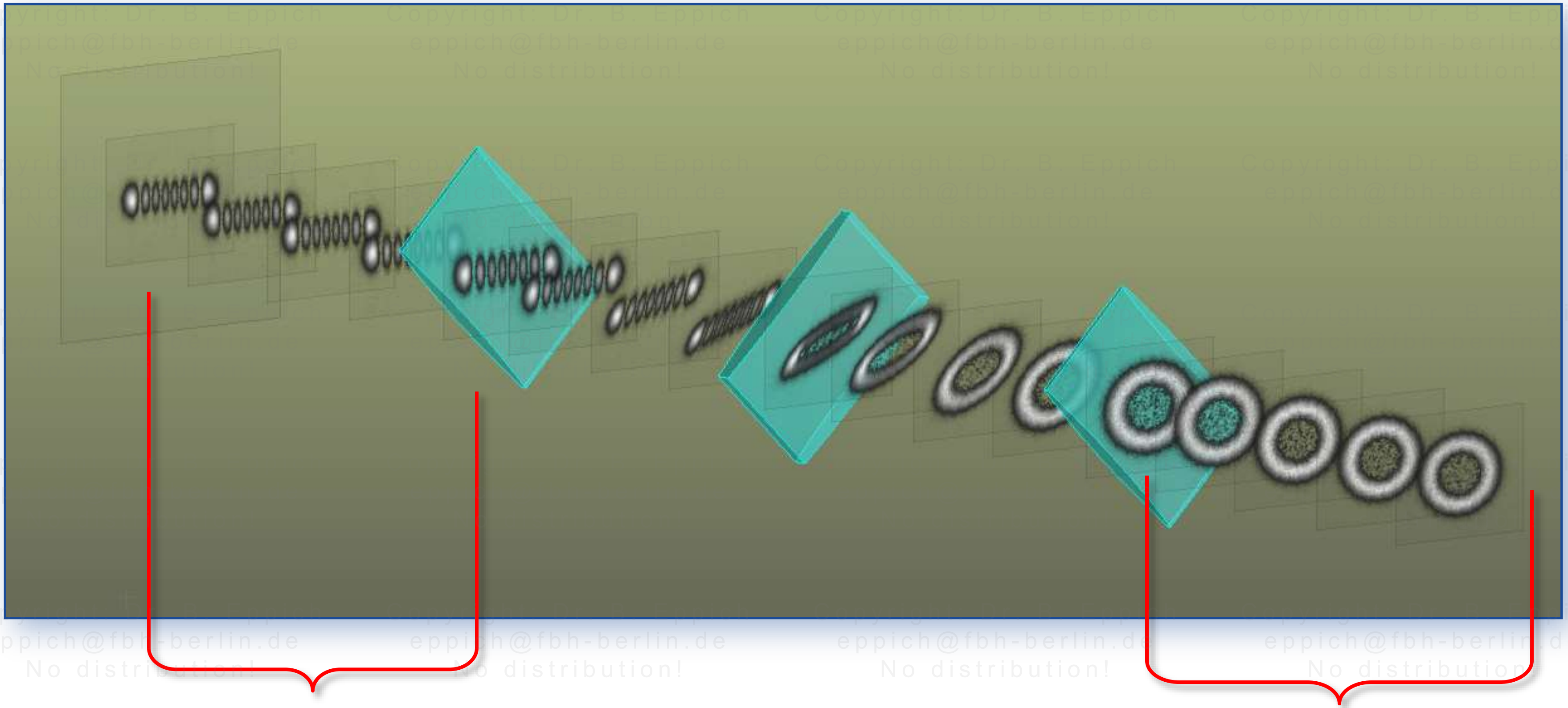
...is a measure of the **focusability** of a beam.

The **intrinsic astigmatism** *a*

$$a=\left\langle x^{2}\right\rangle_{c}\left\langle u^{2}\right\rangle_{c}-\left\langle xu\right\rangle_{c}^{2}+\left\langle y^{2}\right\rangle_{c}\left\langle v^{2}\right\rangle_{c}-\left\langle yv\right\rangle_{c}^{2}+2\left(\left\langle xy\right\rangle_{c}\left\langle uv\right\rangle_{c}-\left\langle xv\right\rangle_{c}\left\langle yu\right\rangle_{c}\right)-2\det\left(\mathbf{P}\right)$$

...is related to the visible and hidden **astigmatism**

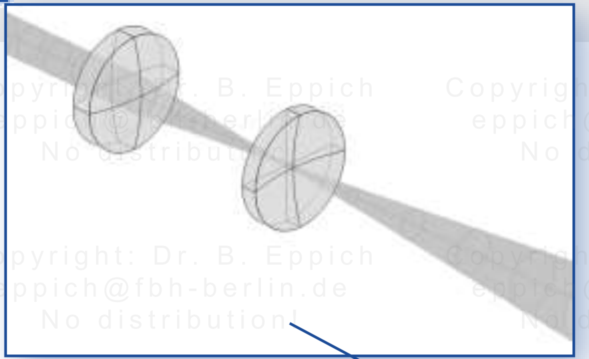
Invariants in general (non-symmetric) systems

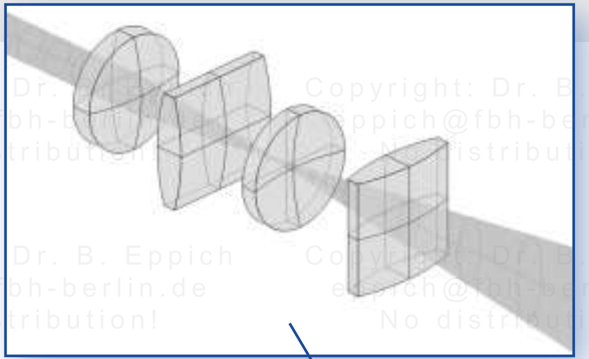


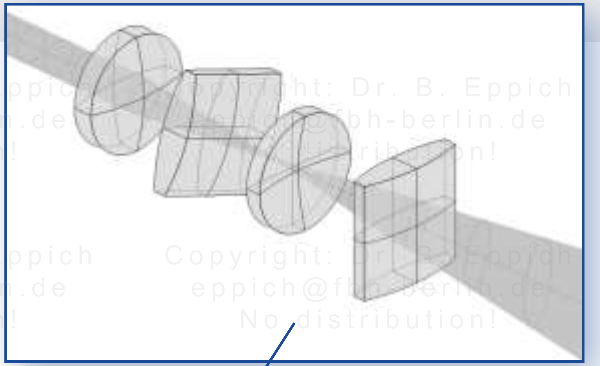
$$M_x^2 > M_y^2$$

$$\tilde{M}_x^2 = \tilde{M}_y^2 = \frac{M_x^2 + M_y^2}{2}$$

Invariants in (non-)symmetric systems





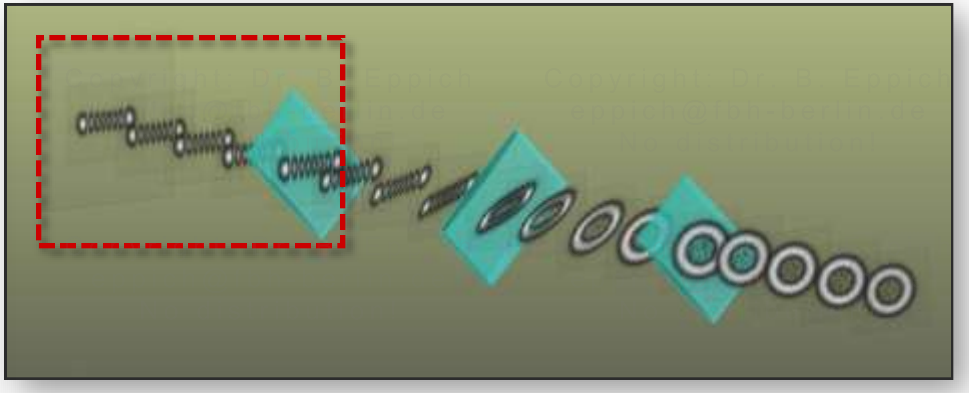


Quantities	Stigmatic systems	Separable systems	General systems
M_x^2, M_y^2	Yes	Yes, if same axes	No
Twist t	Yes	No	No
M_{eff}^2	Yes	Yes	Yes
Intrinsic astigmatism a	Yes	Yes	Yes

Fundamental types (beams of highest symmetry)

Simple astigmatic beam without twist

$$\mathbf{S} = \begin{pmatrix} \langle x^2 \rangle & 0 & 0 & 0 \\ 0 & m^2 \langle x^2 \rangle & 0 & 0 \\ 0 & 0 & \langle u^2 \rangle & 0 \\ 0 & 0 & 0 & m^2 \langle u^2 \rangle \end{pmatrix}$$



$$d_y(z) = m \cdot d_x(z) \quad \forall z$$

$$\theta_y(z) = m \cdot \theta_x(z) \quad \forall z$$

$$t = 0$$

$$M_y^2 = m^2 \cdot M_x^2$$

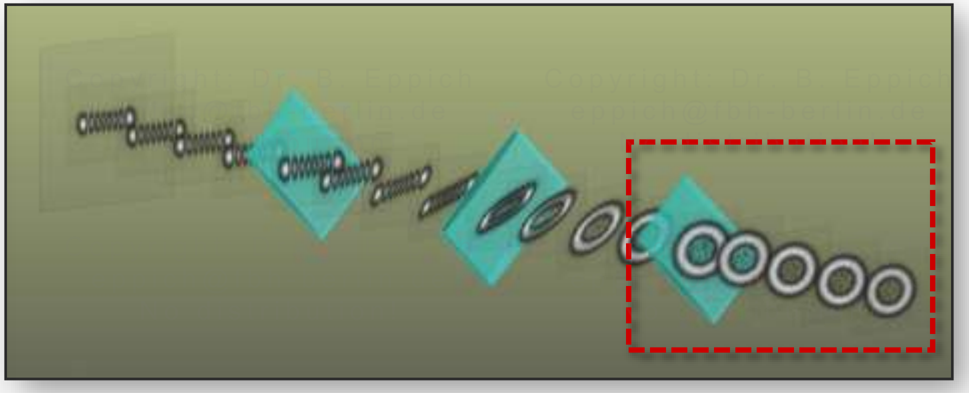
$$M_{eff}^2 = \sqrt{M_x^2 M_y^2} = m M_x^2$$

$$a = \frac{1}{2}\left(m^2 - 1\right)^2 \left(M_x^2\right)^2$$

Fundamental types (beams of highest symmetry)

Pseudo stigmatic beam with twist

$$\mathbf{S} = \begin{pmatrix} \langle x^2 \rangle & 0 & 0 & t/2 \\ 0 & \langle x^2 \rangle & -t/2 & 0 \\ 0 & -t/2 & \langle u^2 \rangle & 0 \\ t/2 & 0 & 0 & \langle u^2 \rangle \end{pmatrix}$$



$$d_y(z) = d_x(z) \quad \forall z$$

$$\theta_y(z) = \theta_x(z) \quad \forall z$$

$$t \neq 0$$

$$M_y^2 = M_x^2$$

$$M_{eff}^2 = \sqrt{\left(M_x^2\right)^2 - \left(\frac{2\pi}{\lambda}\right)^2 t^2}$$

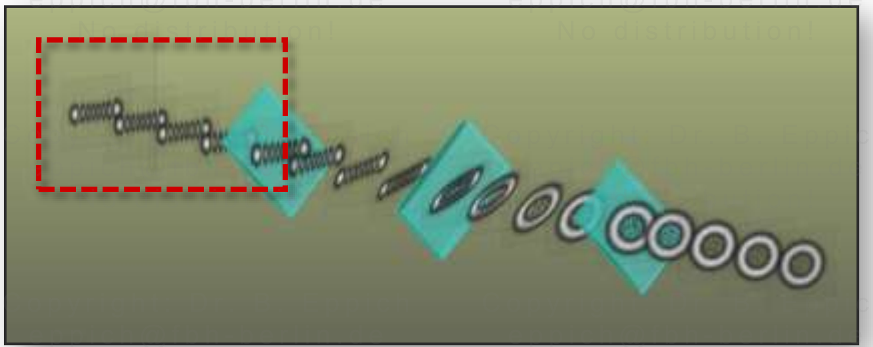
$$a = 2\left(\frac{2\pi}{\lambda}\right)^2 t^2$$

Fundamental types (beams of highest symmetry)

Measured M^2_{eff} and a

Beam may be transformed into….

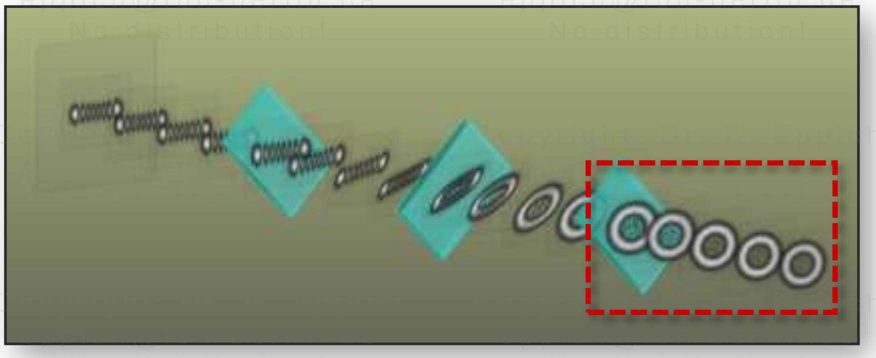
Simple astigmatic beam without twist



$$M^2_{\min} = \sqrt{\left(M^2_{eff}\right)^2 + \frac{a}{2}} - \sqrt{\frac{a}{2}}$$

$$M^2_{\max} = \sqrt{\left(M^2_{eff}\right)^2 + \frac{a}{2}} + \sqrt{\frac{a}{2}}$$

Pseudo stigmatic beam with twist



$$M^2_{\alpha} = \sqrt{\left(M^2_{eff}\right)^2 + \frac{a}{2}}$$

Intrinsic classification

$$\mathbf{S} = \begin{pmatrix} \langle x^2 \rangle & 0 & 0 & 0 \\ 0 & m^2 \langle x^2 \rangle & 0 & 0 \\ 0 & 0 & \langle u^2 \rangle & 0 \\ 0 & 0 & 0 & m^2 \langle u^2 \rangle \end{pmatrix}$$

$$a = \frac{1}{2}\left(m^2 - 1\right)^2 \left(M_x^2\right)^2$$

$$M_{eff}^2 = m M_x^2$$

$$d_y(z) = m \cdot d_x(z)$$

For m ∈ [0.85...1.15] → circular profile!

$$a = \frac{1}{2}\frac{\left(m^2 - 1\right)^2}{m^2}\left(M_{eff}^2\right)^2$$

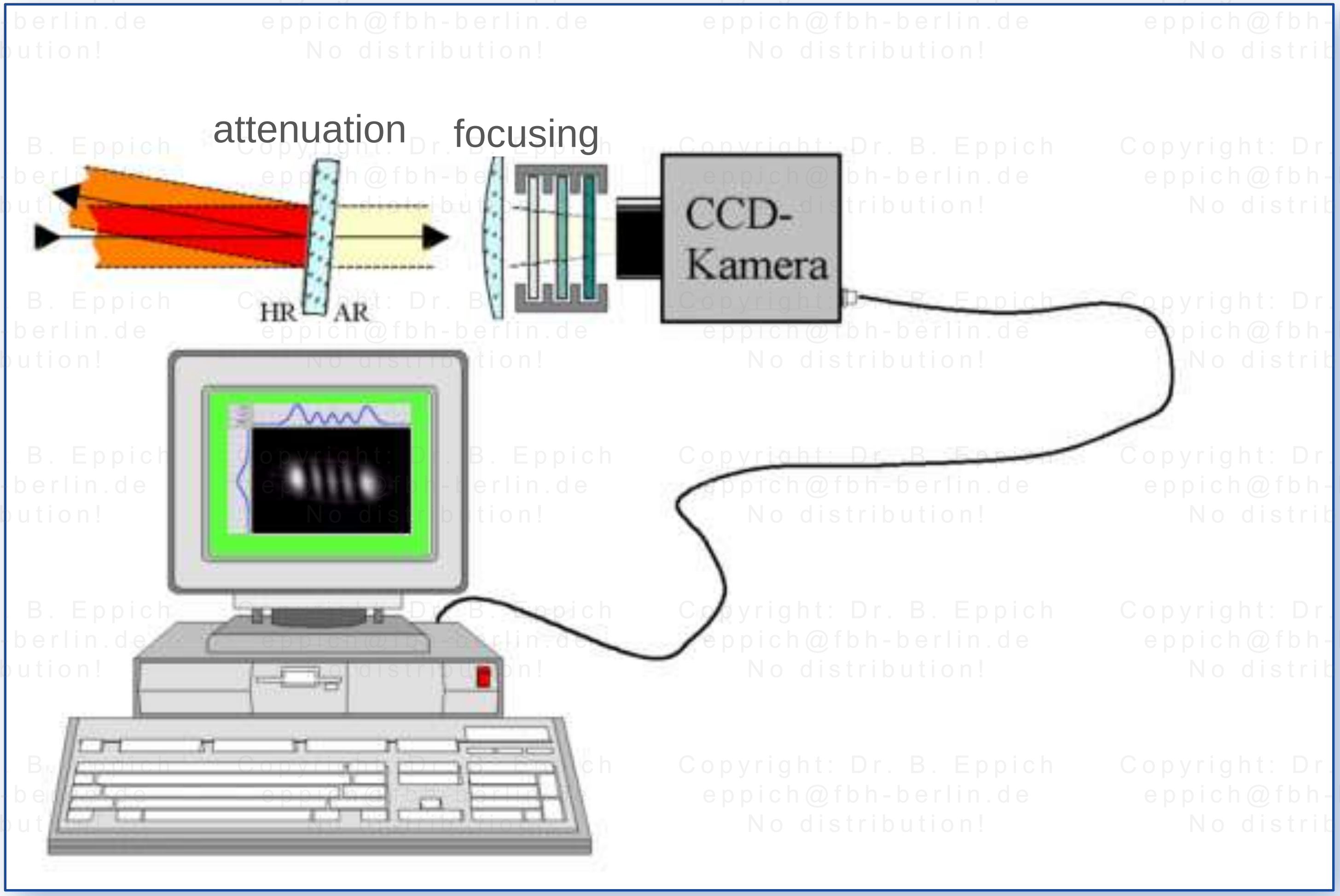
$$\frac{a}{\left(M_{eff}^2\right)^2} = \frac{1}{2}\frac{\left(m^2 - 1\right)^2}{m^2}$$

For $\frac{a}{\left(M_{eff}^2\right)^2} \leq 0.39$ → intrinsic stigmatic beam!

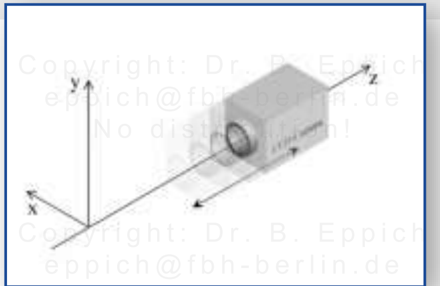
Content

- Motivation
- ISO standard
- Beams and optical systems
- Beam diameter definitions
- Beam classification
- The ten second order moments
- Measurement of the ten second order moments

Beam characterization using matrix detectors



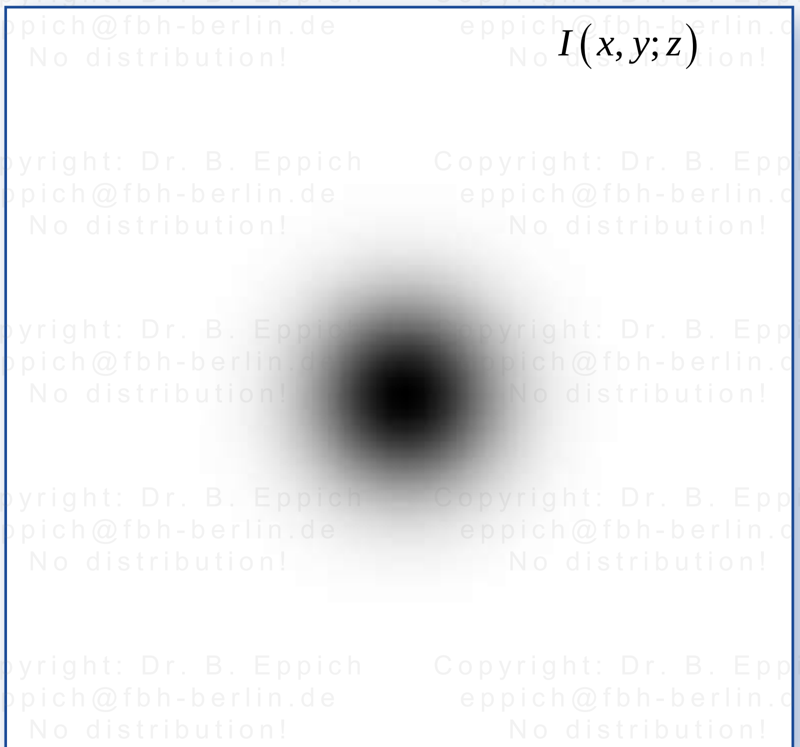
Beam characterization using matrix detectors



$$\left\langle x^2\right\rangle_2(z) = \left\langle x^2\right\rangle_1 + 2z\left\langle xu\right\rangle_1 + z^2\left\langle u^2\right\rangle_1$$

$$\left\langle xy\right\rangle_2(z) = \left\langle xy\right\rangle_1 + z\left(\left\langle xv\right\rangle_1 + \left\langle yu\right\rangle_1\right) + z^2\left\langle uv\right\rangle_1$$

$$\left\langle y^2\right\rangle_2(z) = \left\langle y^2\right\rangle_1 + 2z\left\langle yv\right\rangle_1 + z^2\left\langle v^2\right\rangle_1$$



$$P = \int I(x,y;z)\,dx\,dy$$

$$\langle x\rangle(z) = \frac{1}{P}\int I(x,y;z)\,x\,dx\,dy$$

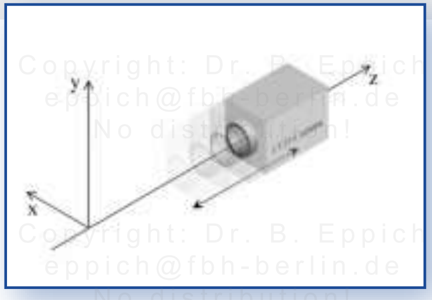
$$\langle y\rangle(z) = \frac{1}{P}\int I(x,y;z)\,y\,dx\,dy$$

$$\left\langle x^2\right\rangle(z) = \frac{1}{P}\int I(x,y;z)\left(x-\langle x\rangle(z)\right)^2 dx\,dy$$

$$\langle x\,y\rangle(z) = \frac{1}{P}\int I(x,y;z)\left(x-\langle x\rangle(z)\right)\left(y-\langle y\rangle(z)\right)dx\,dy$$

$$\left\langle y^2\right\rangle(z) = \frac{1}{P}\int I(x,y;z)\left(y-\langle y\rangle(z)\right)^2 dx\,dy$$

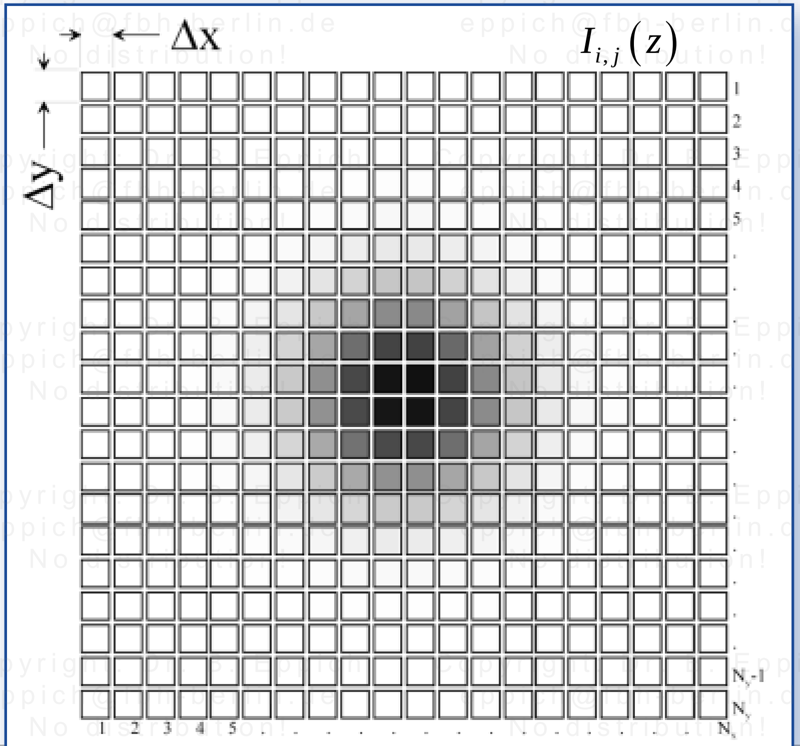
Beam characterization using matrix detectors



$$\langle x^2\rangle_2(z) = \langle x^2\rangle_1 + 2z\langle xu\rangle_1 + z^2\langle u^2\rangle_1$$

$$\langle xy\rangle_2(z) = \langle xy\rangle_1 + z\left(\langle xv\rangle_1 + \langle yu\rangle_1\right) + z^2\langle uv\rangle_1$$

$$\langle y^2\rangle_2(z) = \langle y^2\rangle_1 + 2z\langle yv\rangle_1 + z^2\langle v^2\rangle_1$$



$$P = \Delta x\,\Delta y \sum_{i,j}^{N_x,N_y} I_{i,j}(z)$$

$$\langle x\rangle(z) = \frac{\Delta x^2\,\Delta y}{P}\sum_{i,j}^{N_x,N_y} I_{i,j}(z)\,i$$

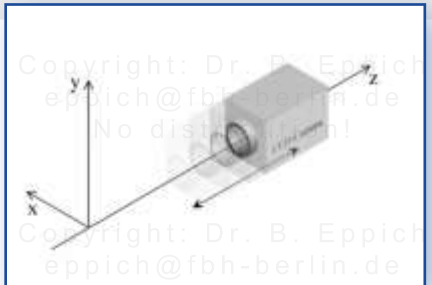
$$\langle y\rangle(z) = \frac{\Delta x\,\Delta y^2}{P}\sum_{i,j}^{N_x,N_y} I_{i,j}(z)\,j$$

$$\langle x^2\rangle(z) = \frac{\Delta x\,\Delta y}{P}\sum_{i,j}^{N_x,N_y} I_{i,j}(z)\left(i\,\Delta x - \langle x\rangle(z)\right)^2$$

$$\langle x\,y\rangle(z) = \frac{\Delta x\,\Delta y}{P}\sum_{i,j}^{N_x,N_y} I_{i,j}(z)\left(i\,\Delta x - \langle x\rangle(z)\right)\left(j\,\Delta y - \langle y\rangle(z)\right)$$

$$\langle y^2\rangle(z) = \frac{\Delta x\,\Delta y}{P}\sum_{i,j}^{N_x,N_y} I_{i,j}(z)\left(j\,\Delta x - \langle y\rangle(z)\right)^2$$

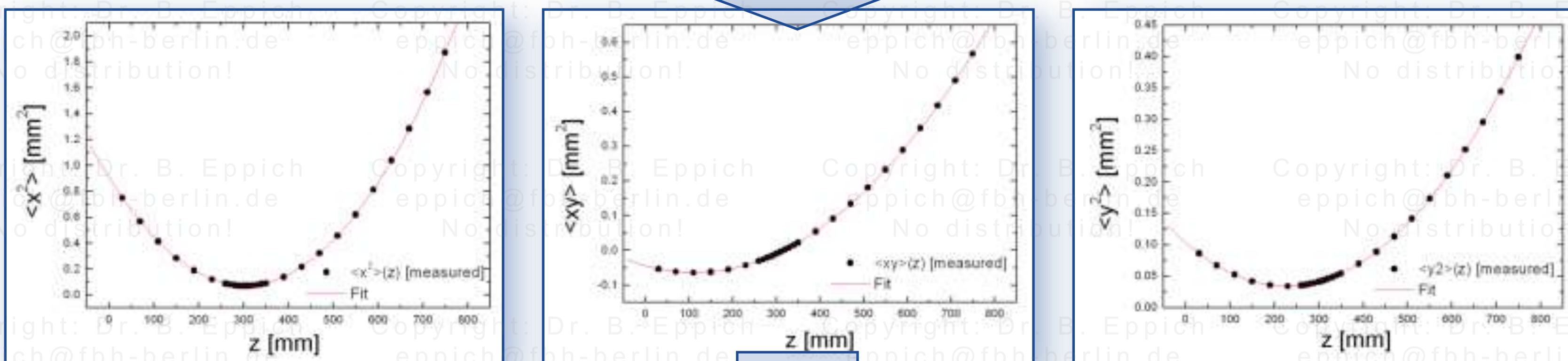
Beam characterization using matrix detectors



$$\langle x^2\rangle_2(z) = \langle x^2\rangle_1 + 2z\langle xu\rangle_1 + z^2\langle u^2\rangle_1$$

$$\langle xy\rangle_2(z) = \langle xy\rangle_1 + z\left(\langle xv\rangle_1 + \langle yu\rangle_1\right) + z^2\langle uv\rangle_1$$

$$\langle y^2\rangle_2(z) = \langle y^2\rangle_1 + 2z\langle yv\rangle_1 + z^2\langle v^2\rangle_1$$

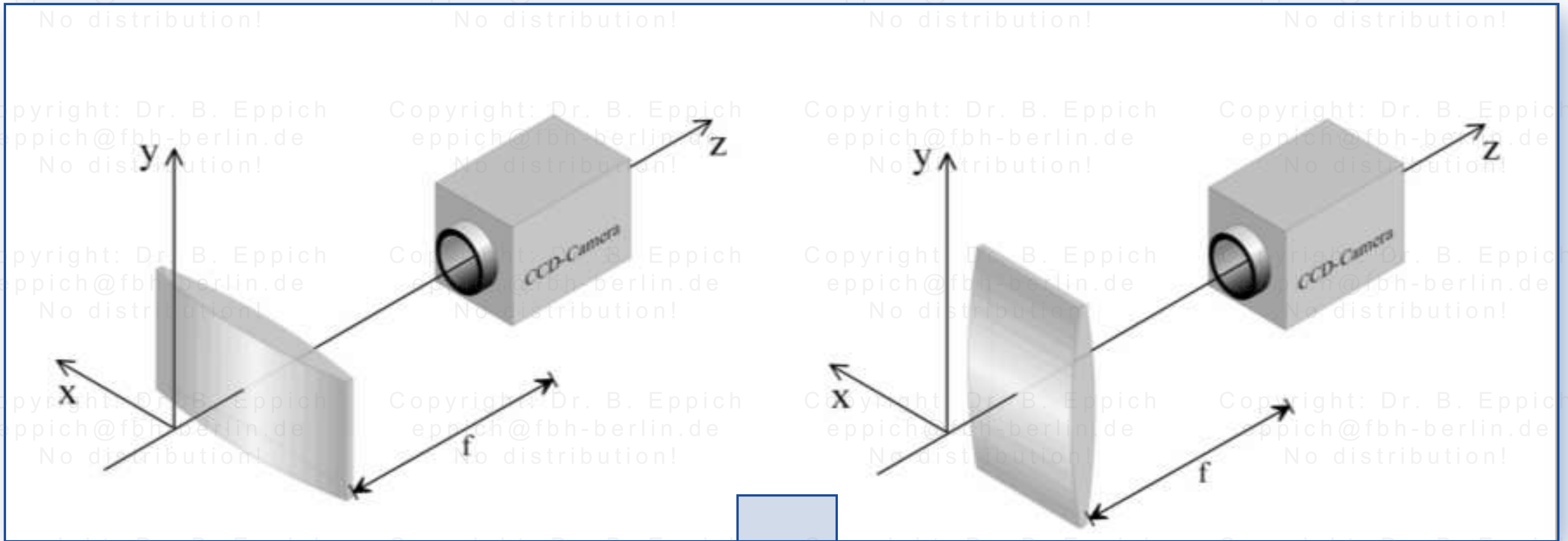


$$\langle x^2\rangle_1, \langle xu\rangle_1, \langle u^2\rangle_1, \langle xy\rangle_1, \langle xv\rangle_1 + \langle yu\rangle_1, \langle uv\rangle_1, \langle y^2\rangle_1, \langle yv\rangle_1, \langle v^2\rangle_1$$

$\langle xv\rangle_1 - \langle yu\rangle_1$ **is missing!**

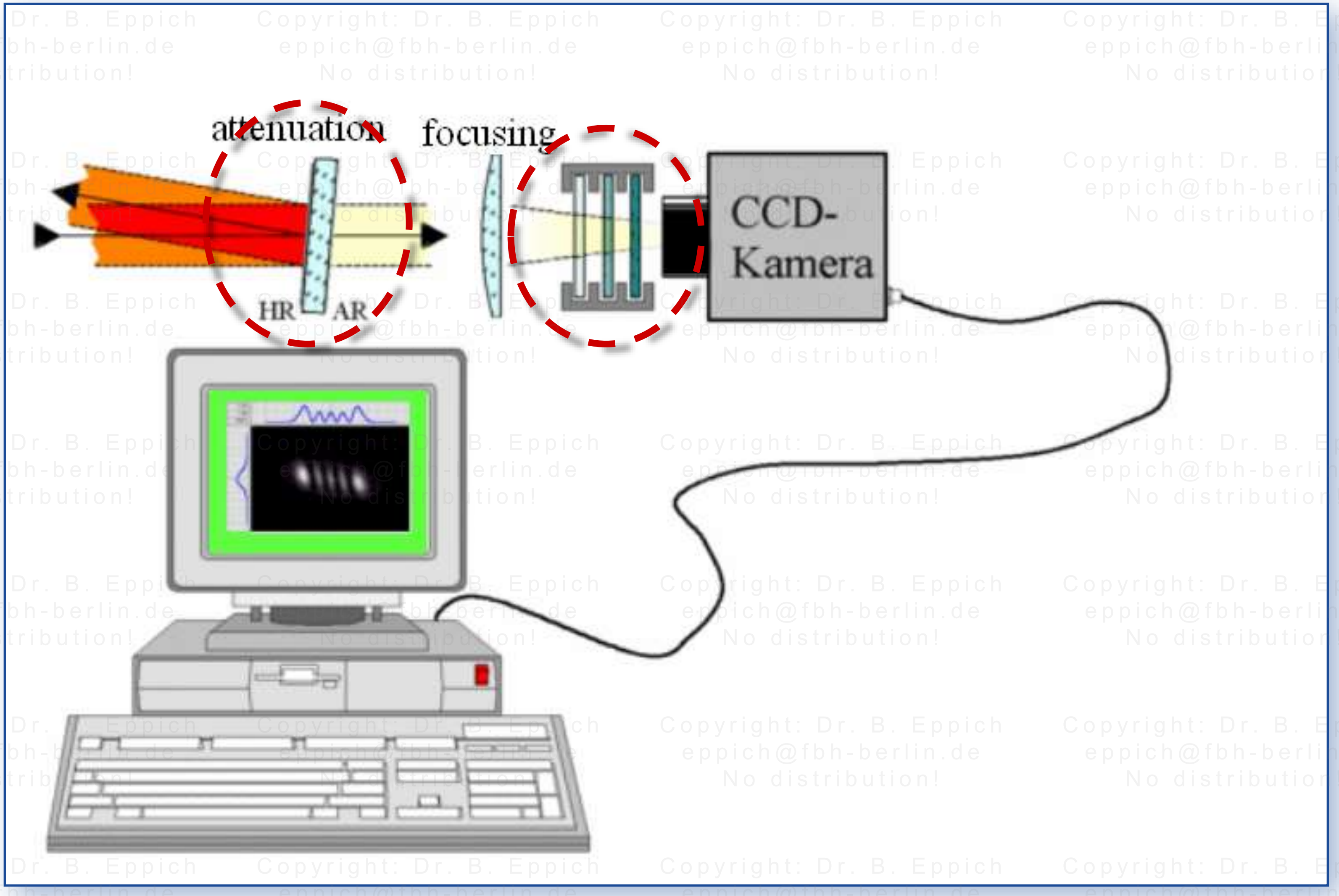
Beam characterization using matrix detectors

Additional measurement necessary:

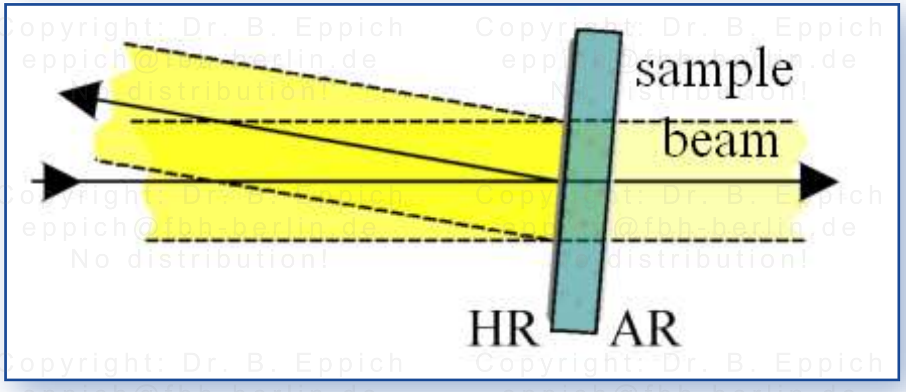


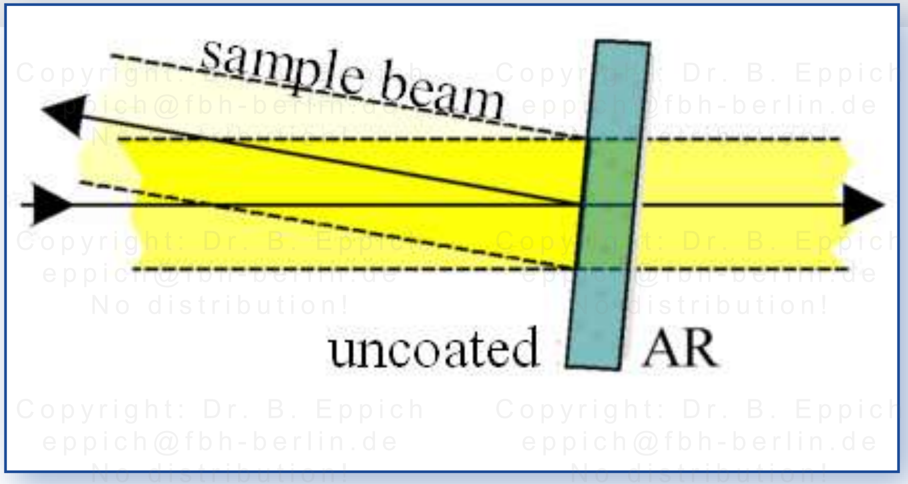
$$\langle xv \rangle_1 - \langle yu \rangle_1 = \frac{\langle xy \rangle_v - \langle xy \rangle_h}{f}$$

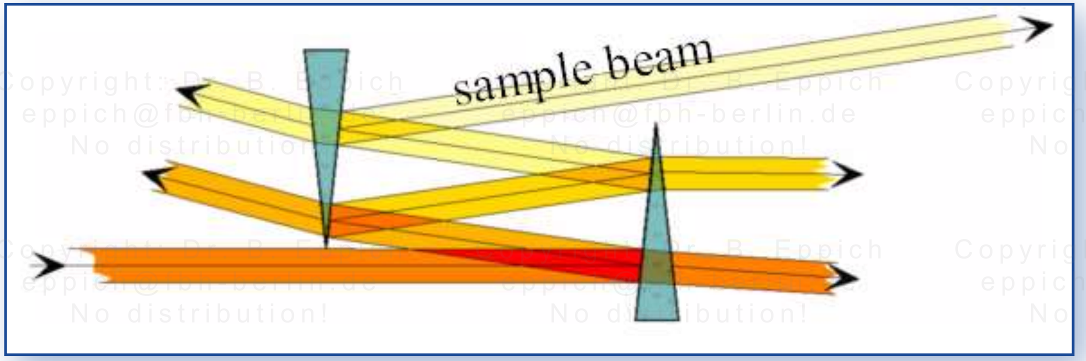
Attenuation

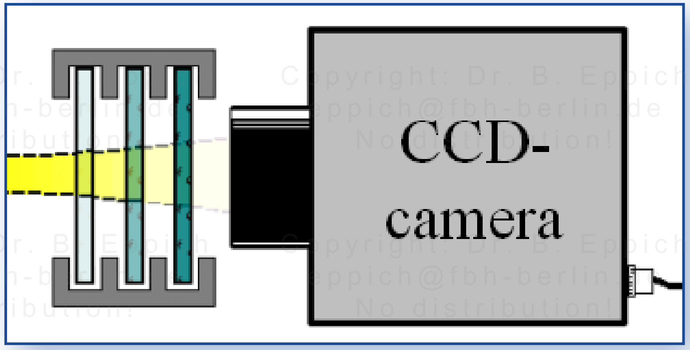


Attenuation

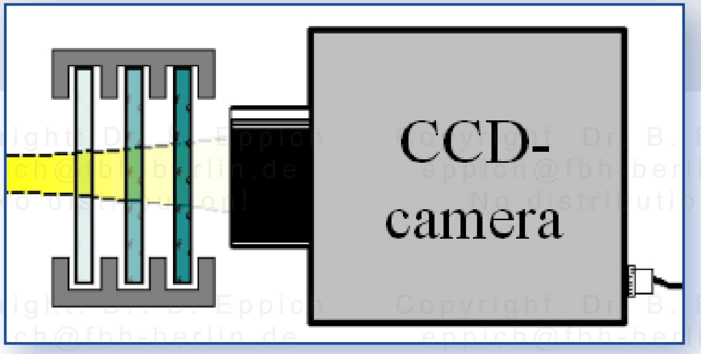








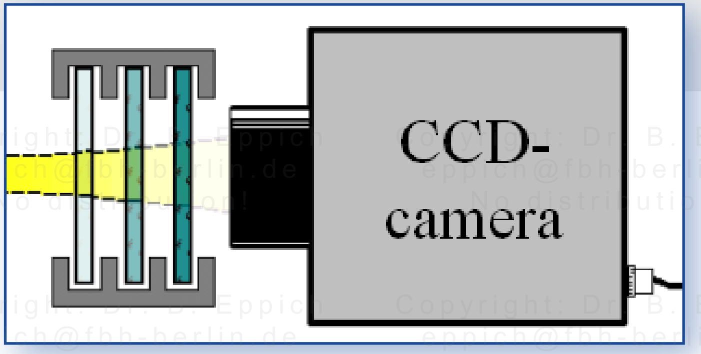
Attenuation



Using absorptive neutral density filters:

- **Only for low power (< 100 mW) → thermal lens**
- **Use close to camera**
- **Order with increasing absorption**
- **If positioned in non-collimated beam segment → keep optical path length constant**
- **AR coating recommended**

Digital cameras

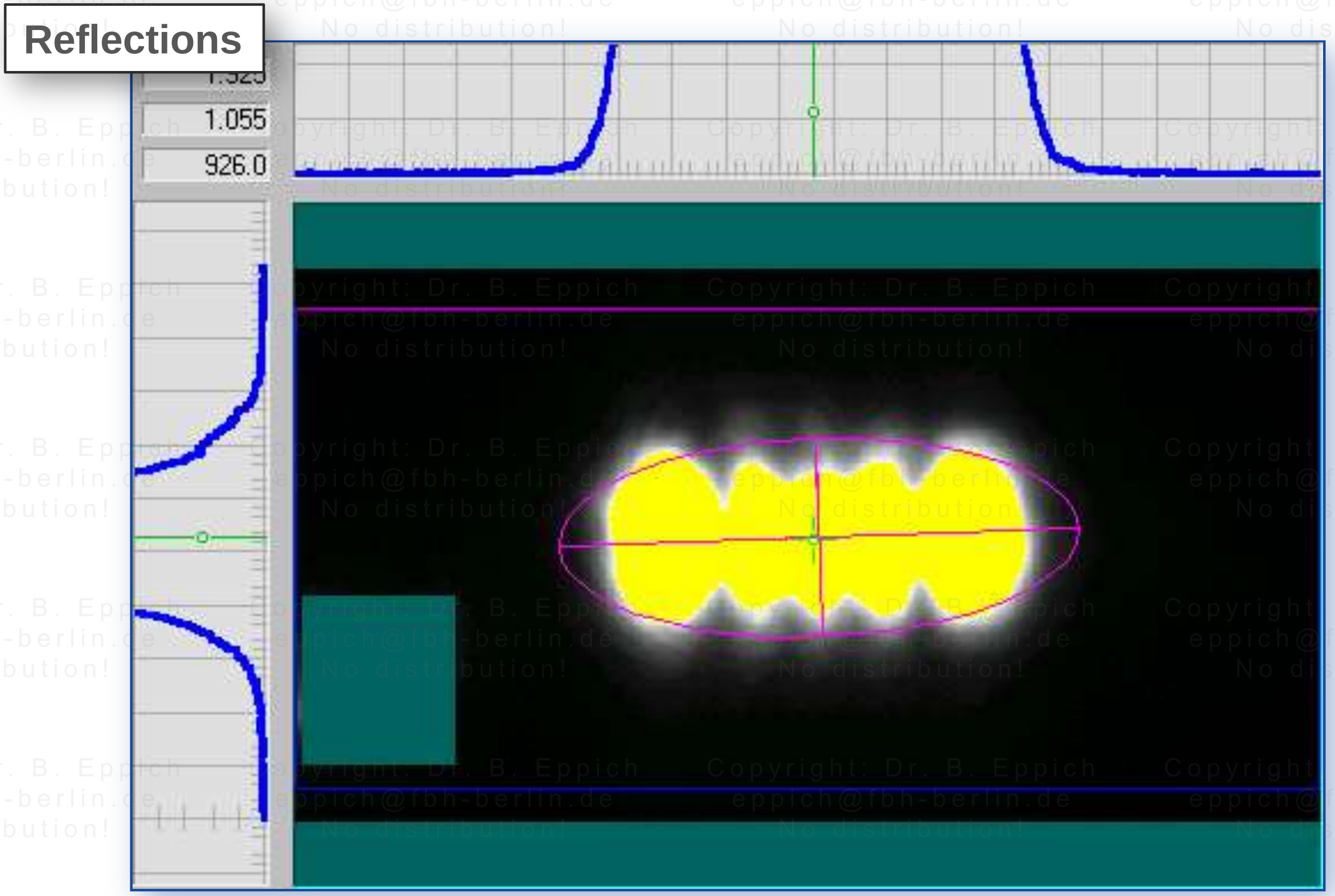


- CMOS may non-linear! Choose CCD or measure linearity!
- Standard have usually two protective glasses:
  - On the housing (often a IR and/or UV filter)

    → get rid of it!
  - One on the CCD chip → ? Get rid of it (may damage)?
- Back reflection of CCD chip may cause artifacts
- Intrinsic “auto-corrections”
- Pixel defects
- Offset homogeneous?
- Offset stable in time?

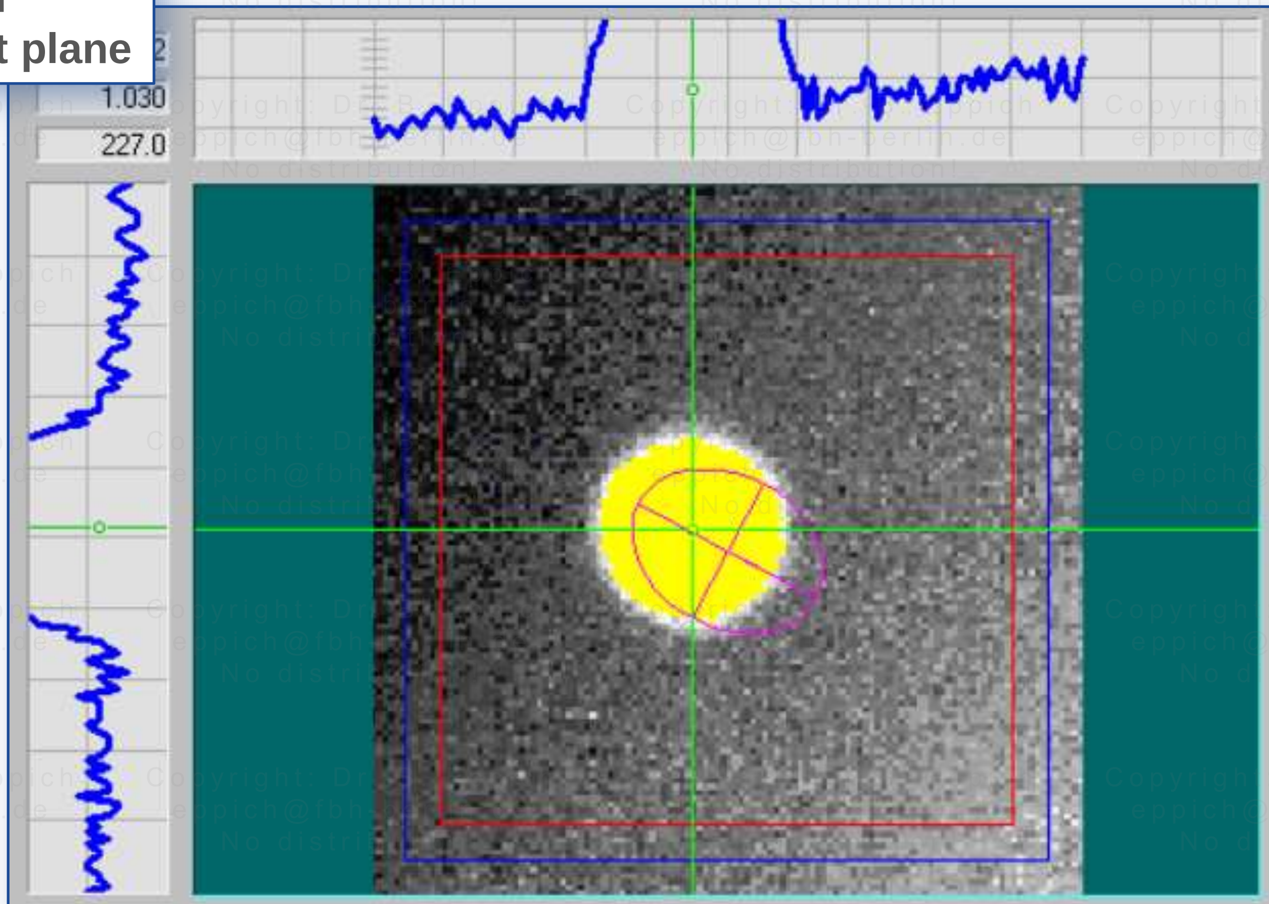
Test Your camera!
Know Your camera!

Common CCD camera problems



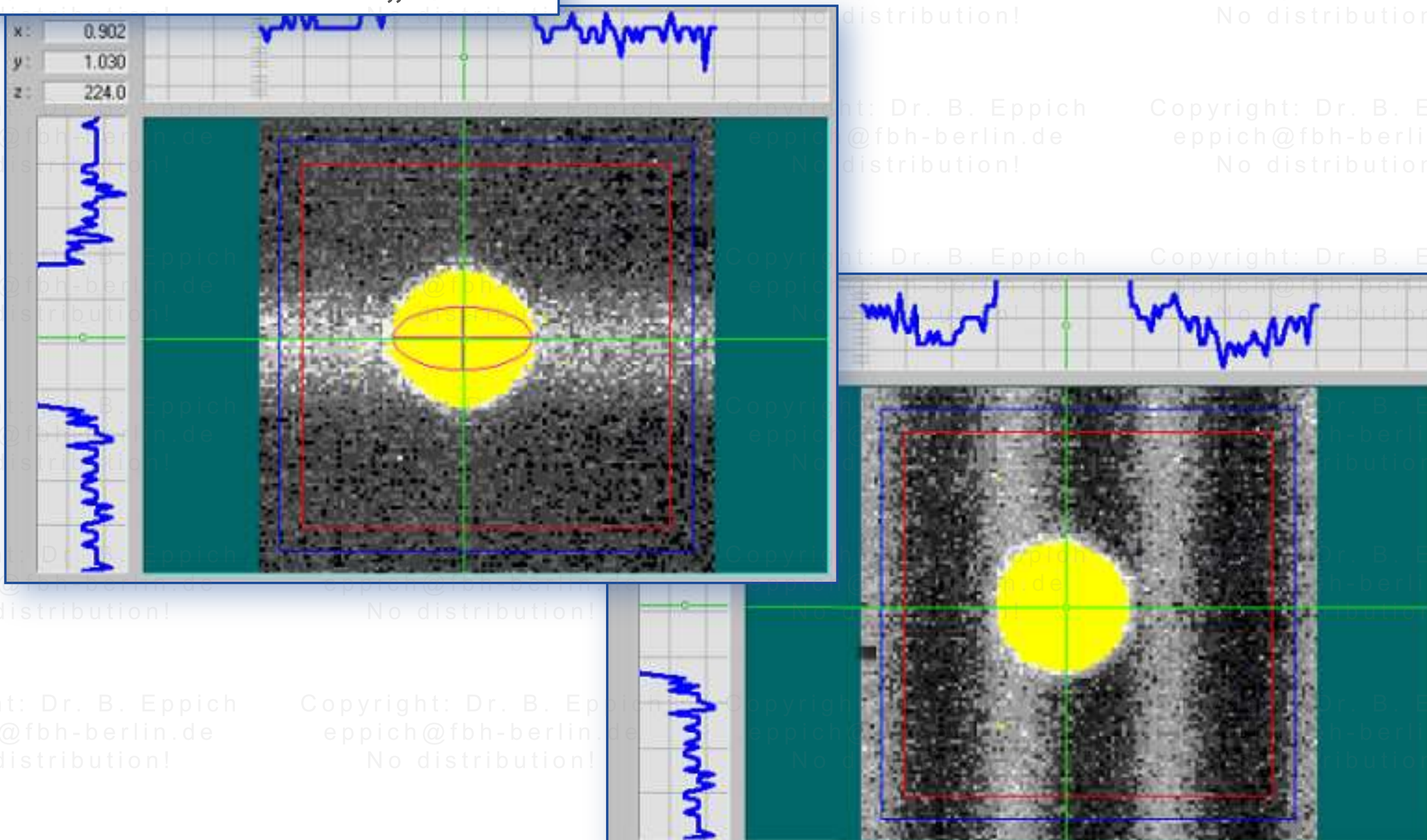
Common CCD camera problems

Tilt of offset plane



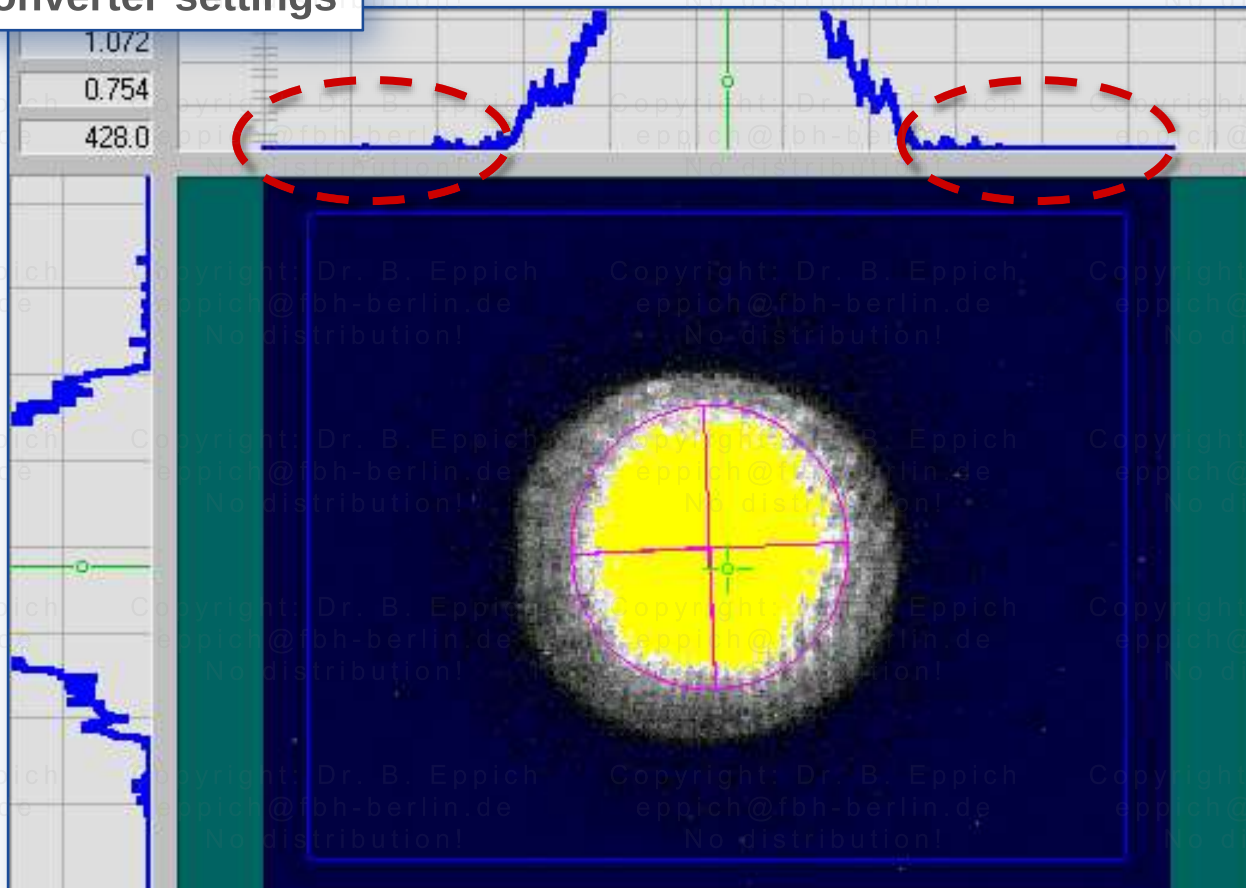
Common CCD camera problems

Horizontal and vertical „waves“



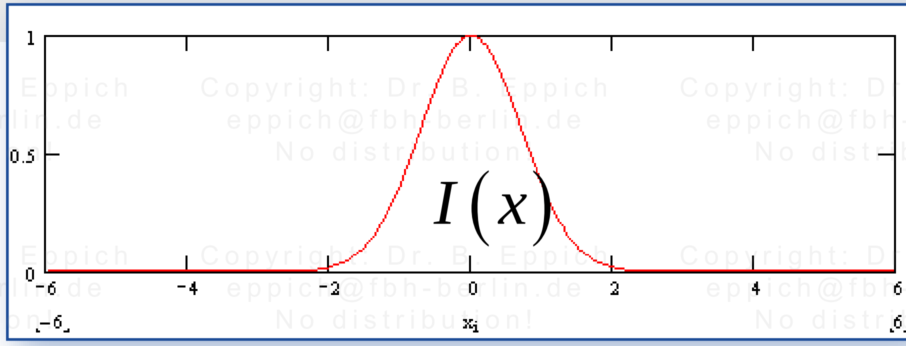
Common CCD camera problems

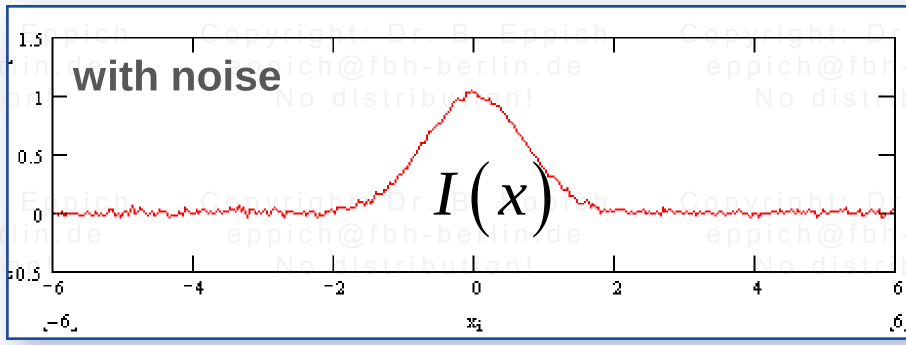
Bad AD-converter settings

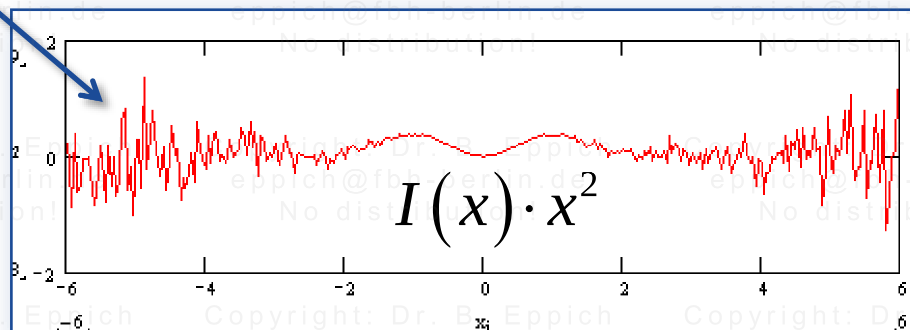


Common CCD camera problems

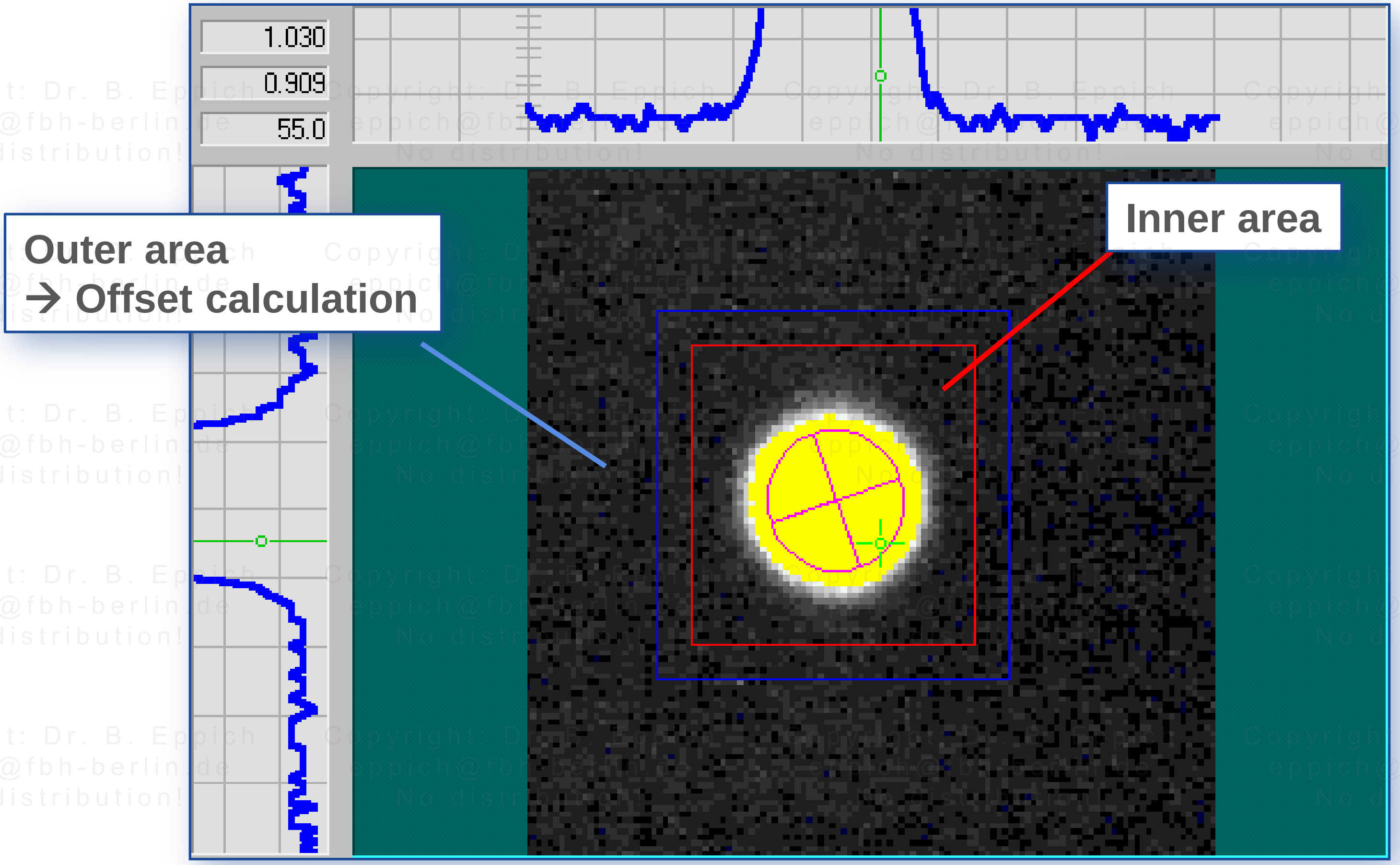
$$d = 4\sqrt{\frac{\int I(x)\,x^2\,dx}{\int I(x)\,dx}}$$



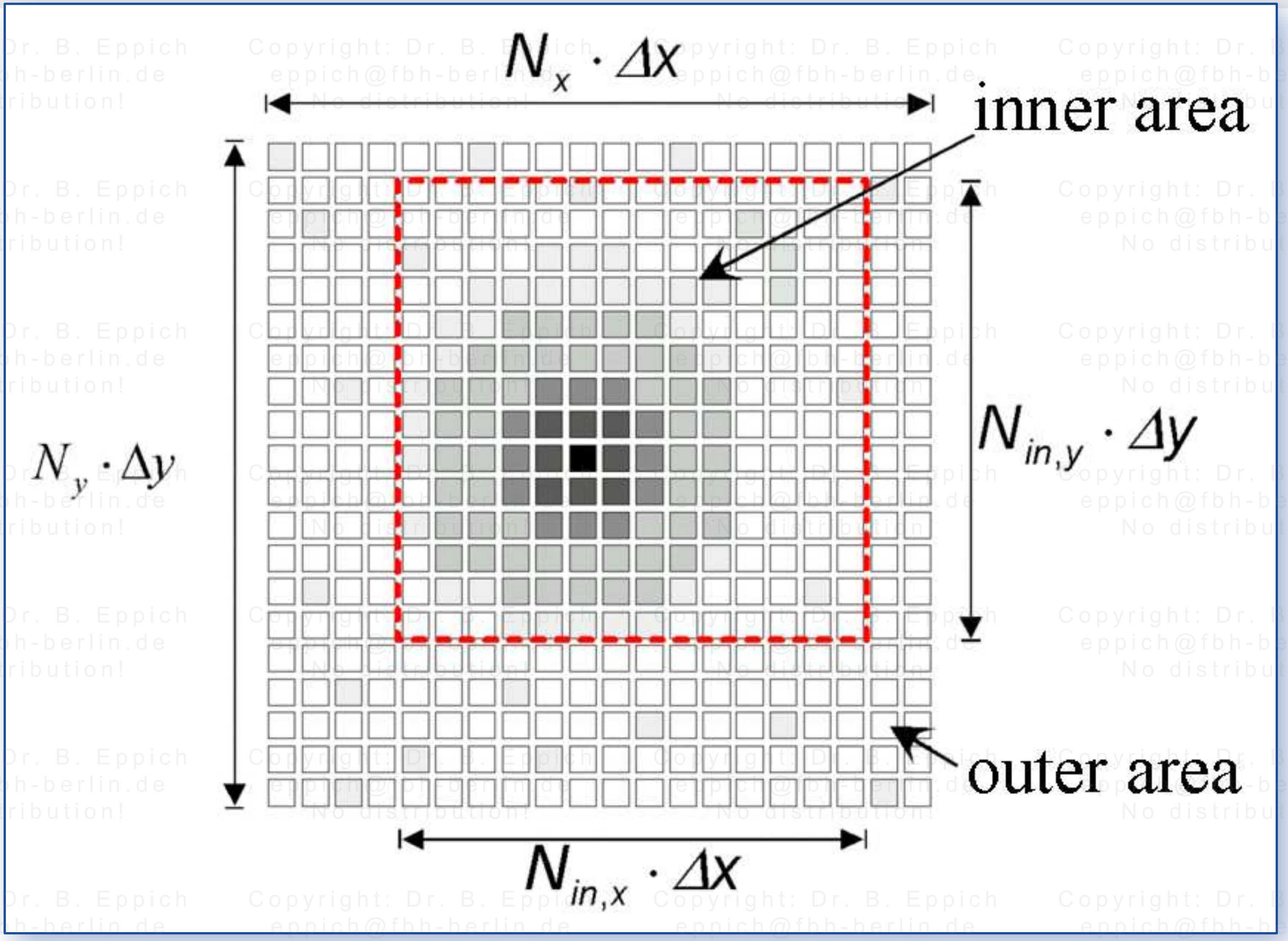




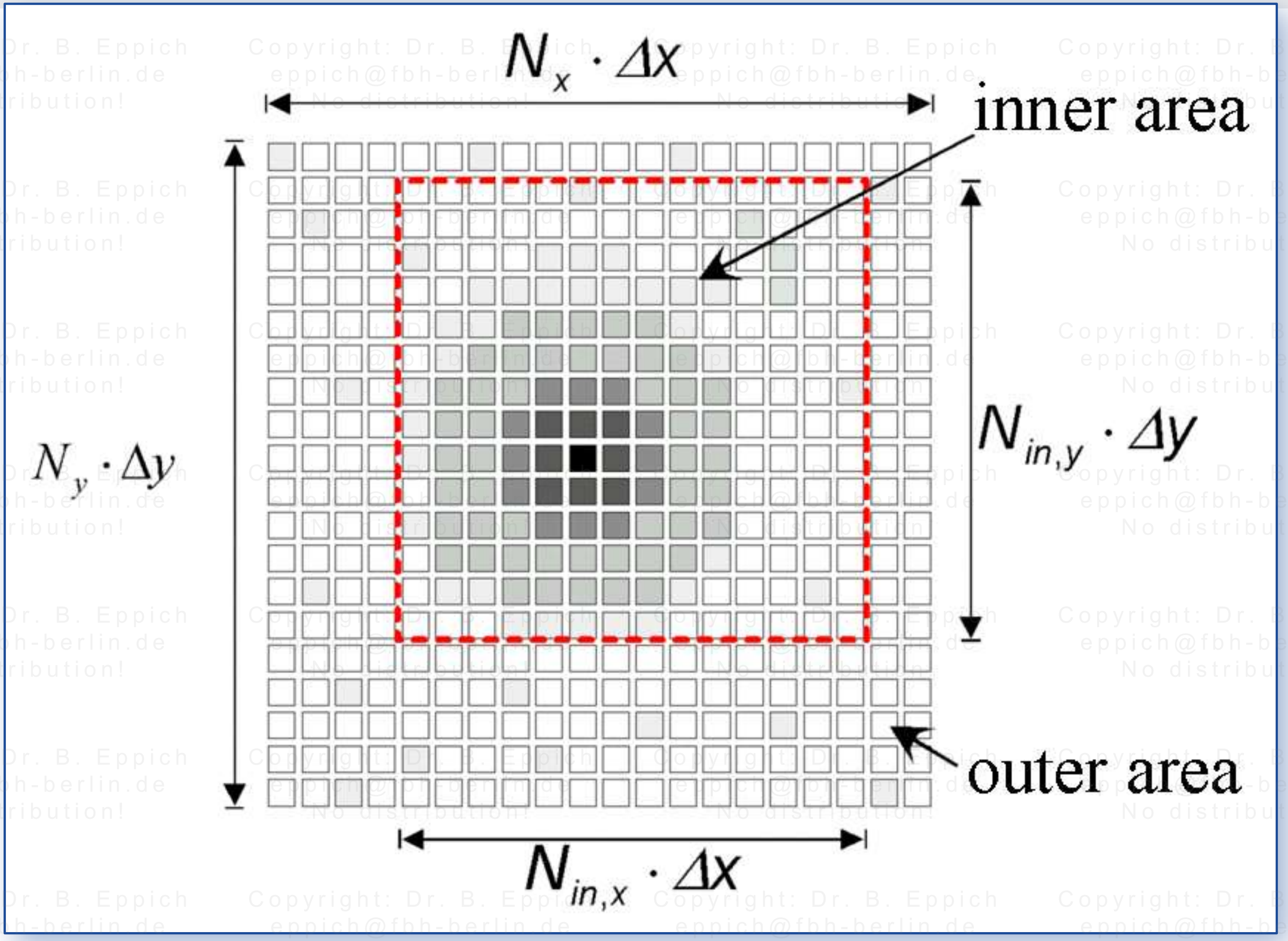
Offset determination and integration area



Error estimation



Error estimation



Error estimation

Offset:

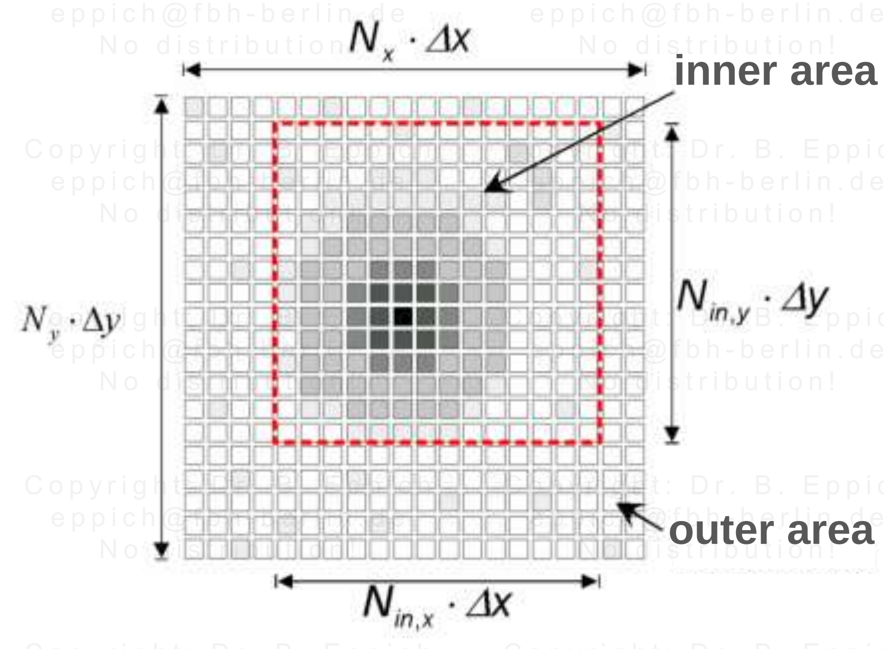
$$Off = \frac{1}{N_{out}} \sum_{i_x, i_y \in A_{out}} I_{i_x, i_y}$$

Power:

$$P = \int I(x,y)\,dx\,dy = \Delta x\,\Delta y \sum_{i_x, i_y \in A_{in}} \left(I_{i_x, i_y} - Off\right)$$

Center of gravity:

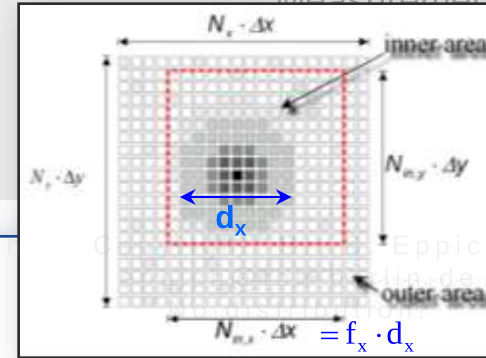
$$\bar{x} = \frac{1}{P}\int I(x,y)\,x\,dx\,dy = \frac{\Delta x\,\Delta y}{P} \sum_{i_x, i_y \in A_{in}} \left(I_{i_x, i_y} - Off\right) x_{ix}$$



Diameter:

$$d_x = 4\sqrt{\frac{1}{P}\int I(x,y)(x-\bar{x})^2\,dx\,dy} = 4\sqrt{\frac{\Delta x\,\Delta y}{P} \sum_{i_x, i_y \in A_{in}} I_{i_x, i_y} (x_{ix} - \bar{x})^2}$$

Error estimation



Error of each pixel value: σ_I

Signal-noise-ratio:

$$D_{Det} = \frac{I_{sat}}{\sigma_I}$$

„average“ intensity:

$$I_{ave} = \frac{\text{beam power}}{\text{beam area}} = \frac{P}{d_x\, d_y}$$

Signal „dynamic“:

$$D_{Sig} = \frac{\text{max.intensity}}{\text{ave.intensity}} = \frac{I_{\max}}{I_{ave}}$$

Width/height of integration area:

$$f_x = \frac{N_{in,x}\Delta x}{d_x} \qquad f_y = \frac{N_{in,y}\Delta y}{d_y}$$

Error estimation

Rel. error of power:

$$\frac{\sigma_P}{P} \propto f_x\ f_y\ \frac{D_{Sig}}{D_{Det}}$$

Rel. error of center of gravity:

$$\frac{\sigma_{\bar{x}}}{d_x} \propto f_x^{\ 2}\ f_y\ \frac{D_{Sig}}{D_{Det}}$$

Rel. error of diameter:

$$\frac{\sigma_{d_x}}{d_x} \propto f_x^{\ 3}\ f_y\ \frac{D_{Sig}}{D_{Det}}$$

Error estimation

Results of error discussion:

- **There is no general error boundary**
- **Statistical error depends on**
  - **- signal dynamic**
  - **- detector dynamic**
  - **- size of integration area**
- **It is important, to choose the integration area as small as possible!**

Integration area determination

Self-converging approach

1. Offset determination in outer area
2. „random“ choice of small central inner area
3. Determination of Diameter in inner area
4. New choice of inner area as a multiple of obtained diameter
5. Continuation at clause 3. until convergence.

Integration area determination

Statistical approach:

- **Choice of outer area**
- **Determination of mean and standard deviation of pixel values in outer area**
- **Determination of inner area as smallest possible rectangle containing all pixel having values larger a multiple of the standard deviation larger than the mean value.**

Averaging and background correction

Advantages of averaging:

- **Increase signal-to-noise ratio**

To be considered:

- **Avoid round-off error → enhance bit depth!**
- **Signal shall be stable (structure and position of profile)**

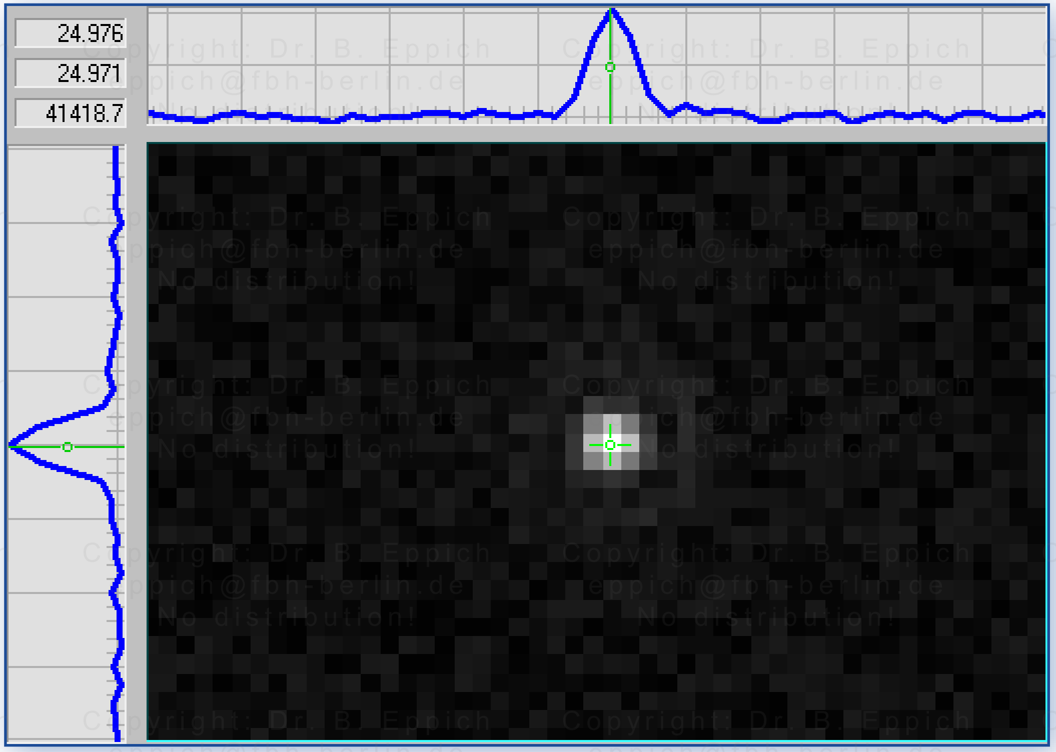
Advantages of background correction:

- **Reduce inhomogeneous offset (black level) values**
- **Eliminate stray light**

To be considered:

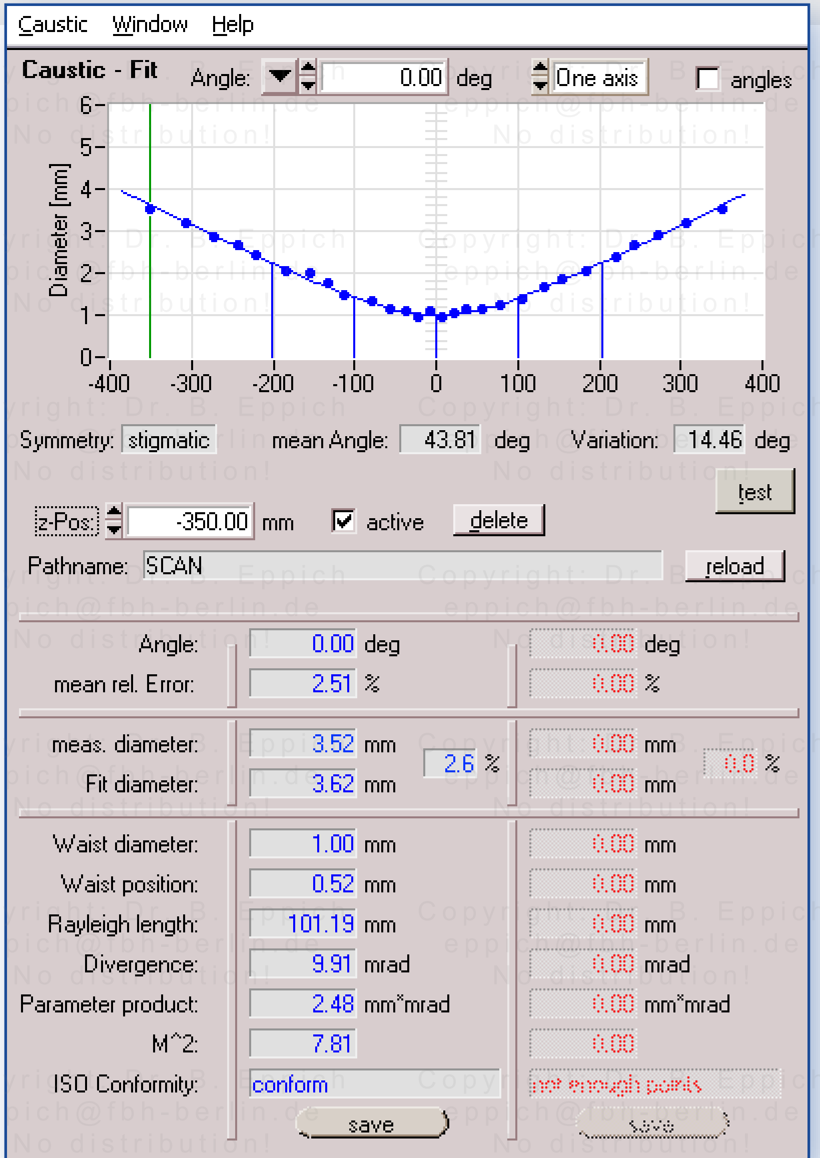
- **Signal-to-noise ratio decreases → use averaging!**
- **Background image may depend on camera settings (integration time, gain, binning,….) → use separate background images!**

Other requirements



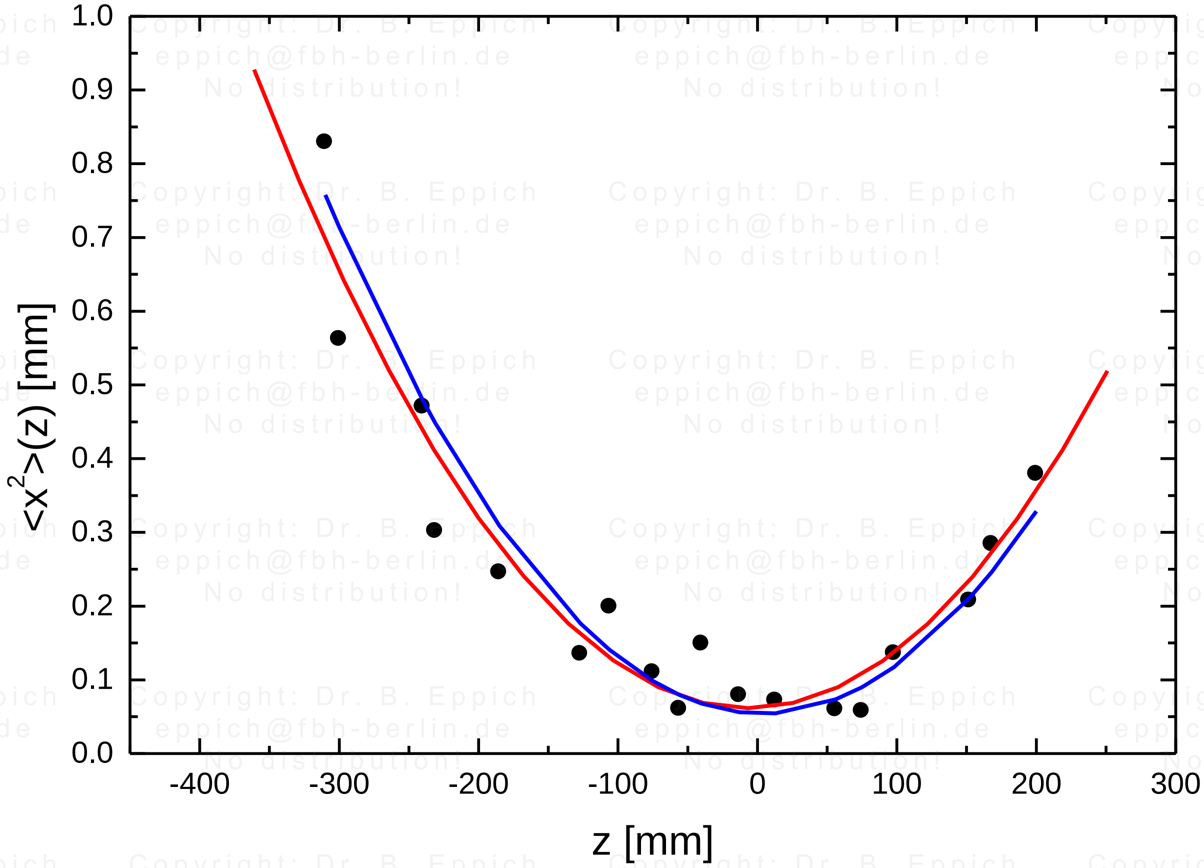
Number of pixels across beam diameter > 50 !

Other requirements



- Number of z-positions ≥ 10
- approx. half number within one Rayleigh length
- approx. half number outside two Rayleigh length

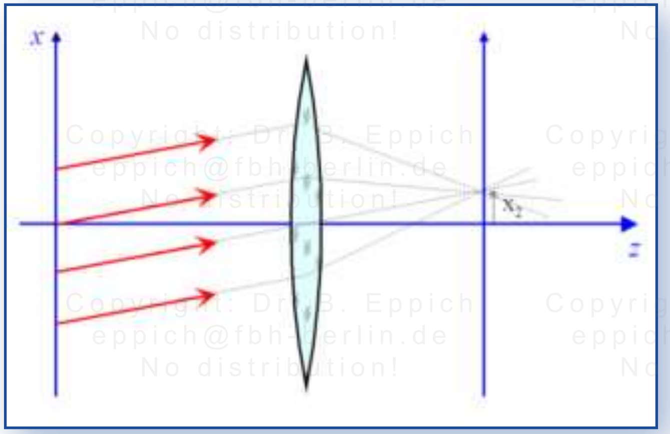
Other requirements



Fitting on poor data may depend on weighing factors!

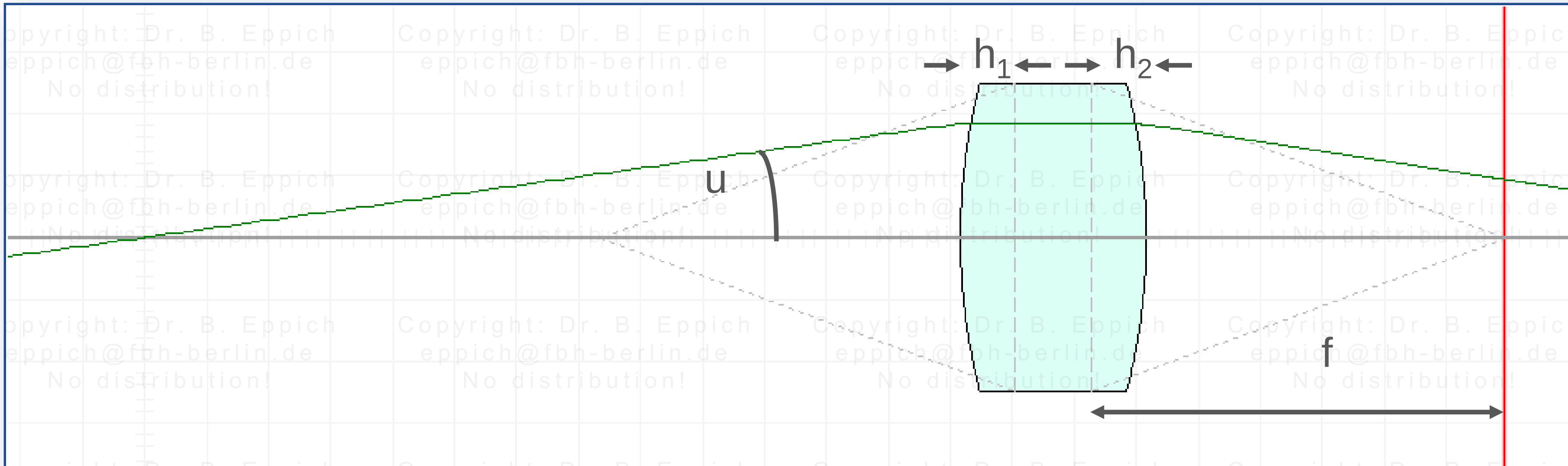
Direct measurement of divergence

Fourier transformer, far field

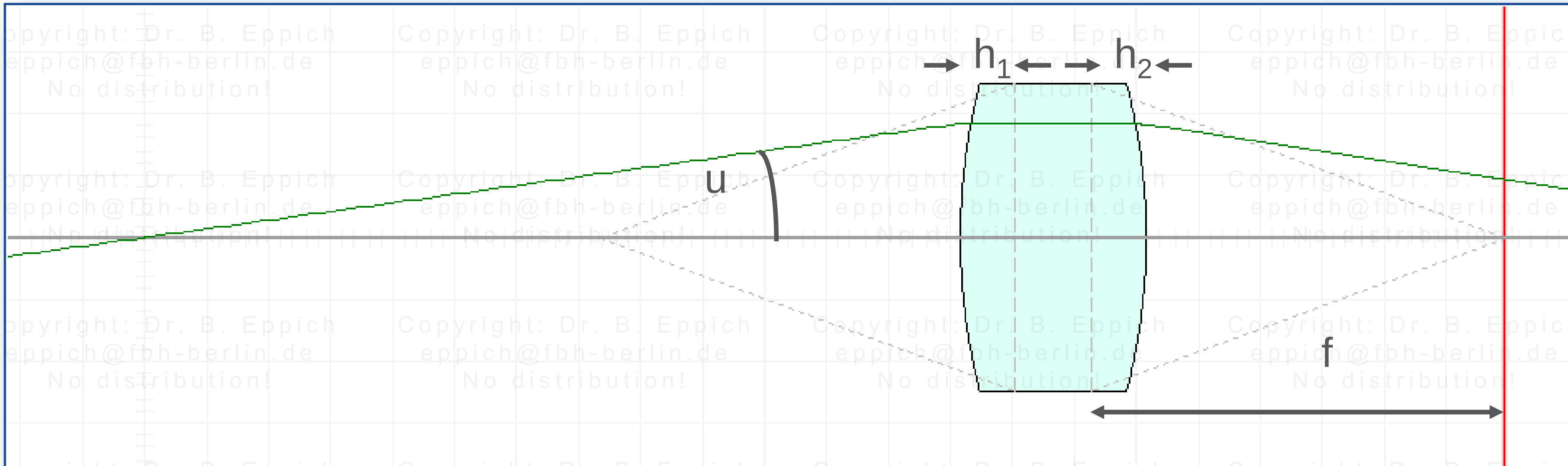


$$\mathbf{S} = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$$

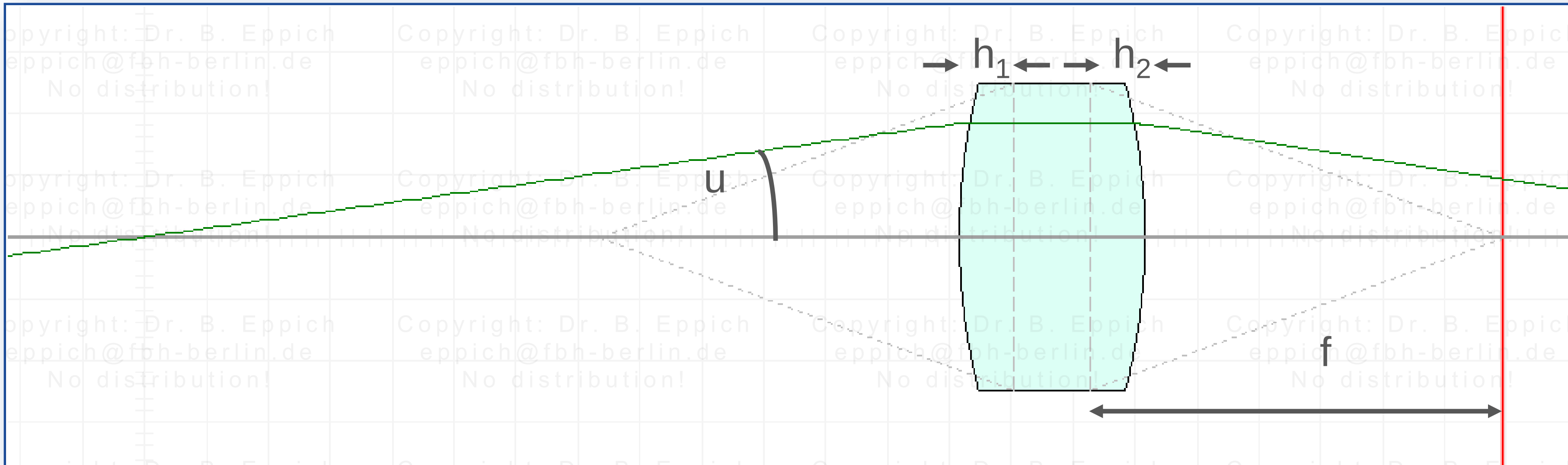
$$\begin{pmatrix} x \\ u \end{pmatrix}_{out} = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} x \\ u \end{pmatrix}_{in} \longrightarrow \begin{pmatrix} x_{out} \\ u_{out} \end{pmatrix} = \begin{pmatrix} B u_{in} \\ C x_{in} + D u_{in} \end{pmatrix}$$



$$\mathbf{S} = \begin{pmatrix} 1 & f + h_2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 - \frac{h_2}{f} & h_1 + h_2 - \frac{h_1 h_2}{f} \\ -\frac{1}{f} & 1 - \frac{h_1}{f} \end{pmatrix}$$

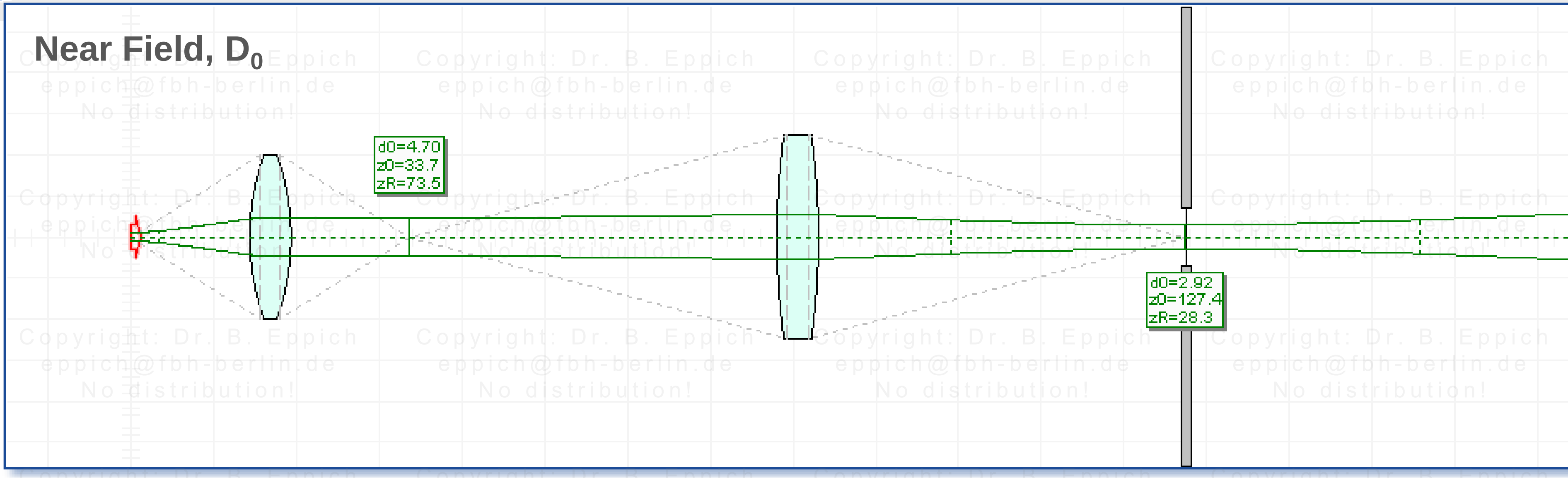


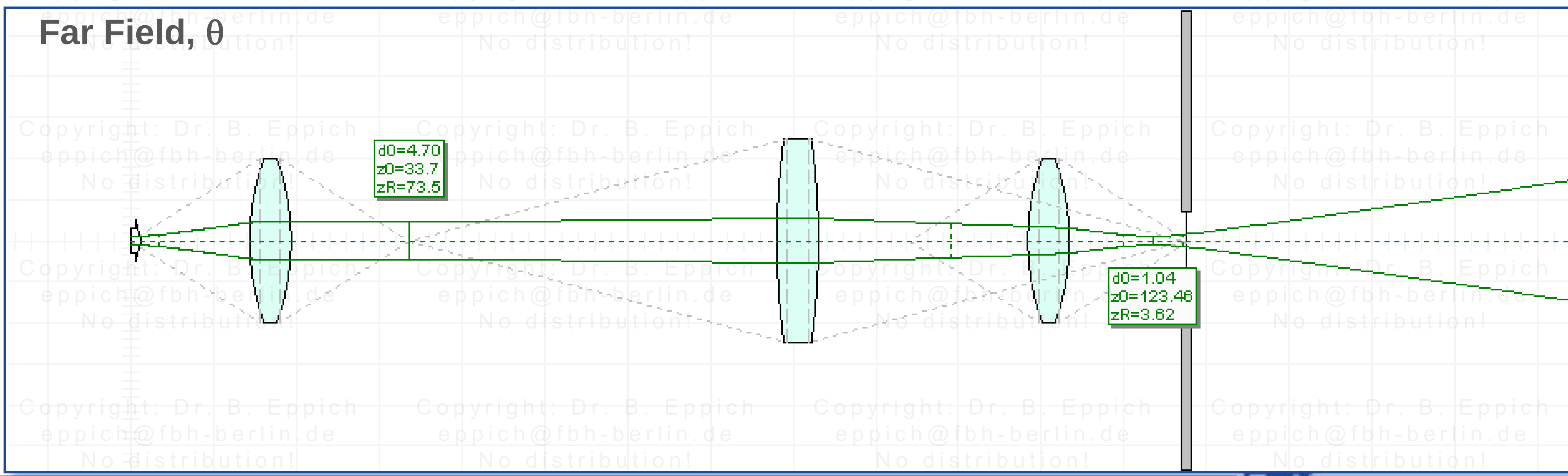
$$\mathbf{S} = \begin{pmatrix} 0 & f \\ -\frac{1}{f} & 1-\frac{h_1}{f} \end{pmatrix}$$



$$\begin{pmatrix} x_{out} \\ u_{out} \end{pmatrix} = \begin{pmatrix} f\, u_{in} \\ -\frac{1}{f} x_{in} + \left(1 - \frac{h_1}{f}\right) u_{in} \end{pmatrix}$$

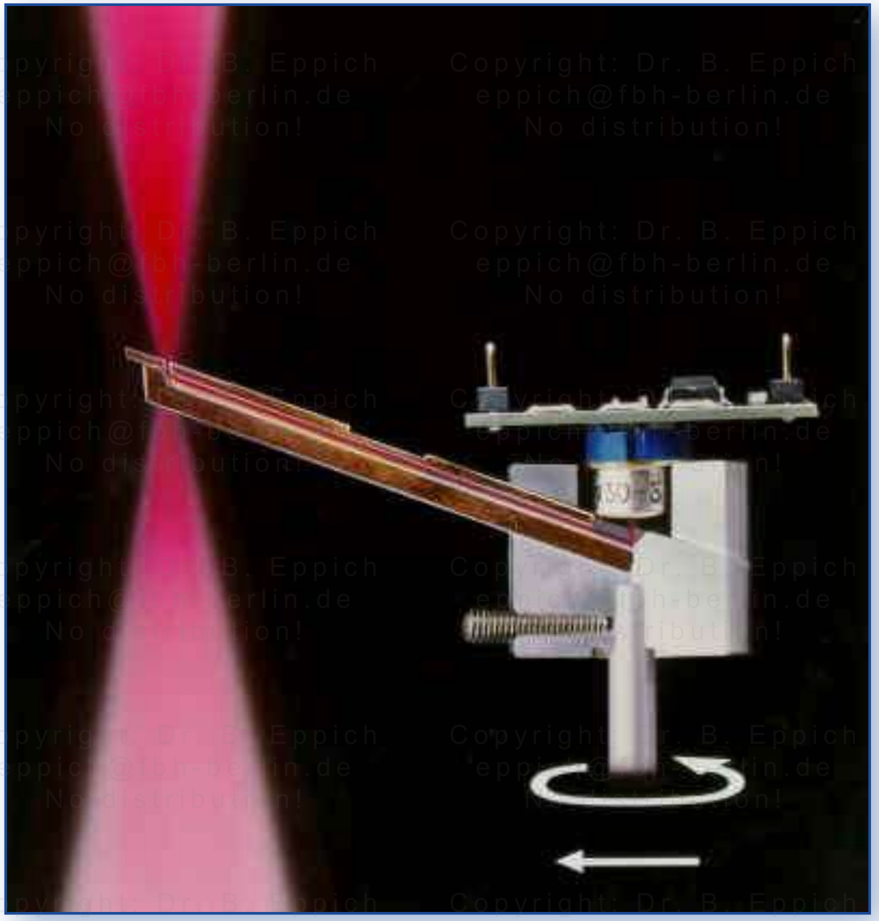
Two-Point measurement of beam quality

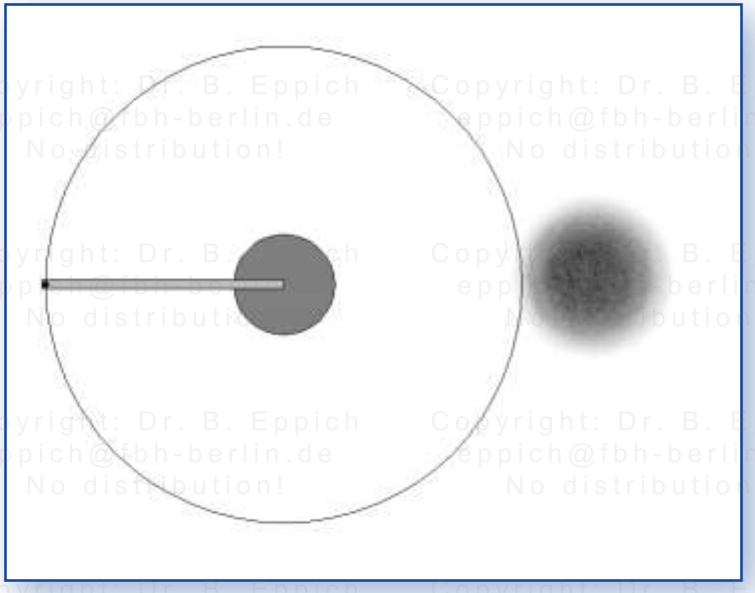




Other types of detectors

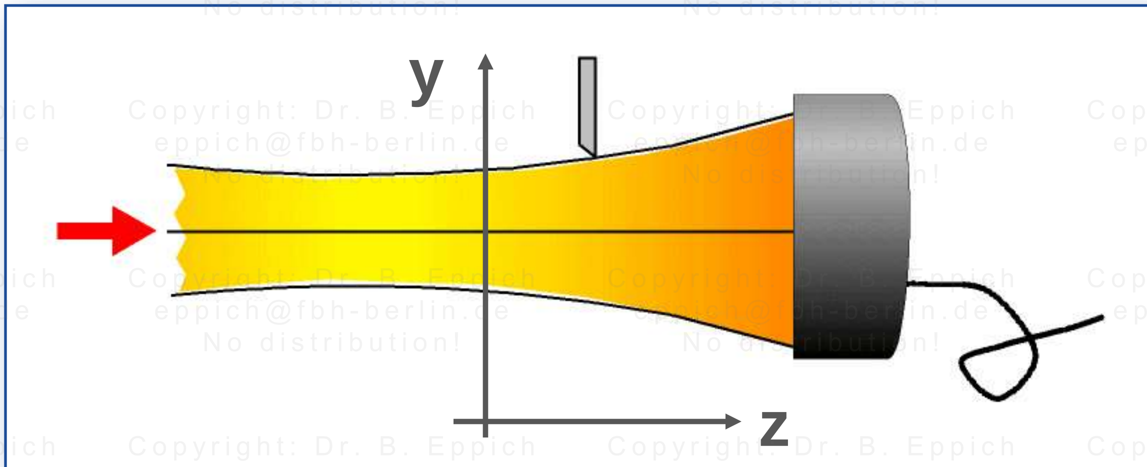
Rotating pin hole





Other types of detectors

Knife edge methods



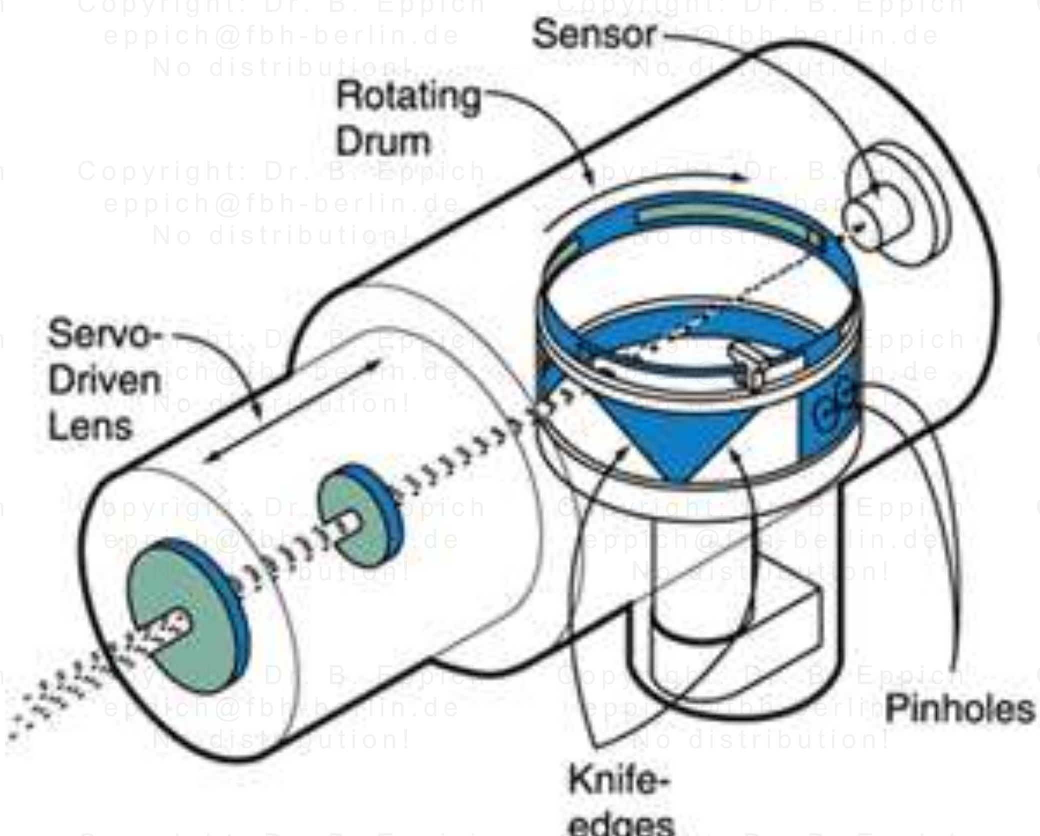
$$P(y) = \int_{-\infty}^{y} \int_{-\infty}^{\infty} I(x', y')\, dx'\, dy'$$

$$\langle y^n \rangle = \frac{\int \partial_y P(y)\, y^n\, dy}{P(\infty)}$$

Other types of detectors

Knife edge methods

Example: Coherents „ModeMaster“

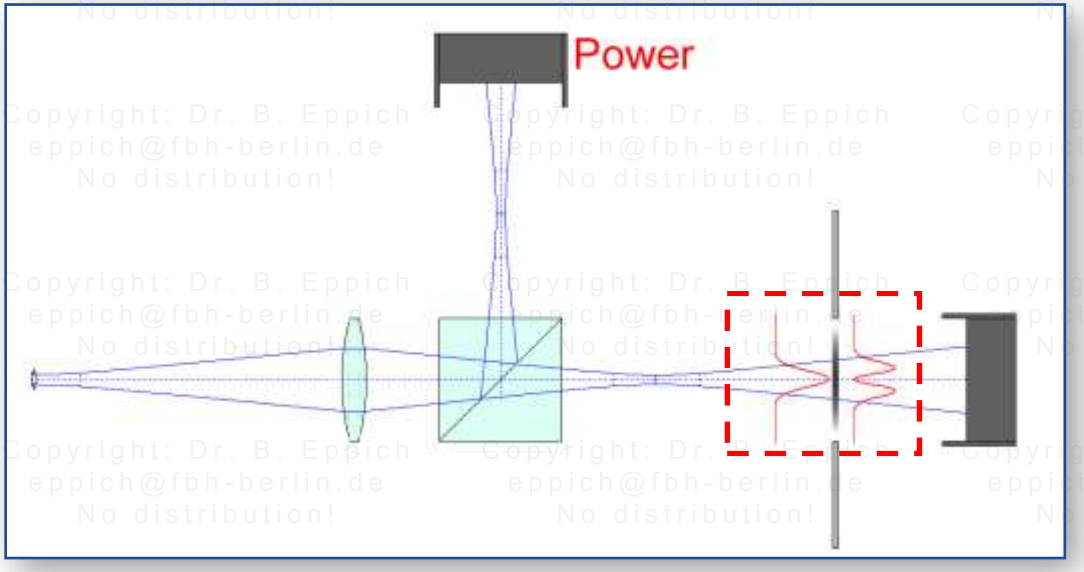


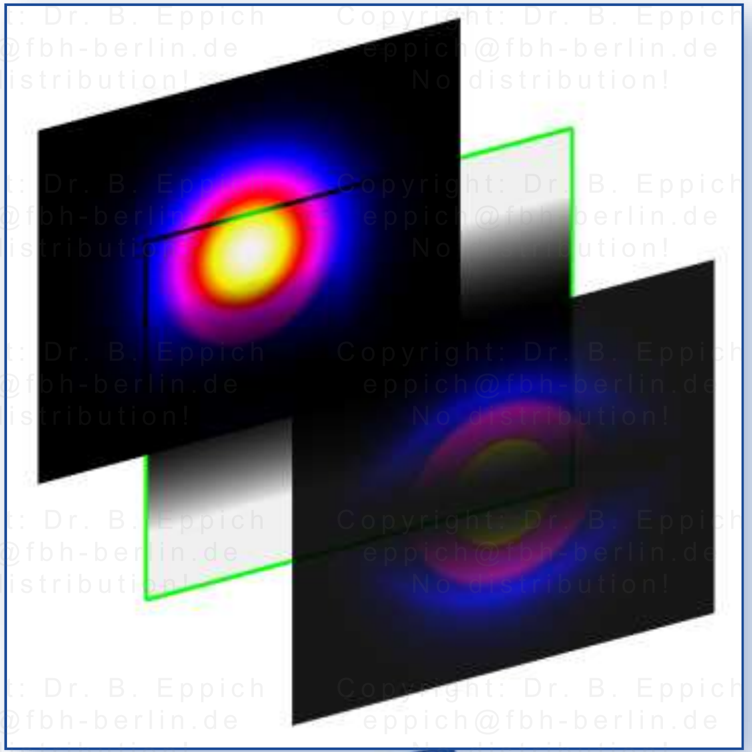
Other types of detectors

Spatial dependent filters

$$\langle x^2 \rangle = \frac{\int I(x,y)\, x^2\, dx\, dy}{\int I(x,y)\, dx\, dy} \longrightarrow \langle x^2 \rangle = \frac{1}{a} \frac{\int I(x,y)\, t(x)\, dx\, dy}{\int I(x,y)\, dx\, dy}$$

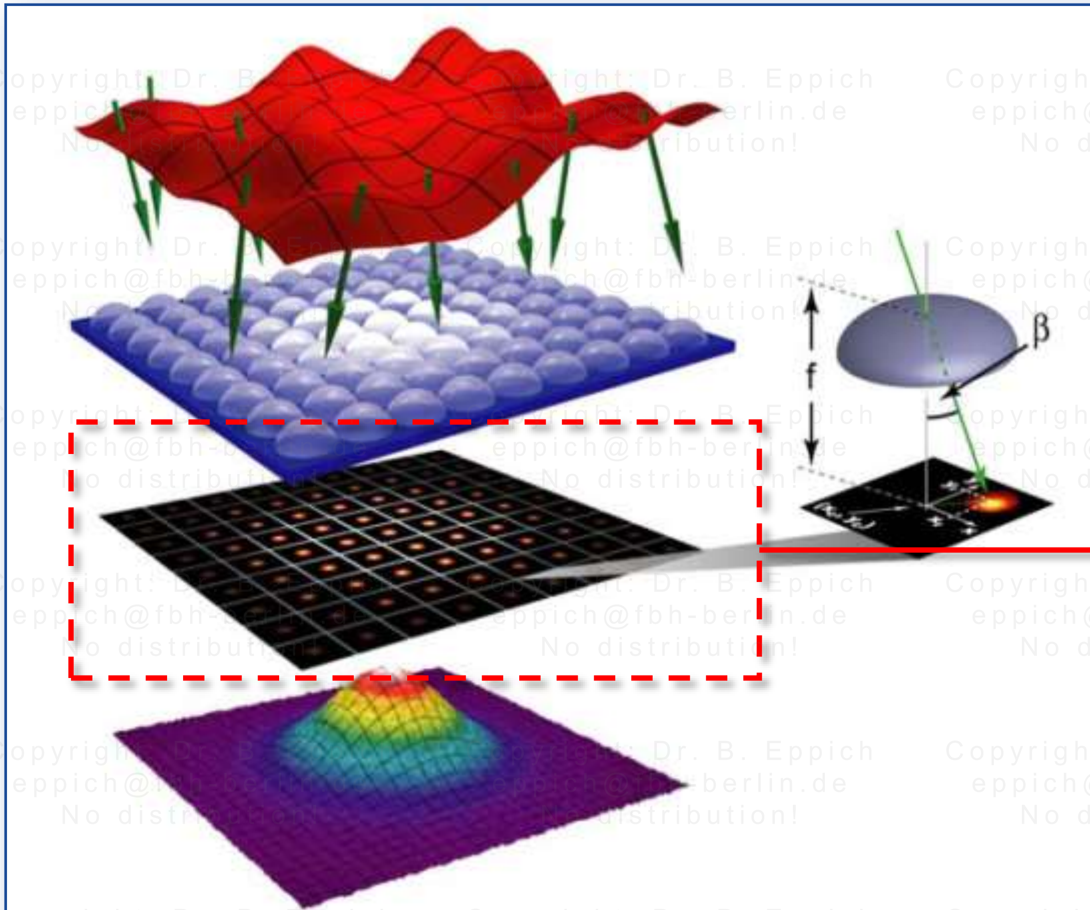
with $t(x) = \begin{cases} a\, x^2 & \text{for} \quad x^2 < 1/a \\ 1 & \text{other} \end{cases}$





Other types of detectors

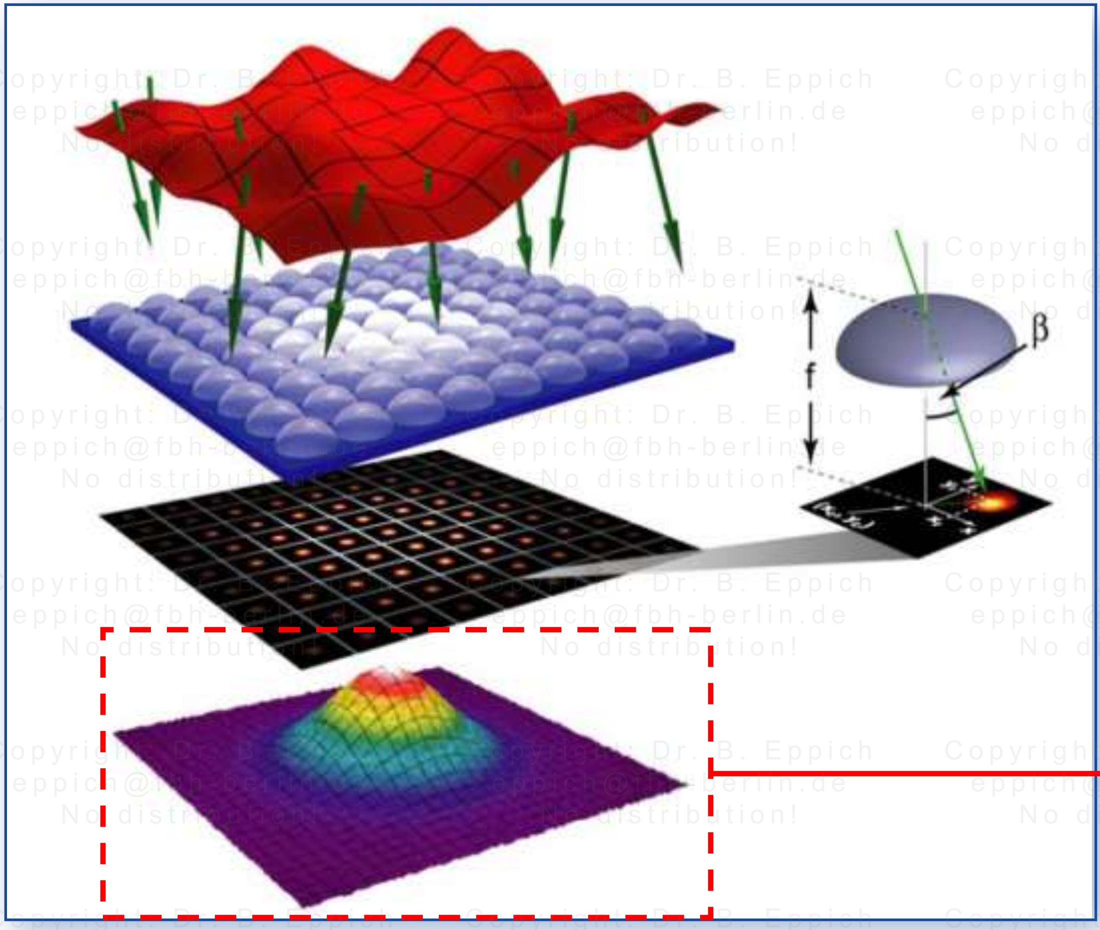
Shack-Hartmann sensors



Phase curvature $R_{x,y,\alpha}$

Other types of detectors

Shack-Hartmann sensors

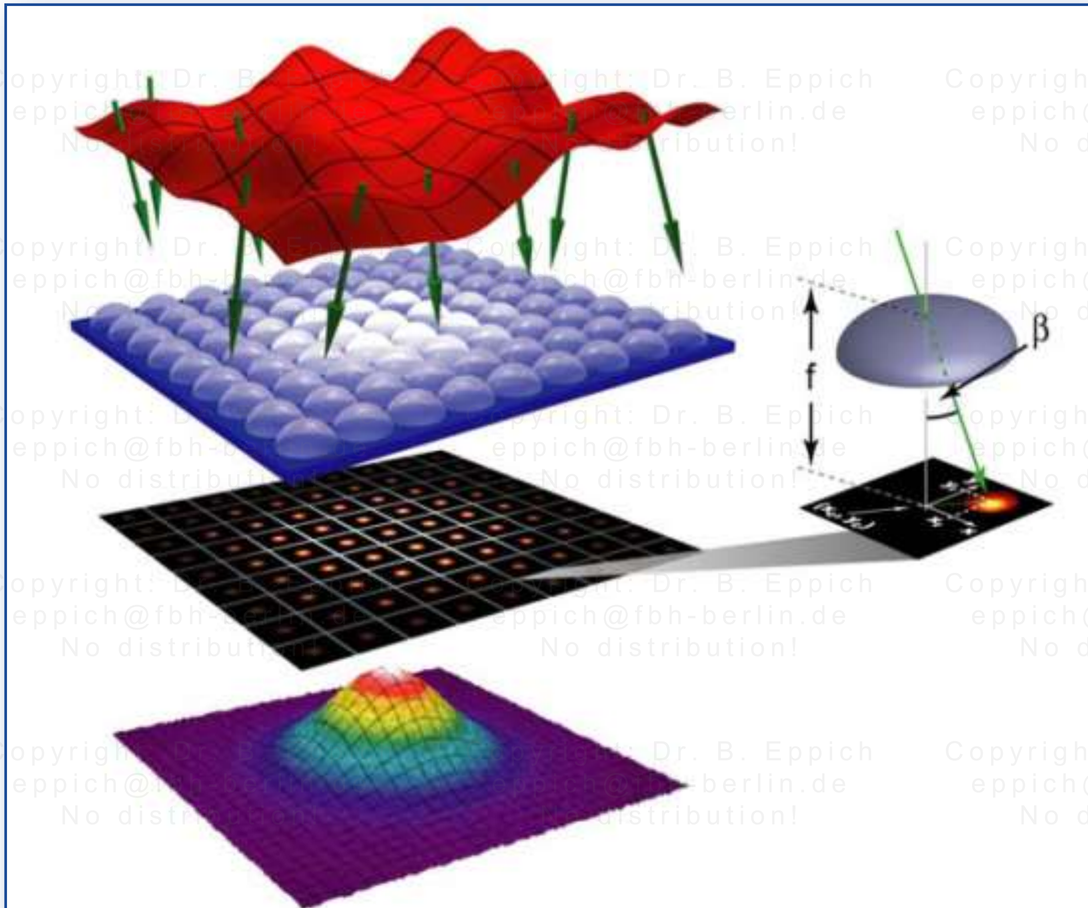


Phase curvature $R_{x,y,\alpha}$

Diameter $D_{x,y,\alpha}$

Other types of detectors

Shack-Hartmann sensors



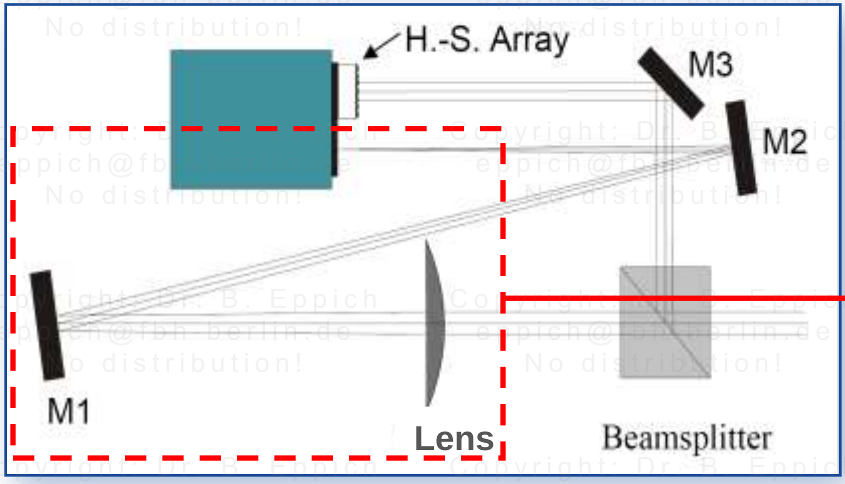
Phase curvature $R_{x,y,\alpha}$

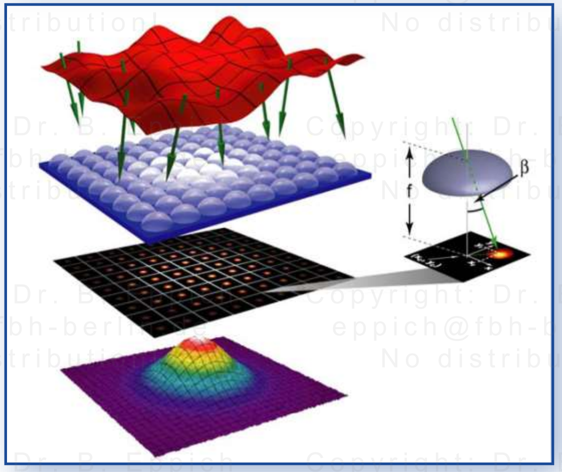
Diameter $D_{x,y,\alpha}$

Third Parameter missing!

Other types of detectors

Shack-Hartmann sensors





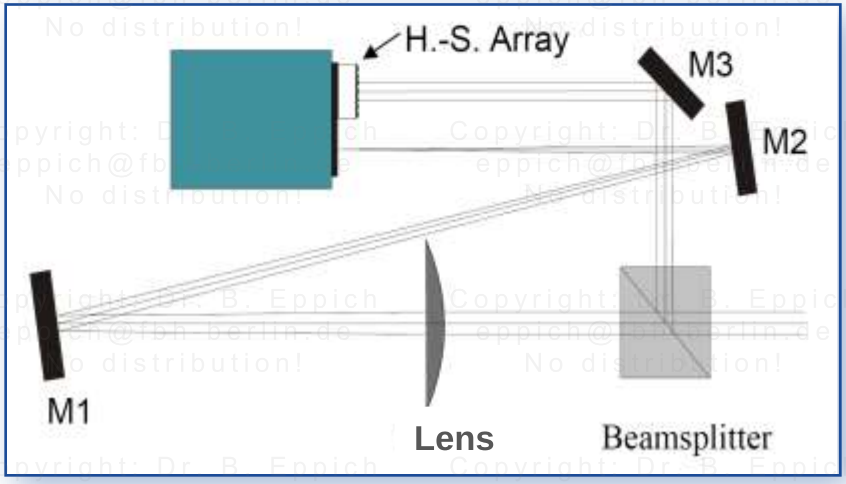
Divergence $\theta_{x,y,\alpha}$

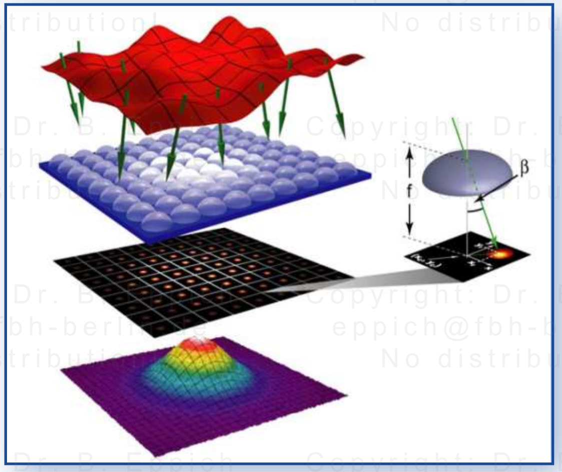
Phase curvature $R_{x,y,\alpha}$

Diameter $D_{x,y,\alpha}$

Other types of detectors

Shack-Hartmann sensors





Divergence $\theta_{x,y,\alpha}$

Phase curvature $R_{x,y,\alpha}$

Diameter $D_{x,y,\alpha}$