

# On the non-linear scale of cosmological perturbation theory

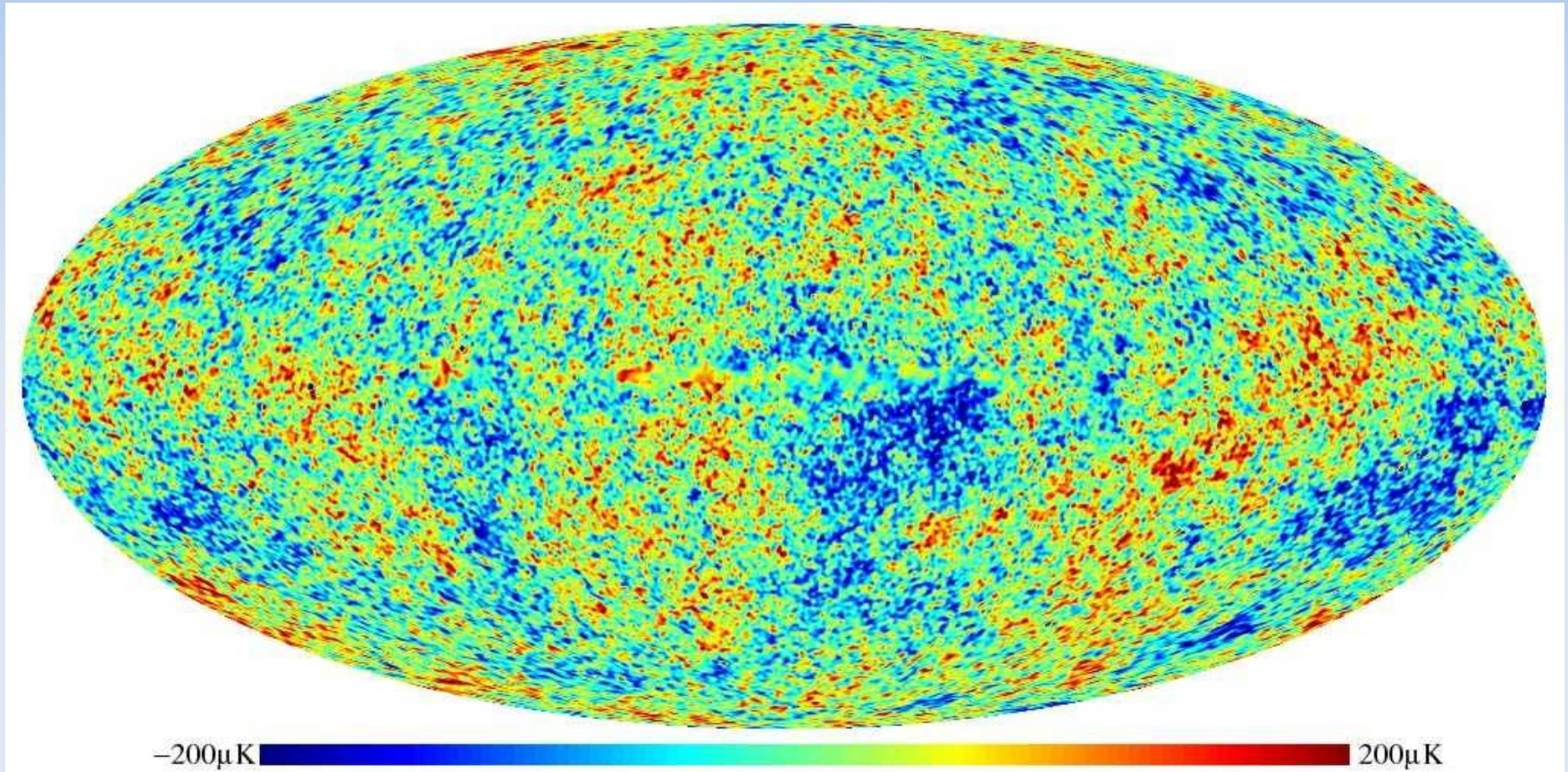
Thomas Konstandin



in collaboration with  
**D. Blas and M. Garny**

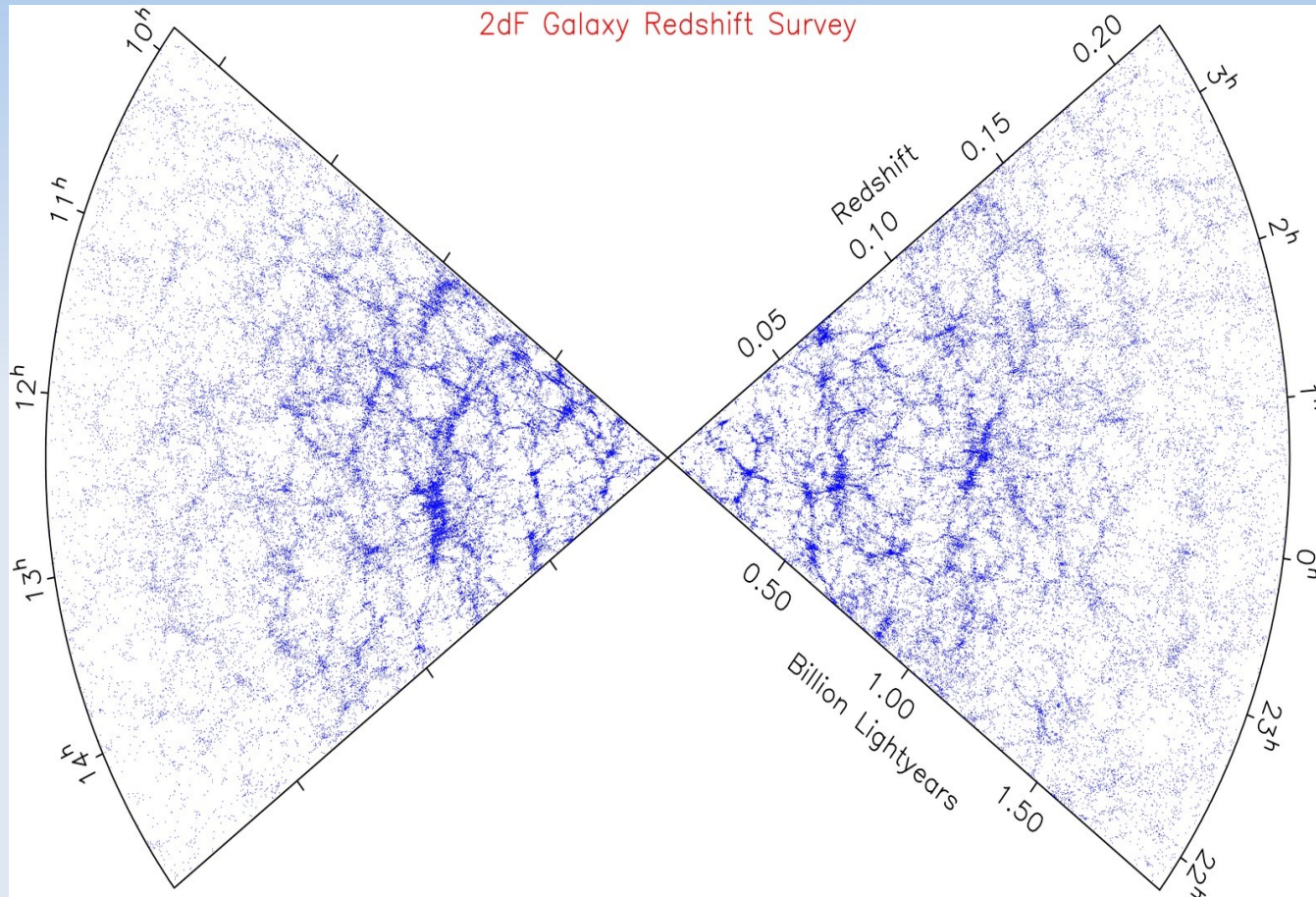
PONT Avignon, April 16, 2014

# How to get from here ...



Initial conditions of structure formation are very well established by CMB measurements

# ... to here ...

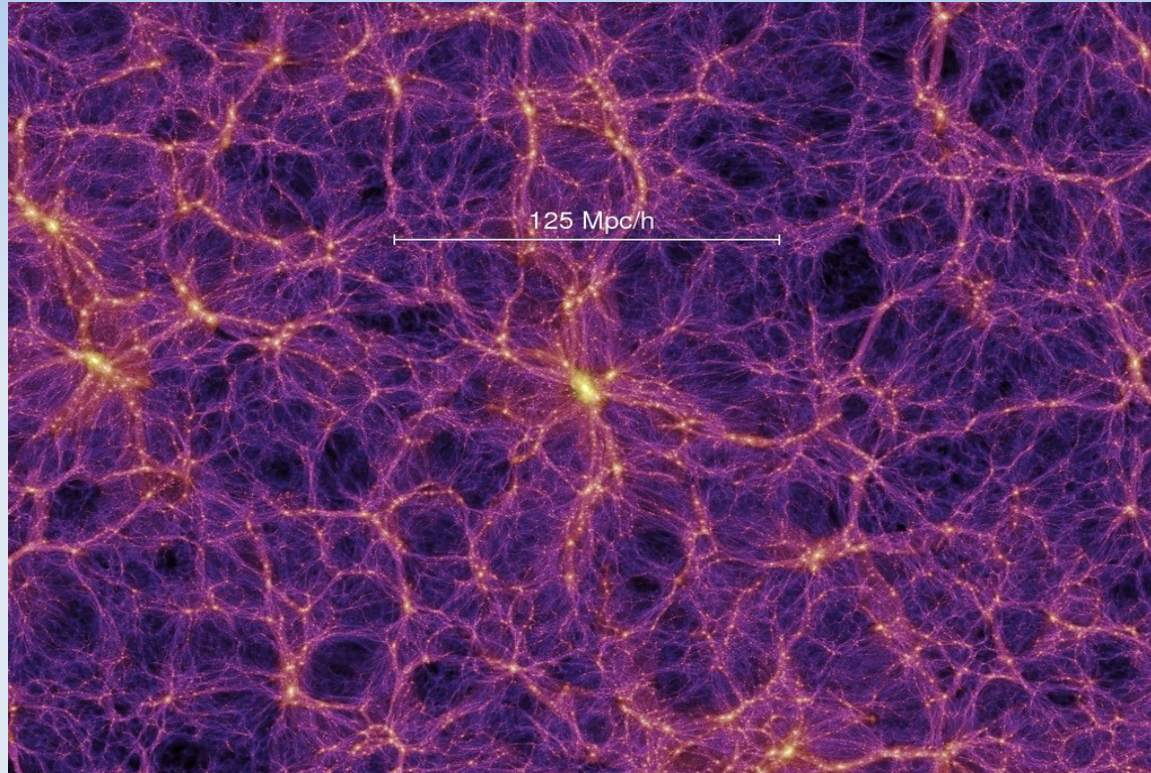


Galaxy surveys determine the matter distribution on large scales.

It is sensitive to different red-shifts and probes the real expansion history and not just a model.



# .. without this.

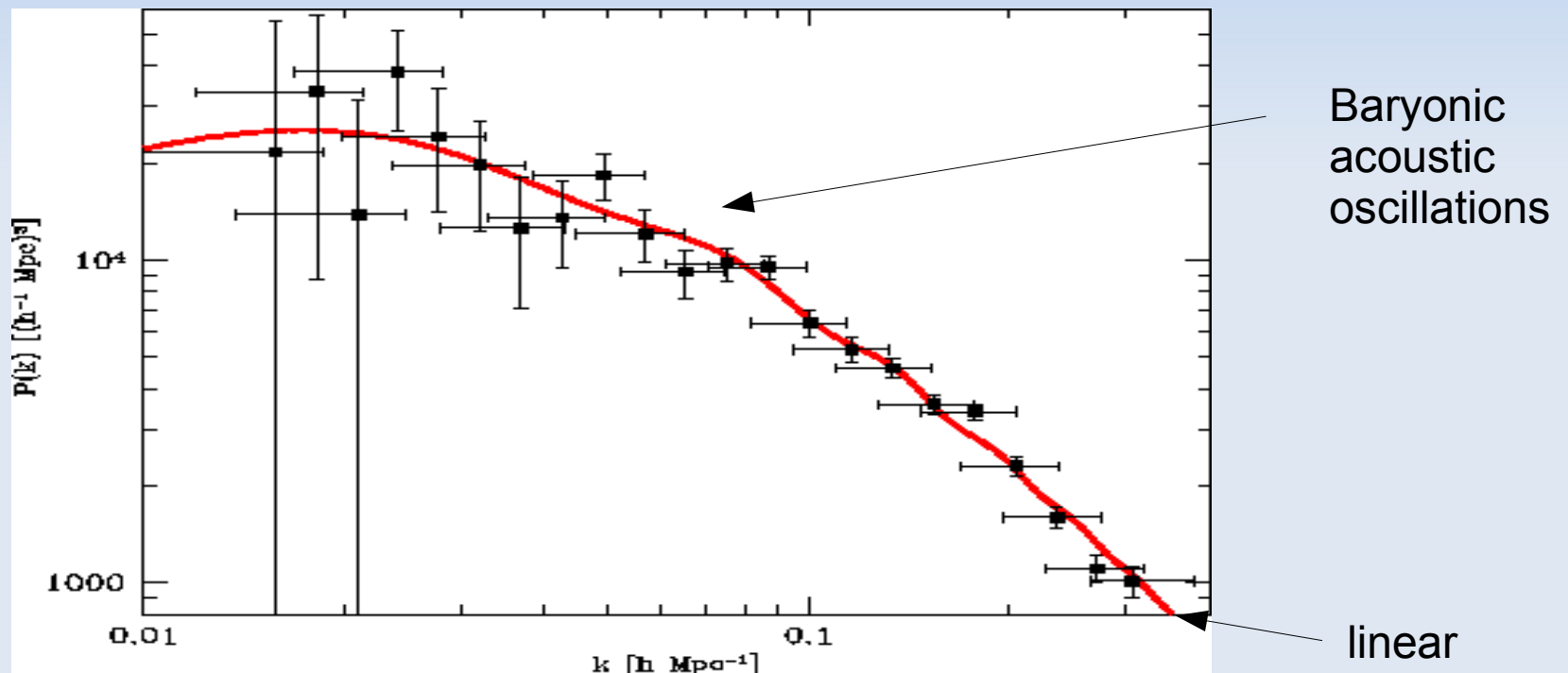


The most reliable predictions for the matter distribution come from n-body simulations. But they are very demanding.

# The linear power spectrum

We are primarily interested in the power spectrum of density fluctuations

Sloan Digital Sky Survey (2006)



This measurement should improve tremendously with Euclid and other LSS measurements. On which **distance scales** can you trust the linear result?

# Outline

Standard perturbation theory (SPT)

Resummation

Conclusions

# Outline

Standard perturbation theory (SPT)

*Feynman rules of large scale  
structure*

# Dark matter as a fluid

Starting point are the hydrodynamic fluid equations in an expanding universe

continuity:	$\frac{\partial \delta}{\partial \tau} + \nabla \cdot (1 + \delta) \vec{v} = 0$	$\delta = \rho / \bar{\rho} - 1$ <div>density contrast</div>
Euler:	$\frac{\partial \vec{v}}{\partial \tau} + H \vec{v} + \vec{v} \cdot \nabla \vec{v} = -\nabla \Phi$	$\vec{v}$ <div>fluid velocity</div> $H$ <div>Hubble parameter</div> $\Phi$ <div>grav. potential</div>

in combination with the Poisson equation

$$\nabla^2 \Phi = 4\pi G a^3 \bar{\rho} \delta(x, \tau)$$

(assumptions: matter is collisionless, pressureless, single-streaming)



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$\vec{v}$  fluid velocity

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(assumptions: matter is collisionless, pressureless, single-streaming)

# Linear solutions: growth

The linearized system can be phrased as

$$\frac{\partial \Psi}{\partial \eta} + \Omega \Psi = 0$$

with

$$\Psi = \begin{pmatrix} \delta \\ \frac{\nabla \vec{v}}{H} \end{pmatrix}$$

$$\eta = \log a(\tau)$$

where (for Einstein-de Sitter)

$$\Omega = \begin{pmatrix} 0 & -1 \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

Newton force

continuity

expansion

# Linear solutions: growth

$$\Psi_L(k, \eta) \equiv g(\eta, \eta_0) \Psi(k, \eta_0)$$

The corresponding Green's function is

$$g(\eta, \eta_0) = \frac{e^{(\eta - \eta_0)}}{5} \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} + \frac{e^{-3(\eta - \eta_0)/2}}{5} \begin{pmatrix} 2 & -2 \\ -3 & 3 \end{pmatrix}$$

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For the power spectrum

$$P_{ab}(k, \tau) \delta_K(k + k') \equiv \langle \Psi_a(k, \tau) \Psi_b(k', \tau) \rangle$$

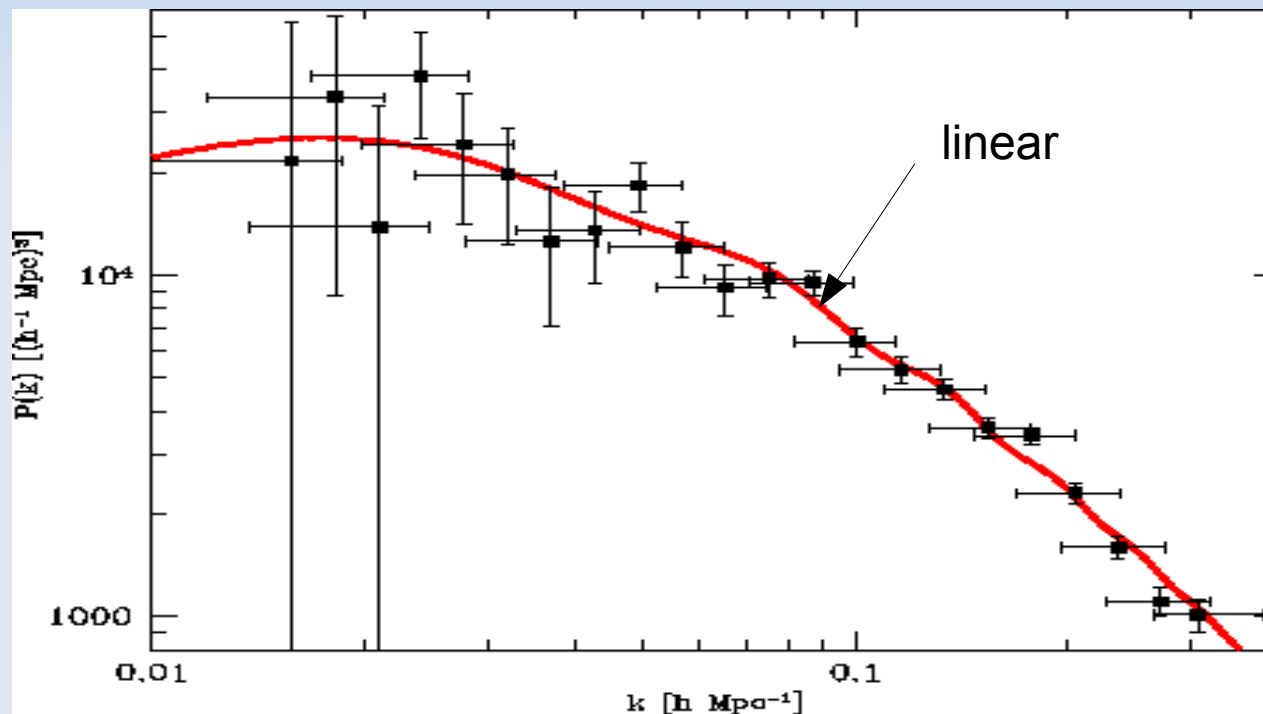
growing mode  $\rightarrow$

$$P_{L,ab}(\tau, k) \propto e^{2(\eta-\eta_0)} P_0(k) \simeq a(\tau)^2 P_0(k)$$

# The linear power spectrum

We are primarily interested in the power spectrum of density fluctuations

Sloan Digital Sky Survey (2006)



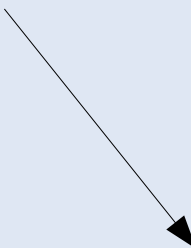


# Perturbative expansion

Non-linearities are determined in SPT by solving the equations iteratively in Fourier space starting from the linear solution

$$\partial_\eta \Psi_a^{(1)}(k, \eta) + \Omega_{ab} \Psi_b^{(1)}(k, \eta) = 0$$

$$\partial_\eta \Psi_a^{(2)}(k, \eta) + \Omega_{ab} \Psi_b^{(2)}(k, \eta)$$


$$= \int d^3 k_1 d^3 k_2 \gamma_{abc}(k, k_1, k_2) \Psi_b^{(1)}(k_1, \eta) \Psi_c^{(1)}(k_2, \eta)$$

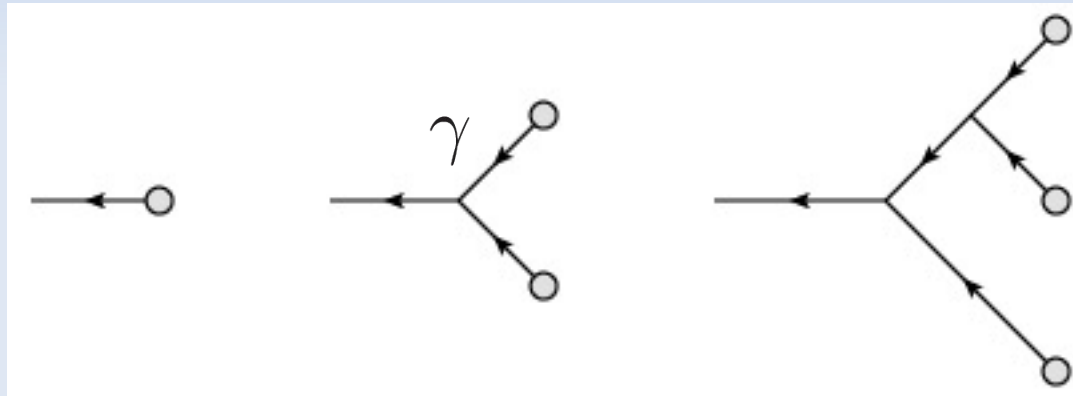
$$\gamma_{abc}(k, k_1, k_2) \ni \delta_K(k - k_1 - k_2) \quad \text{parametrizes the non-linear terms}$$

# Diagrammatic representation

Non-linearities are in SPT included by solving the equations **iteratively**

$$O(\Psi_L) \quad O(\Psi_L^2) \quad O(\Psi_L^3)$$

$$\Psi_a(k, \eta) =$$



The arrows denote the flow of time due to the classical causal structure

[Frieman & Scoccimarro '96]

# Expansion of the power spectrum

This gives for the power spectrum the expansion

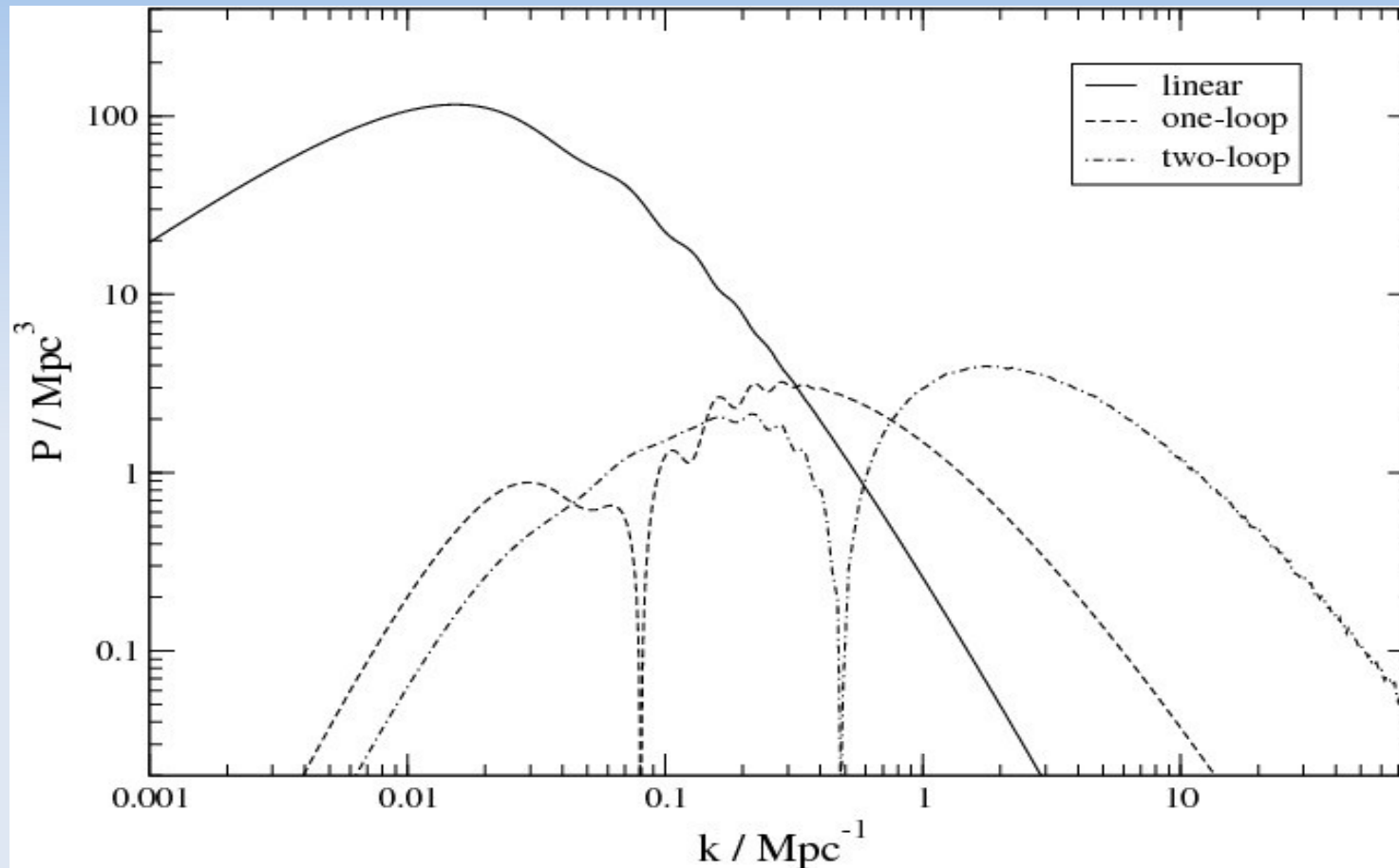
$$\langle \overline{\Psi(k, \tau)} \Psi(k, \tau) \rangle = \text{diagram} = P_L(k)$$

$$P(k, \tau) \delta_K(k + k') \equiv \langle \Psi(k, \tau) \Psi(k', \tau) \rangle$$

$$P(k, \eta) =$$

$$\begin{array}{c} \text{diagram 1} + \text{diagram 2} + \text{diagram 3} \\ \begin{array}{cc} O(P_L) & O(P_L^2) \\ a(\tau)^2 & a(\tau)^4 \end{array} \end{array}$$

# Power spectrum in SPT

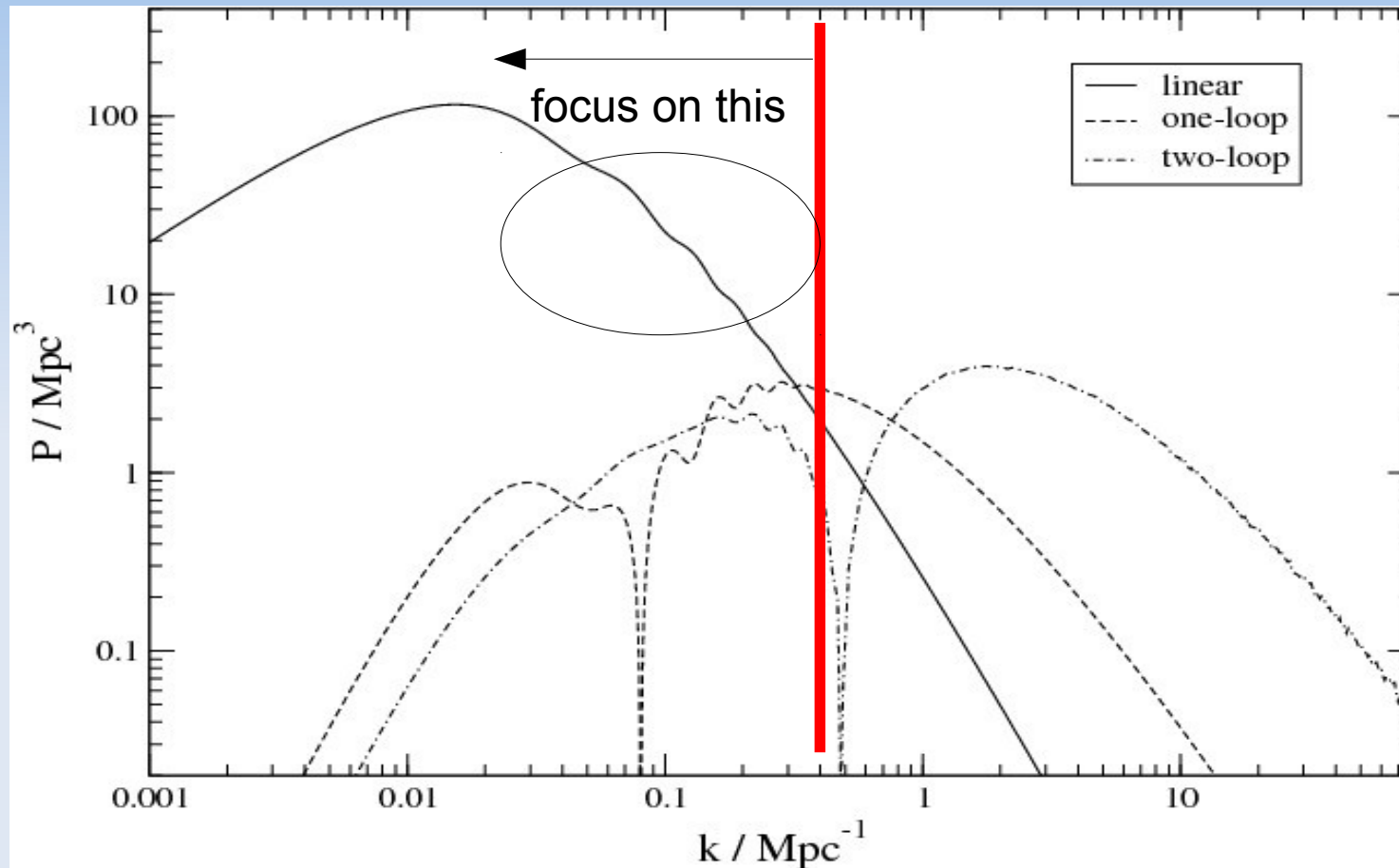


$z = 0$

The convergence is for large momentum very bad for  $z = 0$  but improves for larger redshift, since the  $n$ -loop scales with

$$a(\tau)^{2n} \quad \text{relative to the linear result.}$$

# Power spectrum in SPT



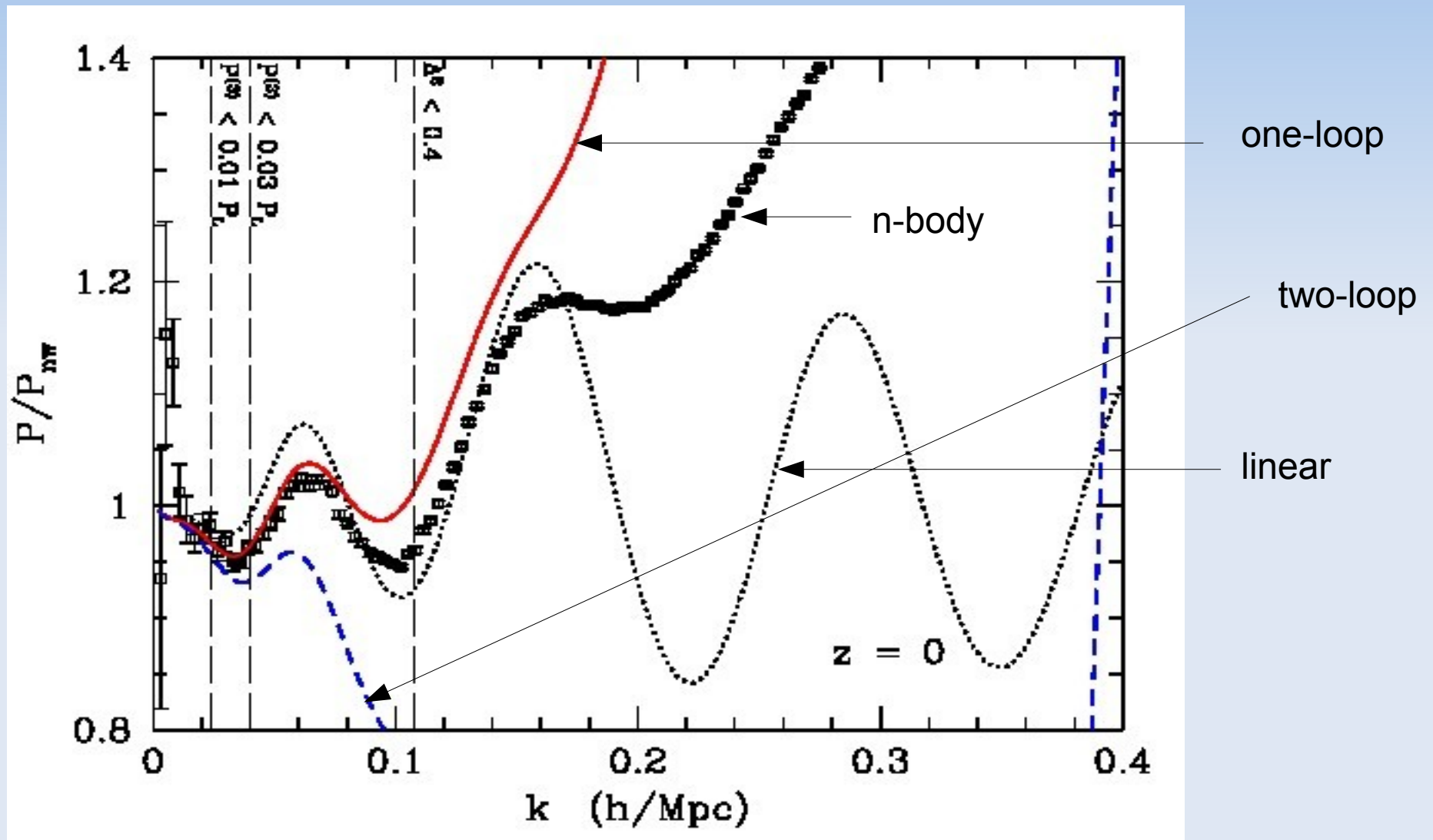
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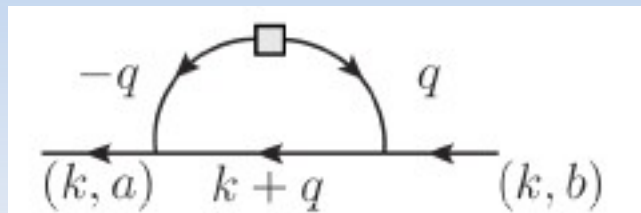
# Deviations $z=0$



[Carlson, White & Padmanabhan '09]

# A problem

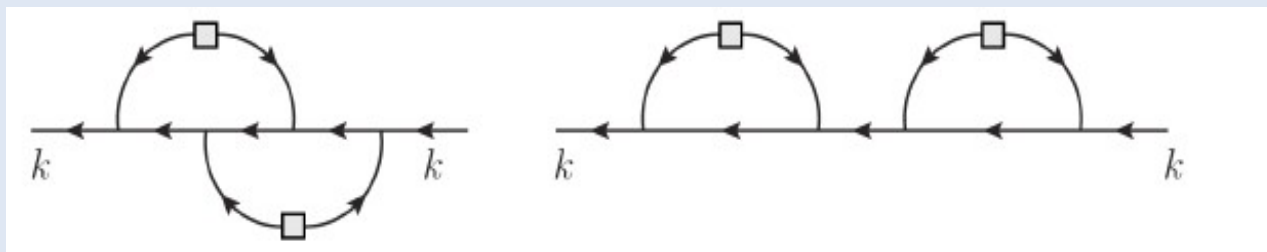
Soft loops are enhanced



$$q \ll k \quad \propto \quad - \int d^3 q \left( \frac{k \cdot q}{q^2} \right)^2 P_L(q) \equiv -\frac{1}{2} k^2 \sigma_d^2$$

$\gamma^2$   
↓

Even though the variance is typically small, this leads potentially to bad convergence for large external momenta



$$\propto k^4 \sigma_d^4$$

# Outline

Resummation

*pictures only*

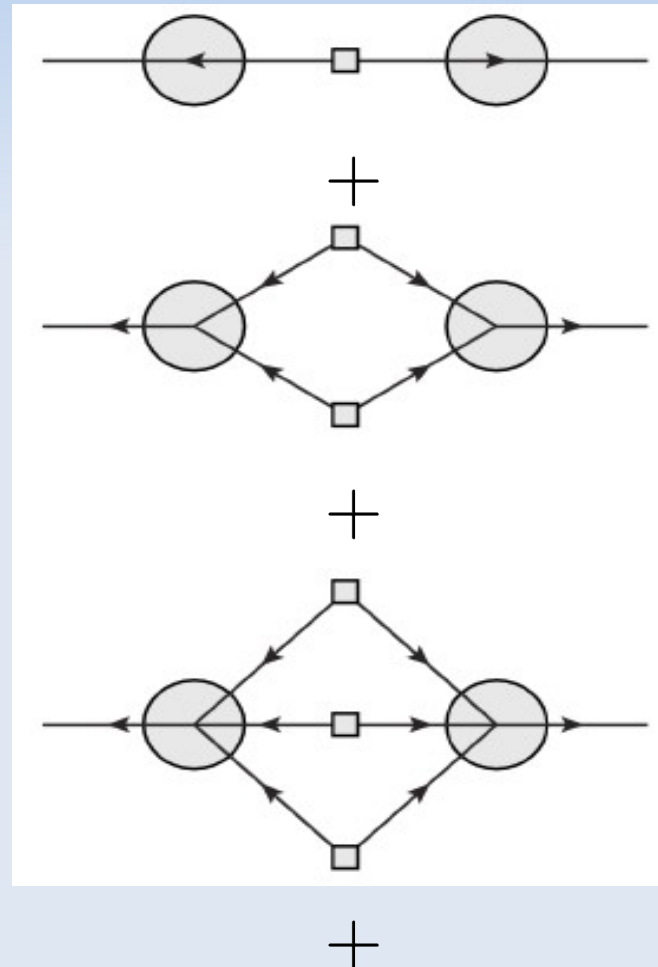
# Many resummation schemes

- Renormalized perturbation theory  
[Crocce and Scoccimarro '05 + '07]
- Simple renormalization group PT [McDonald '06]
- Large-N path-integral methods [Valageas '06]
- Renormalization group techniques  
[Matarrese and Pietroni '07]
- Closure theory [Taruya and Hiramatsu '07]
- Lagrangian resummation [Matsubara '07]
- Time-RG theory [Pietroni '08]
- Multi-point propagators [Bernardeau et al '08]
- Eikonal approximation [Bernardeau et al '11]

# Renormalized PT

In RPT different contributions from SPT are collected in different order (Gamma – expansion)

$$P(k, \eta) =$$



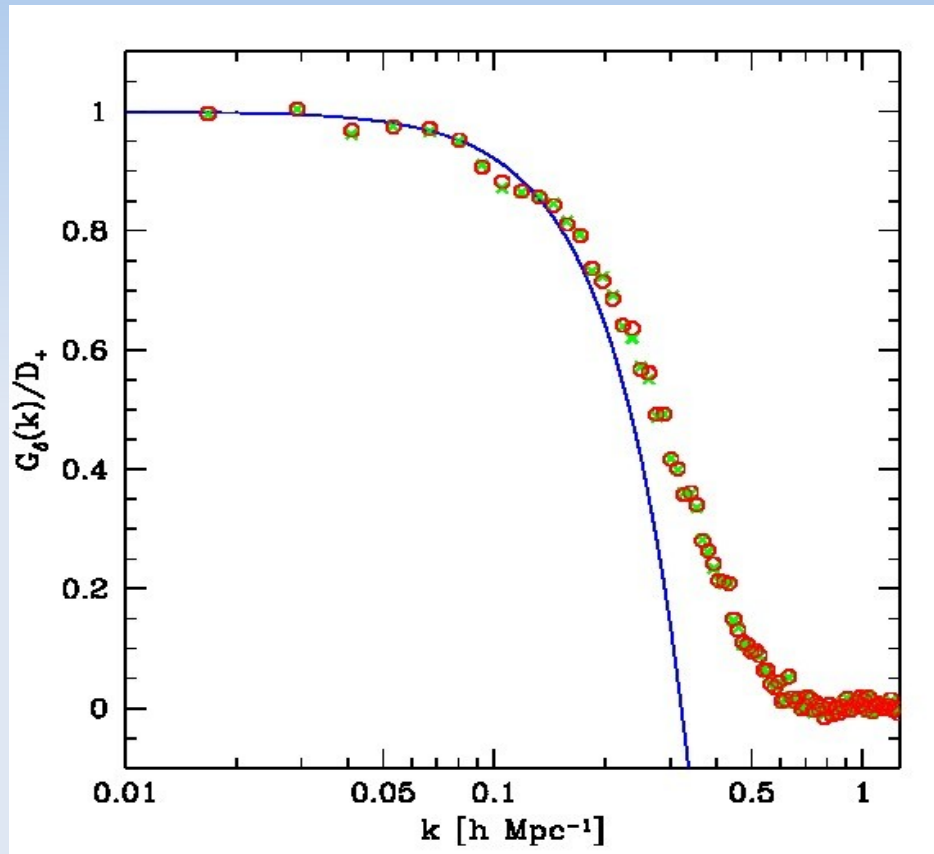
truncate

All contributions **positive**  
which avoids cancellations!

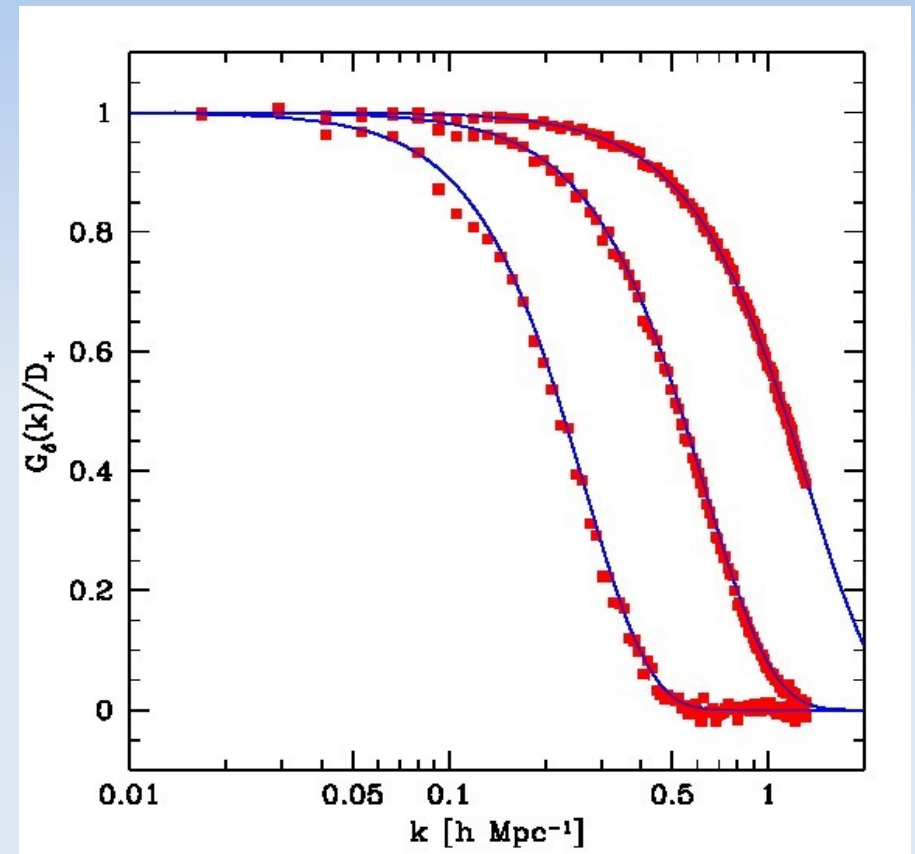
[Crocce and Scoccimarro '05 + '07]



# Full propagator results



SPT (1-loop)

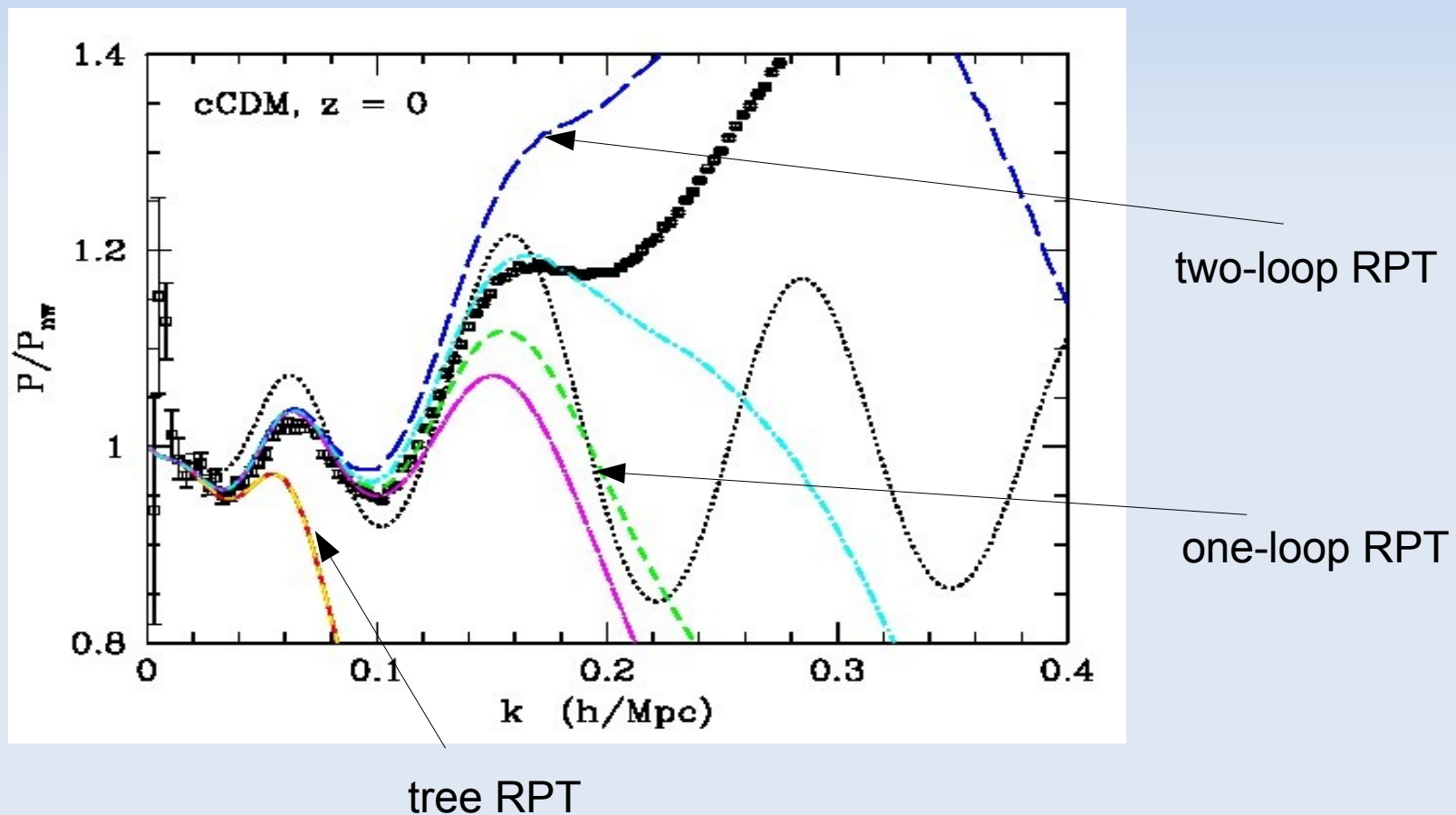


RPT

$$\begin{array}{c} \text{soft} \\ \circlearrowleft \end{array} \sim \leftarrow \times e^{-\frac{1}{2} k^2 \sigma_d^2}$$

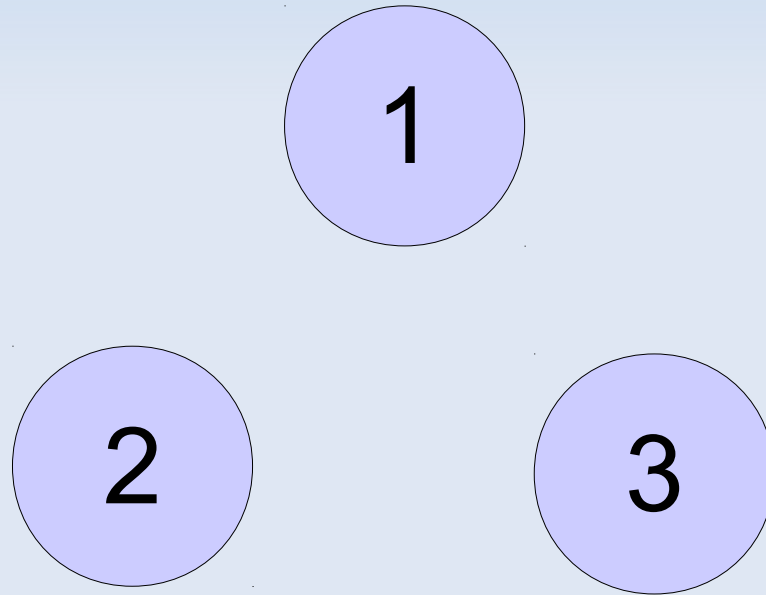
# RPT power spectrum

$$P(k, \eta) \simeq \text{---} \bigcirc \text{---} \square \text{---} \bigcirc \text{---} \simeq P_L(k) e^{-k^2 \sigma_d^2}$$



# Soft loops are harmless...

... to **equal time** correlators



In fact, corrections from soft loops should **not be relevant in equal time correlators** (Galilean invariance – GR equivalence principle).

[Frieman & Scoccimarro '96]

[Jain & Bertschinger '98]

[Creminelli, Gleyzes, Hui, Simonovic, Vernizzi '13]

[Peloso & Pietroni '13]

This is also in the spirit of consistency relations that allow to **absorb soft effects** to certain extent into the **background** cosmology

[Maldacena '02]

[Sherwin & Zaldarriaga '12]

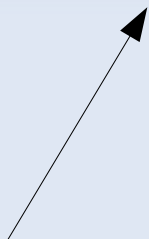
## 2

# Eikonal approximation

The eikonal approximation implements the soft limit on the level of the equations of motion

$$\partial_\eta \Psi_a(k, \eta) + \Omega_{ab} \Psi_b(k, \eta) = \int \gamma_{abc}(k_1, k_2) \Psi_b(k_1, \eta) \Psi_c(k_2, \eta) ,$$

translation invariance:  $\gamma \ni \delta_K(k - k_1 - k_2)$



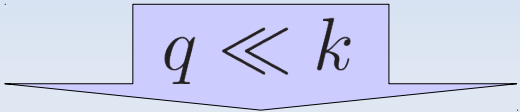


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 $q \ll k$

$$\Psi_a(k, \eta) \int d^3 q \frac{k \cdot q}{q^2} \Psi_2(q, \eta)$$

$$\Psi_a(k, \eta) = g_{ab}(\eta, \eta_0) \xi(\eta, \eta_0, k) \Psi_b(k, \eta_0)$$

$$\xi(\eta, \eta_0, k) = \exp \left[ \int d\bar{\eta} \int d^3 q \frac{k \cdot q}{q^2} \Psi_2(q, \bar{\eta}) \right]$$

# Eikonal results

[Bernardeau et al '11]

In the eikonal approximation, one reproduces the propagator including soft corrections

$$G(k, \eta, \eta_0) = g(\eta, \eta_0) \langle \xi(k, \eta, \eta_0) \rangle$$

$$\text{---} \bigcirc \text{---} \quad \simeq \quad \text{---} \leftarrow \text{---} \quad \times \quad e^{-\frac{1}{2} k^2 \sigma_d^2}$$

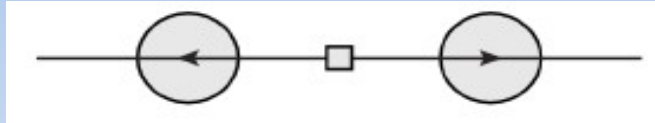
But the power spectrum coincides with the linear result

$$P(k, \eta) = P_L(k, \eta) \langle \xi(k) \xi(-k) \rangle$$

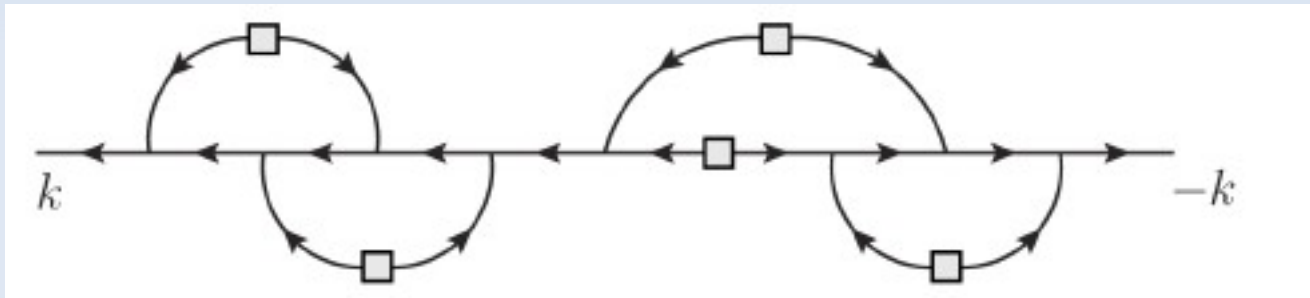
$$P(k, \eta) = \text{---} \leftarrow \square \rightarrow \text{---} = P_L(k, \eta)$$

# More resummations

$$P_{RPT} \simeq$$



In fact RPT is missing something soft

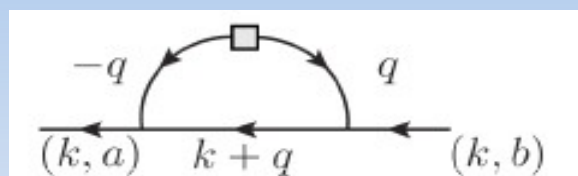


These diagrams can also be resummed if all loops are soft. In RPT they appear in the high multi-point contributions.

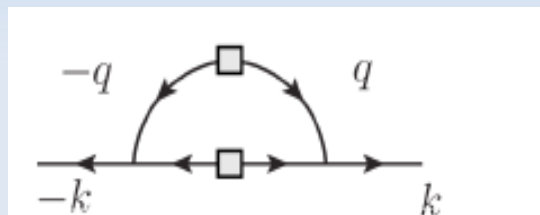
[Blas, Garny & Konstandin '13]

## 3

## Resumming soft modes



$$\simeq -\frac{1}{2}k^2\sigma_d^2$$



$$\simeq k^2\sigma_d^2$$

The relative factor 2 comes from the ordering, the sign from the hard momenta.

$$\dots = \sum_{m,n,l} \frac{1}{m!n!l!} (-\sigma_d^2 k^2 / 2)^{m+n} (\sigma_d^2 k^2)^l = 1$$

There is an **intricate cancellation** of soft effects at **fixed loop order**.  
Breakdown of the Gamma – expansion.

# Integrand manipulations

The cancellation comes from different regions of the loop momentum  $q$

$$P(k, \eta) = \text{[Diagram 1]} + \text{[Diagram 2]} + \text{[Diagram 3]}$$

$$\propto P_L(k) \quad \propto P_L(k)P_L(q) \quad \propto P_L(k-q)P_L(q)$$

The cancellation is only apparent after **integration** or after a judicious **symmetrization**

$$\int d^3q \, I(k, q) \rightarrow \int d^3q \, [I(k, q) + I(k, k - q)] \, \theta(|k - q| - |q|)$$

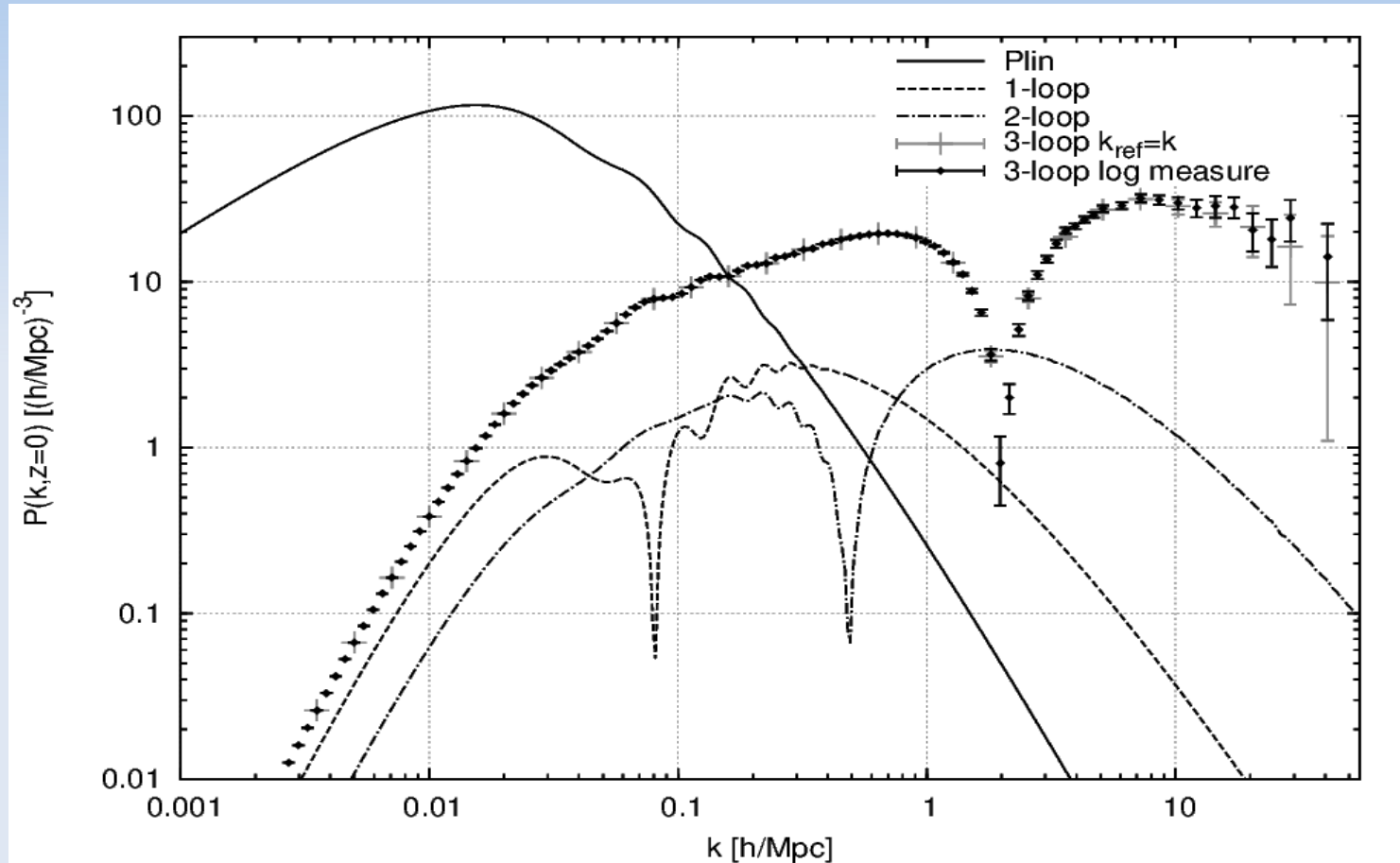
This is important for the numerical stability.

[Blas, Garry &amp; Konstandin '13]

[Senatore et al '13]

# SPT at three loops

[Blas, Garny & Konstandin '13]



$z = 0$

Three loop results in SPT indicate that PT does **not** even **converge** for small momenta at  $z=0$ .

But the result has some features of an **asymptotic series**.

# Asymptotic behavior

And for small  $k$

$$P_{n-loop}(k) \propto k^2 P_L(k) \int_0^\infty d^3 q P_L(q) \sigma_l^{2n-2}(q)$$

Both expressions involve

$$\sigma_l^2(\Lambda, \eta) = \int^\Lambda d^3 q P_L(q, \eta) \neq \sigma_d^2$$

What diverges logarithmically and approaches **unity for  $z=0$**   
(sensitive to hard modes)

[Blas, Garny & Konstandin '13]

# Pade approximant

small k: 
$$P_{n-loop}(k) \propto k^2 P^L(k) \int_0^\infty d^3 q P^L(q) \sigma_l^{2n-2}(q)$$

The Pade approximant represents well complex functions with a branch cut or poles as a rational function

$$f(x) \simeq \frac{a_0 + a_1 x + a_2 x^2 + \dots}{1 + b_1 x + b_2 x^2 + \dots}$$

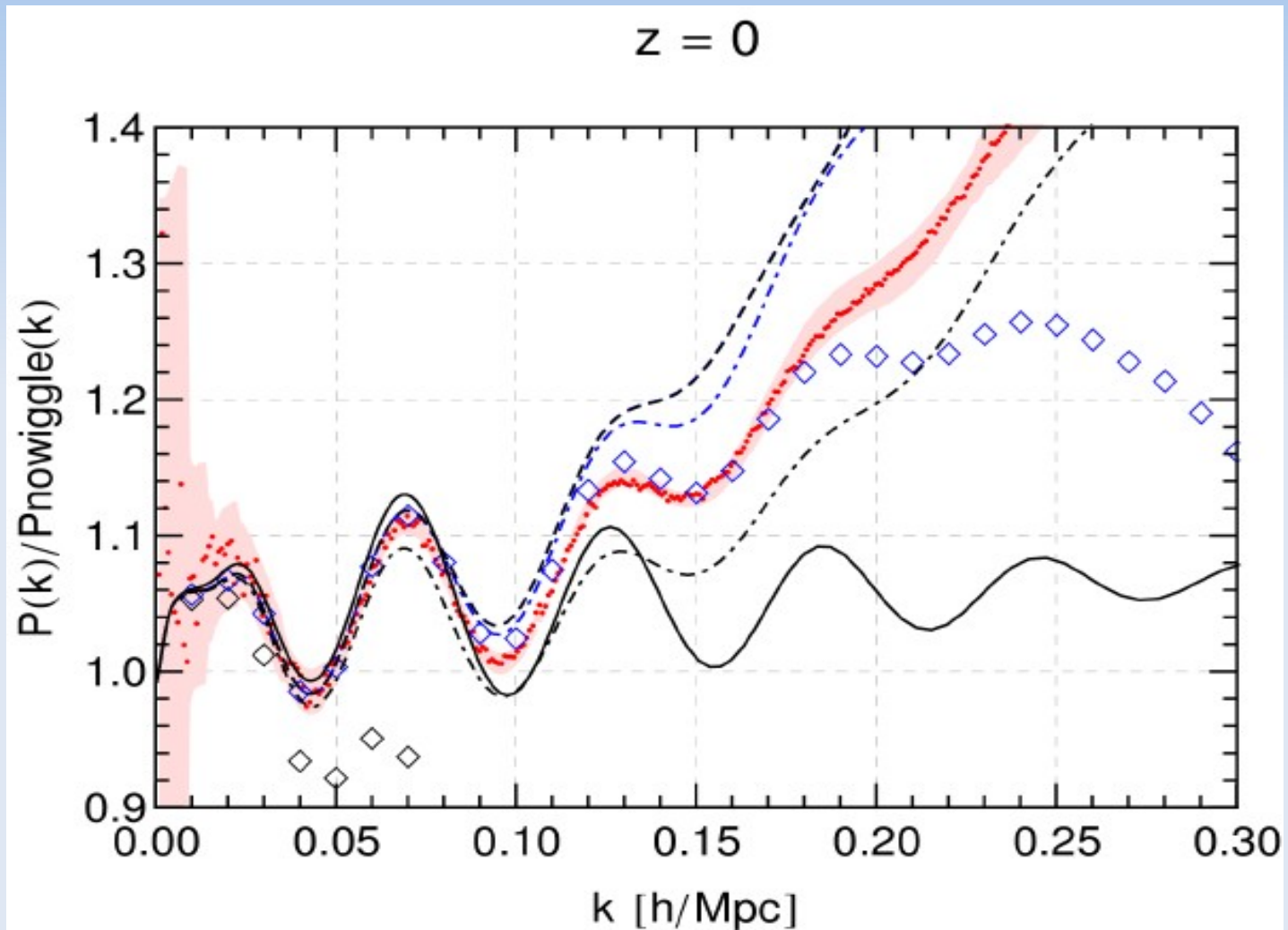
and can be matched to the Taylor expansion.

Here we use the Pade approximant of the integrand with

$$x = \sigma_l^2$$



# Pade power spectrum $z = 0$



No additional degrees of freedom involved

# Conclusions

Most resummation schemes focus on resumming **soft effects** related to the scale

$$\sigma_d^2 \equiv \int \frac{d^3 q}{q^2} P_L(q, \eta)$$

However, in **equal-time** correlators there is no enhancement related to this scale.

# Conclusions

The failure in the convergence of SPT at late times is related to another quantity that is large, namely

$$\sigma_l^2(\Lambda, \eta) = \int^\Lambda d^3q P_L(q, \eta)$$

**Even for small momenta**, SPT converges at best asymptotically. Most resummation schemes share this property.

A Pade resummed SPT result looks promising in this regime.

# Where to go from here?

- Pade resummation is hard to generalize to a larger range of momenta
- Even more resummation
- Effective field theory of large scale structure (taming the UV, predictivity?)
- Lagrangian PT (coordinate space)
- N-body