Relativistic third-order viscous hydrodynamics from kinetic theory

Amaresh Jaiswal

Department of Nuclear and Atomic Physics

Tata Institute of Fundamental Research

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Introduction

- Hydrodynamics: An effective theory describing the long-wavelength, low-frequency limit of the microscopic dynamics of a system.
- Applied quite successfully to study ultra-relativistic heavy-ion collisions; Elegant framework to study the effects of EOS.
- The theory is formulated as an order-by-order expansion in gradients of velocity with ideal hydro being zeroth-order.
- First-order relativistic Navier-Stokes theory has acausal behaviour which is rectified in second-order Israel-Stewart theory.
- Inconsistencies and approximations in IS formulation and application:
  - Use of second moment of Boltzmann equation.
  - Grad's 14 moment approximation.
  - Violation of experimentally observed $1/\sqrt{m_T}$ scaling of the HBT radii.
  - Show disagreement with transport results.

Relativistic viscous hydrodynamics

- The hydrodynamic evolution of a system is governed by the conservation equations for energy and momentum, $\partial_\mu T^{\mu\nu} = 0$.
- In terms of single-particle phase-space distribution function $f(x,p)$,

$$T^{\mu\nu} = \int dp\ p^\mu p^\nu\ f(x,p) = \epsilon u^\mu u^\nu - P\Delta^{\mu\nu} + \pi^{\mu\nu}$$

  - $dp \equiv g d\mathbf{p}/[(2\pi)^3|\mathbf{p}|]$, ($g$: degeneracy factor)
  - p^μ: particle four-momentum
  - u^μ: fluid four-velocity
  - $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$
- For a system close to equilibrium, $f = f_0 + \delta f$,

$$\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \int dp\, p^\alpha p^\beta\, \delta f$$

$$\Delta^{\mu\nu}_{\alpha\beta} \equiv \frac{1}{2}\left[\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\mu_\beta \Delta^\nu_\alpha - (2/3)\Delta^{\mu\nu}\Delta_{\alpha\beta}\right]$$

Viscous evolution equation

- $\delta f = \delta f^{(1)} + \delta f^{(2)} + \cdots$ can be obtained from BE in RTA:

$$p^\mu \partial_\mu f = -(u\cdot p)\frac{f - f_0}{\tau_R} \quad \Rightarrow \quad f = f_0 - \frac{\tau_R}{u\cdot p}\, p^\mu \partial_\mu f$$

- Solving iteratively, [A. Jaiswal, Phys. Rev. C **87**, 051901(R) (2013)]

$$\delta f^{(1)} = -\frac{\tau_R}{u\cdot p}\, p^\mu \partial_\mu f_0 \ ; \qquad \delta f^{(2)} = \frac{\tau_R}{u\cdot p} p^\mu p^\nu \partial_\mu \Big(\frac{\tau_R}{u\cdot p}\partial_\nu f_0\Big)$$

- For $\delta f = \delta f^{(1)}$, we obtain, $\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}, \quad \eta = \tau_R \beta_\pi, \quad \beta_\pi = 4P/5.$

- For $\delta f = \delta f^{(1)} + \delta f^{(2)}$, [A. Jaiswal, Phys. Rev. C **88**, 021903(R) (2013)]

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi \sigma^{\mu\nu} + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \frac{10}{7}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} - \frac{4}{3}\pi^{\mu\nu}\theta$$

$$\tau_\pi = \tau_R, \quad \sigma^{\mu\nu} = \nabla^{\langle\mu}u^{\nu\rangle} \equiv \Delta^{\mu\nu}_{\alpha\beta}\nabla^\alpha u^\beta, \quad \omega^{\mu\nu} \equiv (\nabla^\mu u^\nu - \nabla^\nu u^\mu)/2, \quad \nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu$$

Viscous corrections to the distribution function

- In terms of the dissipative quantities, $\delta f^{(1)}$ and $\delta f^{(2)}$ can be expressed as [R. S. Bhalerao, A. Jaiswal, S. Pal and V. Sreekanth, arXiv:1312.1864]

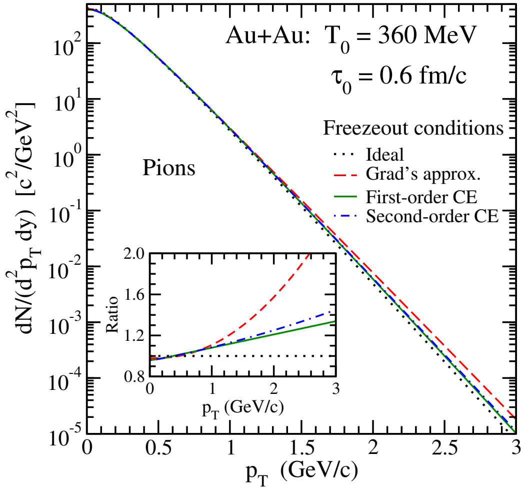
$$\delta f_1 = \frac{f_0\beta}{2\beta_\pi(u\!\cdot\!p)}\, p^\alpha p^\beta \pi_{\alpha\beta}$$

$$\begin{aligned}\delta f_2 = &-\frac{f_0\beta}{\beta_\pi}\bigg[\frac{\tau_\pi}{u\!\cdot\!p}\, p^\alpha p^\beta \pi^\gamma_\alpha\, \omega_{\beta\gamma} - \frac{5}{14\beta_\pi(u\!\cdot\!p)}\, p^\alpha p^\beta \pi^\gamma_\alpha\, \pi_{\beta\gamma} + \frac{\tau_\pi}{3(u\!\cdot\!p)} p^\alpha p^\beta \pi_{\alpha\beta}\theta \\ &- \frac{6\tau_\pi}{5} p^\alpha \dot{u}^\beta \pi_{\alpha\beta} + \frac{(u\!\cdot\!p)}{70\beta_\pi}\pi^{\alpha\beta}\pi_{\alpha\beta} + \frac{\tau_\pi}{5} p^\alpha\Big(\nabla^\beta \pi_{\alpha\beta}\Big) - \frac{3\tau_\pi}{(u\!\cdot\!p)^2}\, p^\alpha p^\beta p^\gamma \pi_{\alpha\beta}\, \dot{u}_\gamma \\ &+ \frac{\tau_\pi}{2(u\!\cdot\!p)^2} p^\alpha p^\beta p^\gamma (\nabla_\gamma \pi_{\alpha\beta}) - \frac{\beta + (u\!\cdot\!p)^{-1}}{4(u\!\cdot\!p)^2\beta_\pi}\Big(p^\alpha p^\beta \pi_{\alpha\beta}\Big)^2\bigg]\end{aligned}$$

- For comparison, Grad's 14-moment approximation for δf:

$$\delta f_G = \frac{f_0\beta^2}{10\beta_\pi}\, p^\alpha p^\beta \pi_{\alpha\beta}$$

Pion transverse momentum spectra

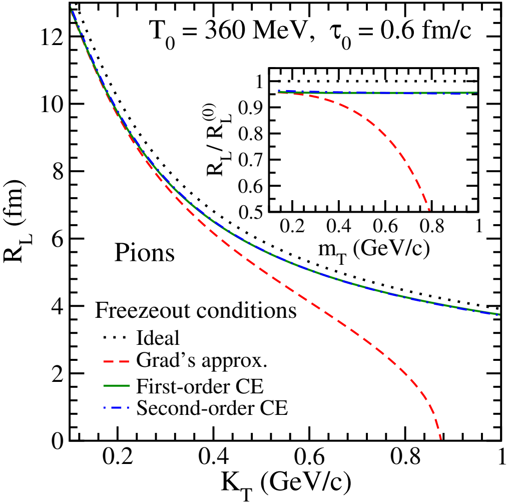


$$\frac{dN}{d^2p_T dy} = \frac{g}{(2\pi)^3}\int p_\mu d\Sigma^\mu f(x,p)$$

- p_T: Particle transverse momentum.
- y: Particle rapidity.
- $d\Sigma^\mu$: Element of the three-dimensional freezeout hypersurface.

First-order and second-order CE shows convergence.

Longitudinal HBT radii



$$R_L^2 = \frac{\int K_\mu d\Sigma^\mu f(x,K) z^2}{\int K_\mu d\Sigma^\mu f(x,K)}$$

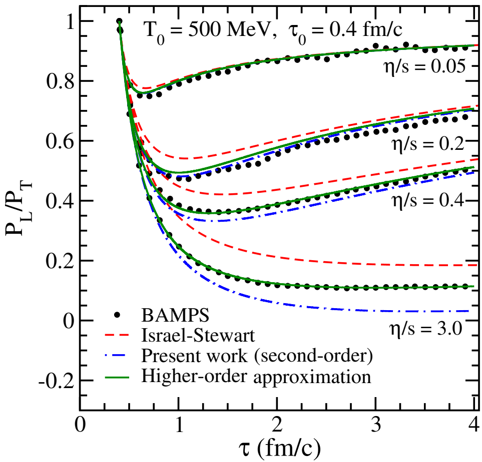
$$\left(R_L^2\right)^{(0)} = \frac{\tau^2 T}{m_T}$$

$$\left(\delta R_L^2\right)^{(1)} = -\frac{5\tau^2 T\pi}{4\beta_\pi m_T}$$

$$\left(\delta R_L^2\right)^{(G)} = -\frac{\tau^2 T\pi}{5\beta_\pi m_T}\left(3 + \frac{m_T}{T}\right)$$

Grad's 14 moment approximation violates $1/\sqrt{m_T}$ scaling while CE does not.

Pressure anisotropy in Bjorken evolution



- $P_L/P_T \equiv (P-\pi)/(P+\pi/2)$

Bjorken expansion

$$\frac{d\pi}{d\tau} = -\frac{\pi}{\tau_\pi} + \beta_\pi \frac{4}{3\tau} - \lambda\frac{\pi}{\tau} - \chi\frac{\pi^2}{\beta_\pi \tau}$$

- $\pi \equiv -\tau^2 \pi^{\eta_s\eta_s}$
- $\tau_\pi = \eta/\beta_\pi, \quad \beta_\pi = 4P/5$
- $\lambda = 38/21$: Second-order coefficient.
- $\lambda^{\mathrm{IS}} = 2$: Second-order IS coefficient.
- $\chi = 36/175$: Heuristic higher-order correction.

First-order and second-order CE shows convergence.

Third-order evolution equation for shear stress tensor

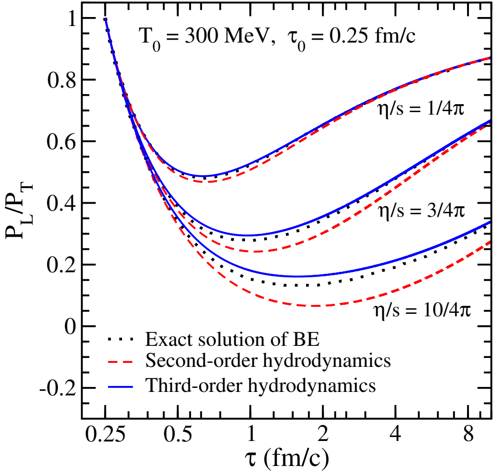
- To obtain the third-order evolution equation of $\pi^{\mu\nu}$,

$$\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \int dp \, p^\alpha p^\beta \, \delta f \quad \Rightarrow \quad \dot{\pi}^{\langle\mu\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta} \int dp \, p^\alpha p^\beta \, \delta \dot{f}$$

- From BE in RTA, we obtain $\delta\dot{f} = -\dot{f}_0 - (p^\gamma \nabla_\gamma f)/u\cdot p - \delta f/\tau_R$
- Substituting $\delta f = \delta f_1 + \delta f_2$ [A. Jaiswal, Phys. Rev. C **88**, 021903(R) (2013)]

$$\begin{aligned}
\dot{\pi}^{\langle\mu\nu\rangle} =& -\frac{\pi^{\mu\nu}}{\tau_\pi} + 2\beta_\pi \sigma^{\mu\nu} + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \frac{10}{7}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} - \frac{4}{3}\pi^{\mu\nu}\theta \\
&+ \frac{1}{\beta_\pi}\left[\frac{25}{7}\pi^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\pi_{\rho\gamma} - \frac{1}{3}\pi_\gamma^{\langle\mu}\pi^{\nu\rangle\gamma}\theta - \frac{38}{245}\pi^{\mu\nu}\pi^{\rho\gamma}\sigma_{\rho\gamma} - \frac{22}{49}\pi^{\rho\langle\mu}\pi^{\nu\rangle\gamma}\sigma_{\rho\gamma}\right] \\
&+ \tau_\pi\left[\frac{26}{21}\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma}\theta - \frac{2}{7}\omega^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\pi_{\rho\gamma} - \frac{2}{7}\pi^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\omega_{\rho\gamma} - \frac{10}{63}\pi^{\mu\nu}\theta^2\right] \\
&- \frac{24}{35}\nabla^{\langle\mu}\left(\pi^{\nu\rangle\gamma}\dot{u}_\gamma\tau_\pi\right) + \frac{4}{35}\nabla^{\langle\mu}\left(\tau_\pi\nabla_\gamma\pi^{\nu\rangle\gamma}\right) - \frac{2}{7}\nabla_\gamma\left(\tau_\pi\nabla^{\langle\mu}\pi^{\nu\rangle\gamma}\right) \\
&+ \frac{12}{7}\nabla_\gamma\left(\tau_\pi\dot{u}^{\langle\mu}\pi^{\nu\rangle\gamma}\right) - \frac{1}{7}\nabla_\gamma\left(\tau_\pi\nabla^\gamma\pi^{\langle\mu\nu\rangle}\right) + \frac{6}{7}\nabla_\gamma\left(\tau_\pi\dot{u}^\gamma\pi^{\langle\mu\nu\rangle}\right)
\end{aligned}$$

Pressure anisotropy in Bjorken evolution: Third-order



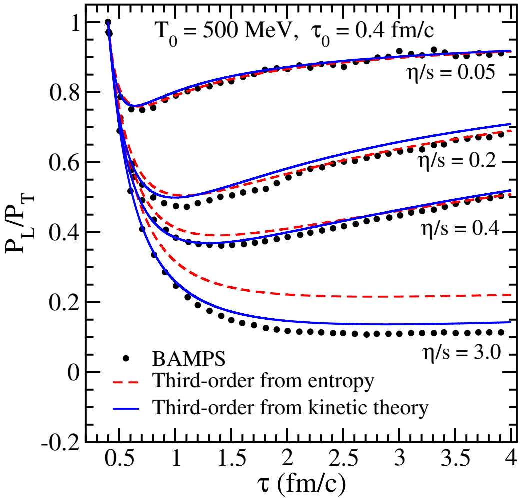
Bjorken expansion

$$\frac{d\pi}{d\tau} = -\frac{\pi}{\tau_\pi} + \beta_\pi \frac{4}{3\tau} - \lambda\frac{\pi}{\tau} - \chi\frac{\pi^2}{\beta_\pi \tau}$$

- $\tau_\pi = \eta/\beta_\pi$: Shear relaxation time.
- $\beta_\pi = 4P/5$
- $\lambda = 38/21$: Second-order coefficient.
- $\chi = 72/245$: Third-order correction.
- $P_L/P_T \equiv (P-\pi)/(P+\pi/2)$

Third-order shows Improved agreement with exact solution of BE.

Comparison of pressure anisotropy with BAMPS



Third-order from entropy:
[A. El, Z. Xu and C. Greiner, Phys. Rev. C **81**, 041901(R) (2010)]

- $\tau'_\pi = \eta/\beta'_\pi$: Shear relaxation time.
- $\beta'_\pi = 2P/3$
- $\lambda' = 4/3$: Second-order coefficient.
- $\chi' = 3/4$: Third-order coefficient.

Third-order from kinetic theory shows better agreement with BAMPS.

Conclusions

- Derived the expression for the viscous corrections to the distribution function up to second-order (alternative to Grad's approximation).
- These corrections does not violate the experimentally observed $1/\sqrt{m_T}$ scaling of the HBT radii and shows convergence.
- Derived evolution equation for shear stress tensor, directly from its definition, up to third-order (without using second moment of BE).
- Second-order hydrodynamic equations derived here results in better agreement with transport calculation (BAMPS) compared to IS.
- Third-order hydrodynamic evolution shows improved agreement with exact solution of BE compared to second-order.
- Third-order hydrodynamic equation derived here also shows better agreement with BAMPS compared to third-order from entropy.

THANK YOU

Backup slide: Bjorken Flow [J. D. Bjorken, Phys. Rev. D **27**, 140 (1983)]

- In Milne coordinates: proper time $\tau = \sqrt{t^2 - z^2}$, spacetime rapidity $\eta_s = \tanh^{-1}(z/t)$, $t = \tau\cosh\eta_s$, $z = \tau\sinh\eta_s$ and the metric is given by $g_{\mu\nu}$ =diag$(1, -1, -1, -\tau^2)$.
- Boost invariance ($v^z = z/t$) for hydro translates into

$$u^t = \frac{t}{\tau}, \;\; u^z = \frac{z}{\tau}, \;\; u^{\eta_s} = -u^t\frac{\sinh\eta_s}{\tau} + u^z\frac{\cosh\eta_s}{\tau} = 0 \;\Rightarrow\; u^\mu = (1, 0, 0, 0)$$

- In centre of the fireball, stress energy tensor in local comoving frame has the form: $T^{\mu\nu} = diag(\epsilon,\ P_T,\ P_T,\ P_L)$.

$$P_T = P + \pi/2 \quad ; \quad P_L = P - \pi \quad ; \quad \frac{P_L}{P_T} = \frac{P - \pi}{P + \pi/2}\ .$$

- The evolution equations for ϵ and $\pi \equiv -\tau^2\pi^{\eta_s\eta_s}$ becomes

$$\frac{d\epsilon}{d\tau} = -\frac{1}{\tau}\left(\epsilon + P - \pi\right),$$

$$\frac{d\pi}{d\tau} = -\frac{\pi}{\tau_\pi} + \beta_\pi\frac{4}{3\tau} - \lambda\frac{\pi}{\tau} - \chi\frac{\pi^2}{\beta_\pi\tau}$$