On the realizability of relativistic acoustic geometry under a generalized perturbative scheme for matter flow onto a Schwarzschild black hole

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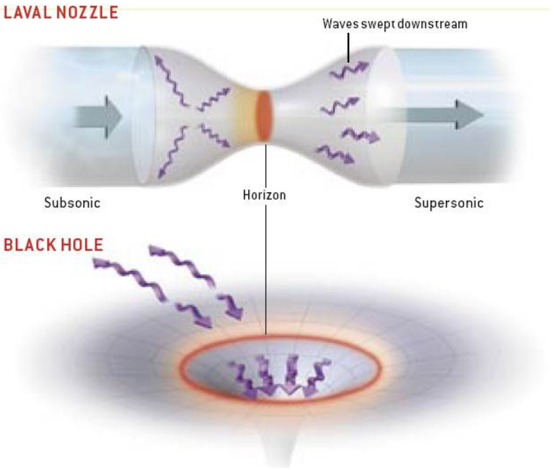
• In recent years, strong analogies have been established between the physics of acoustic perturbations in an inhomogeneous dynamical fluid system, and some kinematic features of the space time in general relativity.

• An effective metric, referred to as the 'acoustic metric', can be constructed, which describes the geometry of a manifold in which the acoustic perturbation propagate. This effective geometry can capture the properties of curved space-time in general relativity.

• Physical models constructed utilizing such analogies are called 'analogue gravity models'.

Laboratory experiments with general relativistic black holes are basically impossible. There is now a lot of interests in stimulating black holes by using fluid dynamics/condensed matter analogues.

A Laval Nozzle - found at the end of rockets - makes a ready analogue to a black hole. The incoming fluid is subsonic; the constriction forces it to accelerate to the speed of sound, so that the outgoing fluid is supersonic. Sound waves in the subsonic region can move upstream, whereas waves in the supersonic region cannot. The constriction thus acts just like the horizon of a black hole: sound can enter but not exit the supersonic region. Quantum fluctuations in the constriction should generate sound analogue to Hawking radiation.



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- Eq. of state of an ideal ultrarelativistic fluid $\Rightarrow$
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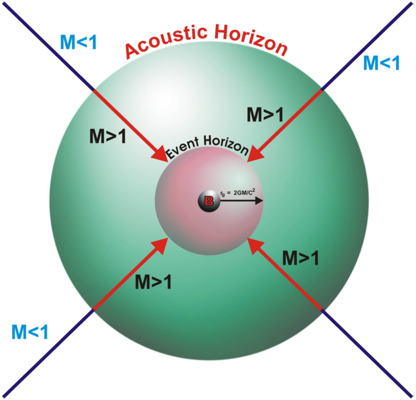
- Astrophysically relevant scenarios - black holes, early big-bang cosmology, relativistic collisions of elementary particles and ions.

Spherically symmetric black hole accretion



- $M = \frac{v}{c_s}$
- The exact location of R_H is computed as a non-linear function of only two accretion parameters.
- The analogue Hawking temperature is also computed as a non-linear function of accretion parameters.

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- Stationary solution of the continuity equation $\rightarrow$ *matter inflow/accretion rate.*
- Explains stability of both the velocity and density fields.
- *Our goal* $\rightarrow$ find acoustic metric for fluid moving around a Schwarzschild black hole in linear perturbation over stationary background flow.

Axisymmetric flow in Schwarzschild Space time

- The metric :

$$ds^2 = -fdt^2 + f^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \qquad (1)$$

where $f = \left(1 - \frac{2M}{r}\right)$.

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$$T^{\mu\nu} = (\epsilon + p)\, v^\mu v^\nu + p g^{\mu\nu}. \quad (2)$$

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- Assumptions : *inviscid, isentropic* $(\Rightarrow \frac{\partial s}{\partial \rho} = 0)$ *and irrotational.*
- Thermodynamic identity

$$dh = Td\left(\frac{s}{\rho}\right) + \frac{1}{\rho}dp, \qquad (3)$$

where, $h = \frac{\epsilon + p}{\rho}$.

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- For an Isentropic irrotational flow

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- Stationary Solutions, Eq. (5) $\Rightarrow$

$$\rho v r^2 = \Psi = \text{constant}. \qquad (7)$$

- $\mu = t$ and $\nu = r \rightarrow$

$$\frac{1}{f}\partial_t v + \partial_r (fvt) + \frac{a2}{\rho}\left[\frac{v}{f}\partial_t \rho + v^t f \partial_r \rho\right] = 0 \qquad (8)$$

- $\mu = t$ and $\nu = \phi \rightarrow$

$$\partial_t \left[v^\phi g_{\phi\phi}\frac{\epsilon + p}{\rho}\right] = 0 \qquad (9)$$

- $\mu = r$ and $\nu = \phi \rightarrow$

$$f\left\{\partial_r \left(v^\phi r^2\right) + v^\phi r^2 \frac{a^2}{\rho}\partial_r \rho\right\} = 0 \qquad (10)$$

-

$$\partial_t \left(\rho v^t\right) + \frac{1}{r^2}\partial_r \left(\rho v r^2\right) = 0 \qquad (11)$$

- $\mu = t \rightarrow$

$$v^t \partial_t v^t + v\partial_r v^t + \frac{v^t v}{f}\partial_r f + \frac{a^2}{\rho}\left[\left((v^t)^2 - \frac{1}{f}\right)\partial_t \rho + v v^t \partial_r \rho\right] = 0 \qquad (12)$$

- $\mu = r \rightarrow$

$$v^t \partial_t v + v\partial_r v + \frac{1}{2}\partial_r f\left(1 + (v^\phi)^2 r^2\right) - (v^\phi)^2 fr + \left(v^2 + f\right)\frac{a^2}{\rho}\partial_r \rho + \frac{v v^t a^2}{\rho}\partial_t \rho = 0 \qquad (13)$$

- $\mu = \phi \rightarrow$

$$v^t \partial_t v^\phi + v\partial_r v^\phi + \frac{2vv^\phi}{r} + \frac{v^\phi a^2}{\rho}\left[v^t \partial_t \rho + v\partial_r \rho\right] = 0 \qquad (14)$$

Axisymmetric flow . . .

- Linear perturbation

$$
\begin{aligned}
v &= v_0 + v', \; v^\phi = v_0^\phi + v^{\phi'} \\
\rho &= \rho_0 + \rho' \\
\Psi &= \Psi_0 (= \rho_0 v_0 r^2) + \Psi' (= \rho_0 v' r^2 + \rho' v_0 r^2). \qquad (15)
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- Substituting above in Eq.(11)

$$\frac{-1}{r^2}\frac{\partial \Psi'}{\partial r} = \left(v_0^t - \frac{{v_0^\phi}^2 r^2 a_0^2 f}{v_0^t f^2} \right) \partial_t \rho' + \left(\frac{\rho_0 v_0}{f^2 v_0^t} \right) \partial_t v' \quad (16)$$

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- Taking time derivative of $\Psi' = \rho_0 v' r^2 + \rho' v_0 r^2$ and using above eqn.

$$
\partial_t v' = \frac{v_0^t f^2}{\rho_0 r^2 \left[f + v_0^{\phi^2} r^2 f (1 - a_0^2) \right]} \left[\left(v_0^t - \frac{v_0^{\phi^2} r^2 a_0^2 f}{v_0^t f^2} \right) \partial_t \Psi' + v_0 \partial_r \Psi' \right] \quad (17)
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\partial_t \rho' = \frac{-1}{r^2 \left[f + v_0^{\phi^2} r^2 f (1 - a_0^2) \right]} \left[v_0 \partial_t \Psi' + v_0^t f^2 \partial_r \Psi' \right] \quad (18)
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- Substituting in Eq. (13)

$$\frac{1}{v_0^t f^2}\partial_t v^{'} + \partial_r \left[\frac{v_0 v^{'} + v_0^{\phi} v^{\phi^{'}} r^2 f}{v_0^t f^2} + \frac{a_0^2 \rho^{'}}{\rho}\right] + \frac{a_0^2 v_0}{\rho_0 v_0^t f^2}\partial_t \rho^{'} = 0 \quad (19)$$

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- Wave equation

$$\partial_\mu \left(f^{\mu\nu} \partial_\nu \Psi^{'} \right) = 0, \qquad (20)$$

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$$f^{\mu\nu} = \frac{1}{\rho_0 r^2 v_0^t x}\begin{bmatrix} \frac{v_0^{t^2} f^2 - a_0^2(v_0^2 + v_0^{\phi^2} r^2 f)}{f^2} & v_0 v_0^t(1 - a_0^2) \\ v_0 v_0^t(1 - a_0^2) & v_0^2 - a_0^2(v_0^2 + f) \end{bmatrix}$$

$x = (f + v_0^{\phi^2} r^2 f(1 - a_0^2))$ and $v_0^t = \frac{\sqrt{v_0^2 + f + v^{\phi^2} r^2 f}}{f}$

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- In Lorentzian manifold, d'Alembertian $\rightarrow$

$$\Delta\psi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\psi\right). \tag{21}$$

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- Acoustic metric :

$$G_{\mu\nu} \equiv \frac{-1}{a_0}\begin{bmatrix} -v_0^2 + a_0^2(v_0^2 + f) & v_0 v_0^t(1 - a_0^2) \\ v_0 v_0^t(1 - a_0^2) & \frac{-v_0^{t^2} f^2 + a_0^2(v_0^2 + v_0^{\phi^2} r^2 f)}{f^2} \end{bmatrix}$$

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A rotating Rindler spacetime

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$$G^{\mu\nu} = \frac{-1}{c_{s_0}} \begin{bmatrix} -v_s^2 + c_{s_0}^2(v_s^2 + 1) & v_s v_s^0(1 - c_{s_0}^2) \\ v_s v_s^0(1 - c_{s_0}^2) & \dfrac{-v_s^{0^2} a^2x^2 + c_{s_0}^2(v_s^2 + \rho^2 v_s^{\phi^2})}{f^2} \end{bmatrix}$$

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- Generalize for Kerr Geometry - metric has off-diagonal elements.

General review articles/ books on analogue gravity:

1) *Artificial Black Holes* by Mario Novello(Ed.), Matt Visser and G. E. Volovik (Ed.), World Scientific Publishing Company; 1st edition, October 2002.

2) *Analogue Gravity* by Carlos Barcelo, Stefano Liberati, Matt Visser, 2005.

3) *Quantum Analogues: From Phase Transition to Black Holes and Cosmology*by William G. Unruh(Ed.), Ralf Schutzhold(Ed.), Springer; 1st ediiton, June 11, 2007.

Review article on application in astrophysics:

1) *Astrophysical Accretion as an Analogue Gravity Phenomena* by Tapas K Das. Indian Journal of Physics(invited review article).

Other references for this talk.

1) W.G. Unruh, Phys. Rev. Lett,**46**,1351-53(1981).

2) M. Visser, Class. Quant. Grav. **15**, 1765-91(1998).

3) N. Bilic, Class. Quant. Grav. **16**, 3953-64(1999).

4) T. Naskar, N. Chakravarty, J. K. Bhattacharjee, A. K. Ray, Phys.Rev. D**76**, (2007).

5) *An Echo of Black Holes* by T. A. Jacobson, R. Parentani.