Beam stability without eigenvalues

(ongoing work)

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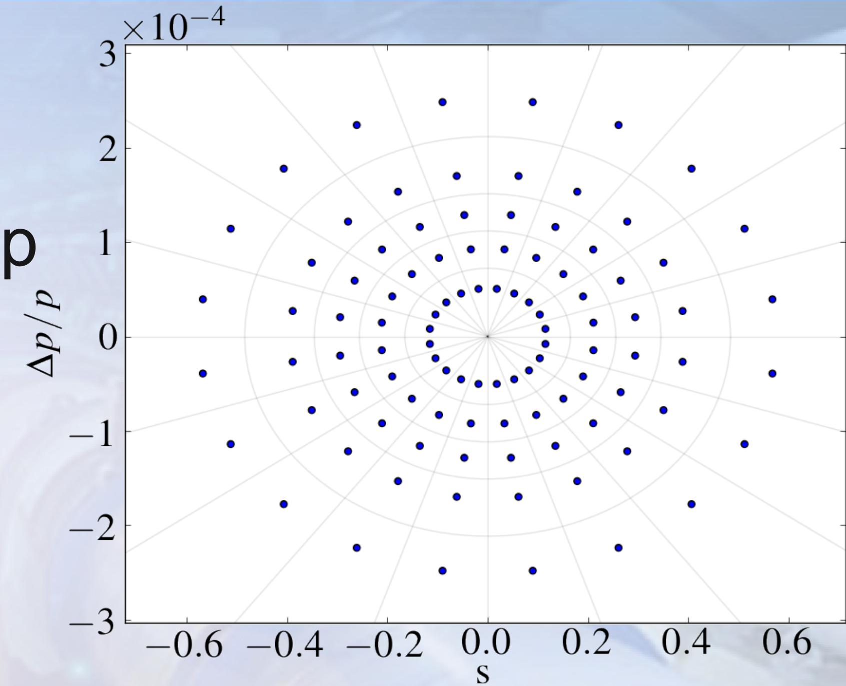
- Remainder on the circulant matrix model
- Multibunch beam breakup / coupled bunch instabilities
- Non-diagonalizable system and pseudo-spectrum
- First results for the LHC case

Circulant matrix model

- Discretized distribution in longitudinal plane
- Build and diagonalize one turn map
- Especially usefull to study beam-beam and impedance (S. White, et al, ”Beam beam and Impedance”, BB2013)



- Python module ”BimBim”
  - Single/multi bunch Impedance based on wake tables
  - 4D/6D Beam-beam interactions
  - Perfect BbyB damper
  - Any filling scheme / IP configuration

Multibunch beam breakup
2 bunch model

- For simplicity, let us consider 2 bunches with 1 slice, 1 ring (i.e. Rigid bunches)
- One turn map for a single bunch : B
- One turn map for the two bunches, assuming that the distance from b1 to b2 << b2 to b1 (e.g. Train of two bunches in the LHC)

$$M = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ Z & 1 \end{pmatrix}$$

- This matrix is not diagonalizable !

  → Eigenvectors do not provide a complete basis for $\mathbb{C}^{dim(M)}$

  → The dynamic of the system is not fully described by the eigenvalues/eigenvectors

Jordan normal form

- One can always find a basis such that the matrix has the Jordan normal form →

$$\begin{bmatrix} \begin{matrix}\lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1\end{matrix} & & & & \\ & \begin{matrix}\lambda_2 & 1 \\ & \lambda_2\end{matrix} & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \begin{matrix}\lambda_n & 1 \\ & \lambda_n\end{matrix} \end{bmatrix}$$

- Vectors of this basis, which are not eigenvectors, are generalized eigenvectors
- Behavior under nth power of M :
  - Eigenvectors : $\vec{v}_n = e^{2\pi i Q n}\vec{v}_0$
  - Generalized eigenvectors : $\vec{v}_n = e^{2\pi i Q n}\vec{v}_0 + \ldots$

Powers of M

2 bunch model

$$M_d = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}$$

Consider a vector in the subspace associated to λ_i :

$$\vec{V} = a_1 \vec{e}_1 + a_2 \vec{e}_2 \quad \text{with} \quad \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\longrightarrow \quad \vec{V}_n = \lambda_i^n a_1 \vec{e}_1 + \lambda_i^n a_2 \vec{e}_2 + \sum_{k=0}^{n-1} (\lambda_i a_2)^k \vec{e}_1$$

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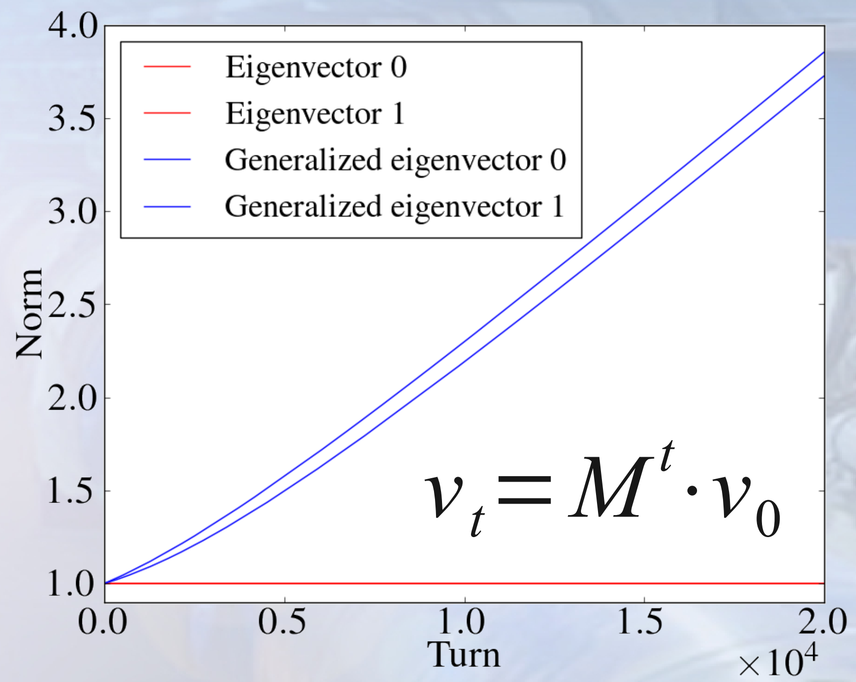
- Linear growth which depends on the initial condition
- The behavior of the system under a small perturbation is no longer independant of the perturbation
- More complicated behavior expected for higher number of bunches

Multibunch beam breakup

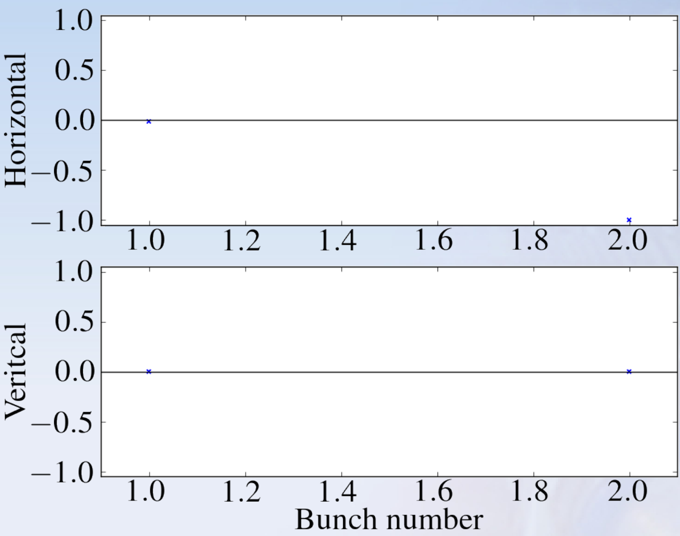
2 bunch model

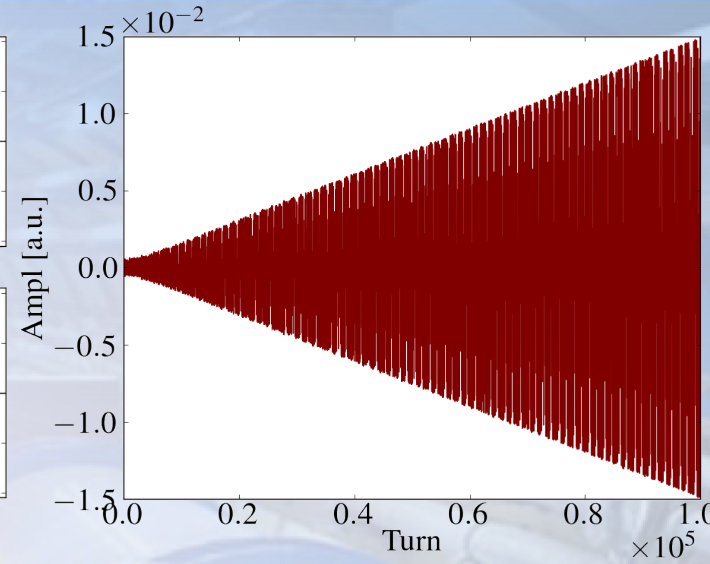
- Matrix model confirms this behavior
- However, mathematically, and physically, the matrix can be rendered diagonalizable
  - Multiturn wake
  - Equidistant bunches
  - Beam-beam
- Does the physcis change ?

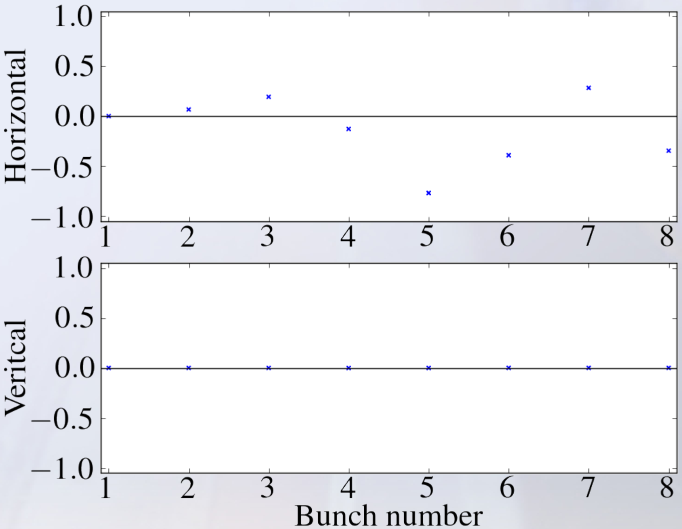


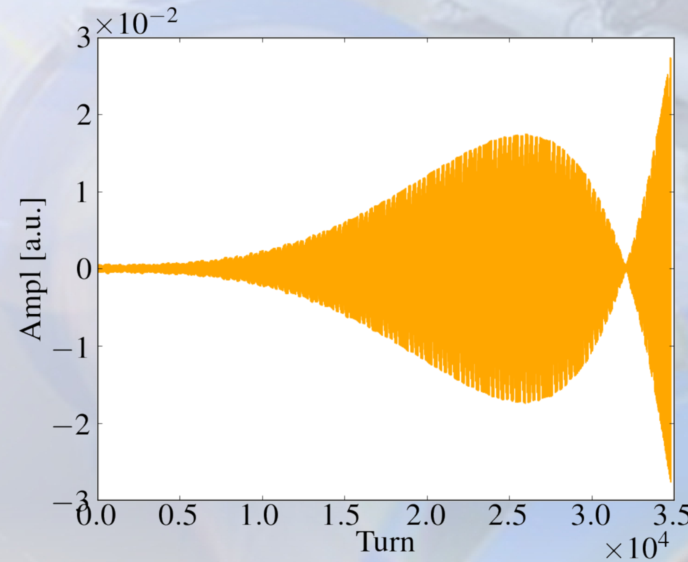
Coupled bunch instabilities

- Simulation with COMBI
- LHC impedance model
  - Dipolar wake
  - Quadrupolar wake
  - Multiturn wake
- Long term behavior is well described by eigenvalue approach, but short term is dominated by transient effects









Pseudo spectrum

- This phenomenon can be described by the pseudo spectrum

$$Spectrum(M) = \{\lambda \in \mathbb{C} \mid \exists \vec{v} : (M - \lambda I) \cdot \vec{v} = 0\}$$

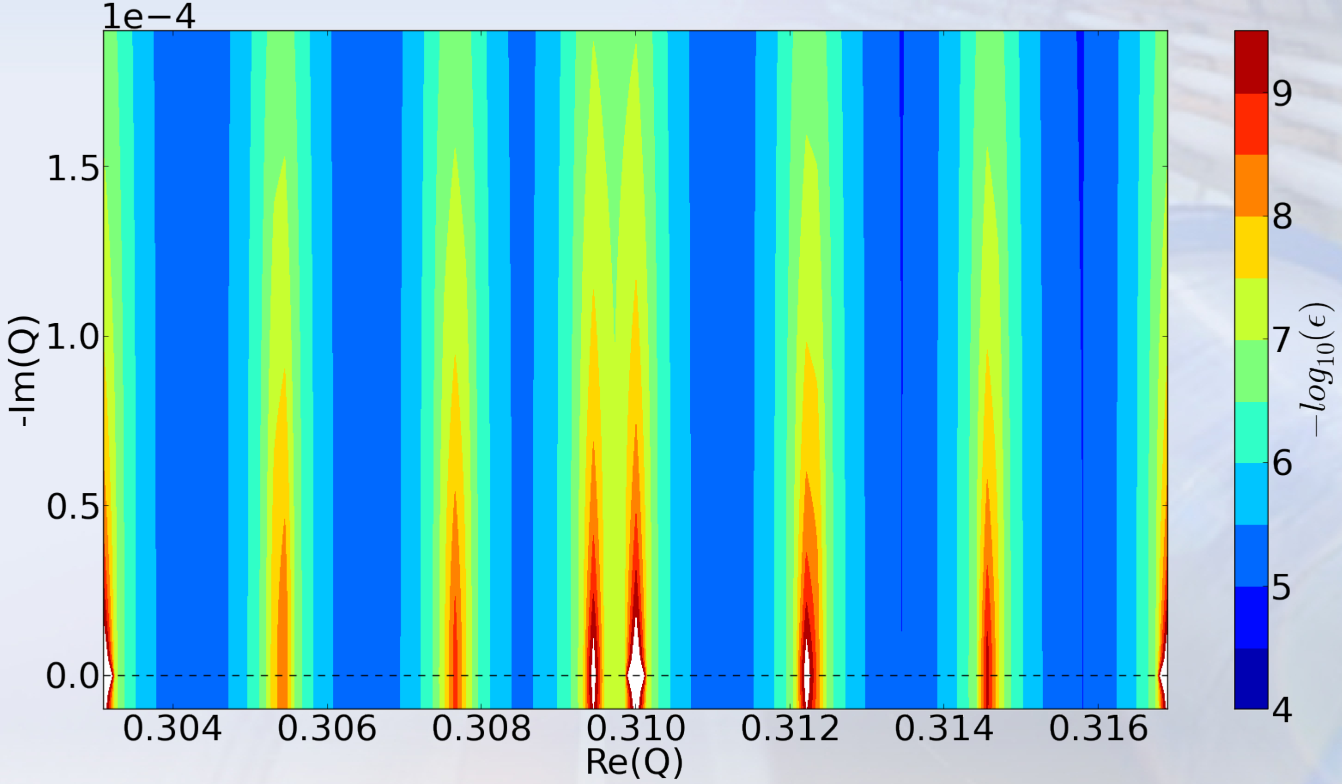
$$Pseudo\ spectrum(M, \epsilon) = \{\lambda \in \mathbb{C} \mid \exists \vec{v} : \|(M - \lambda I) \cdot \vec{v}\| < \epsilon\}$$

- For a given point of the complex plane z, the corresponding ε is given by the smallest singular value of $(M - zI)$

Method applied in many field, for example :
L. Trefethen, et al, *"Hydrodynamic stability without eigenvalues"*, Science, New Series, Vol. 261, No. 5121. (Jul. 30, 1993), pp. 578-584

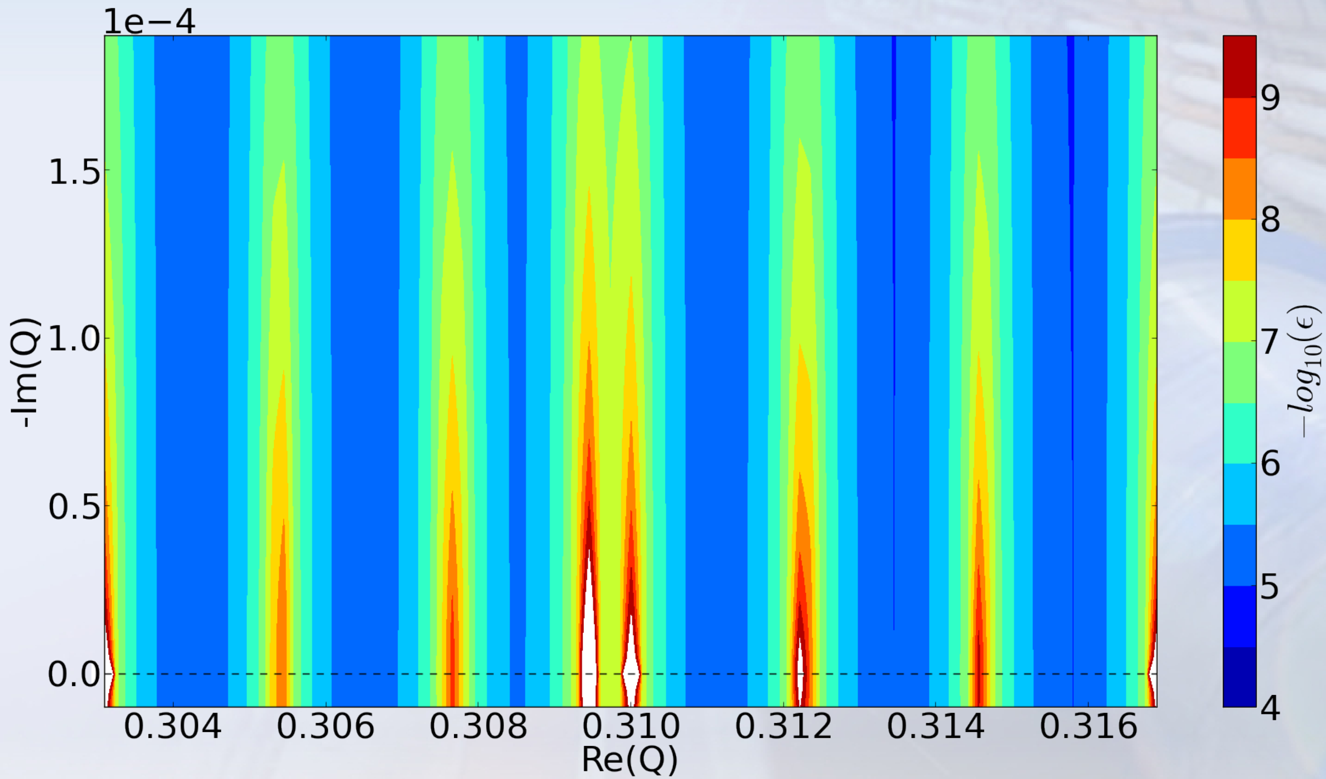
First results

- 1 bunch
- 0.0 chromaticity, no damper



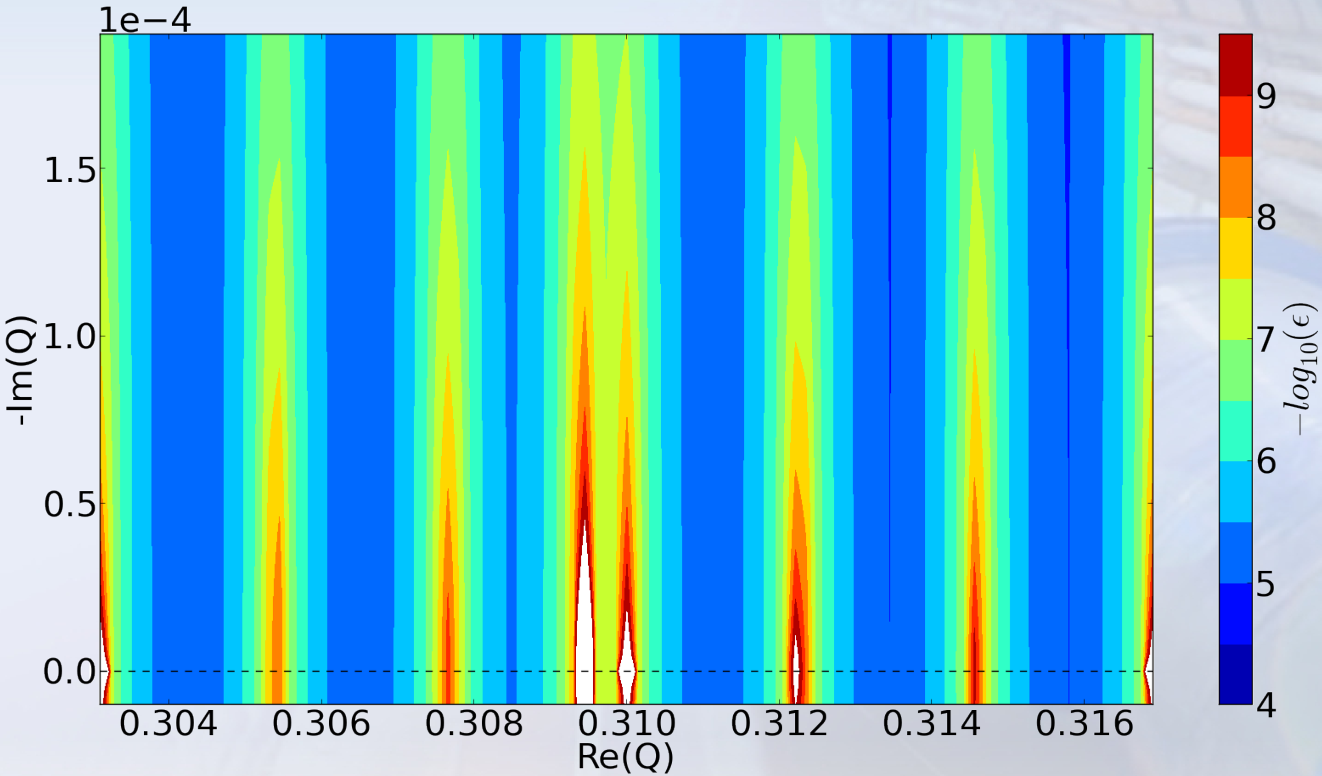
First results

- 4 bunches
- 0.0 chromaticity, no damper



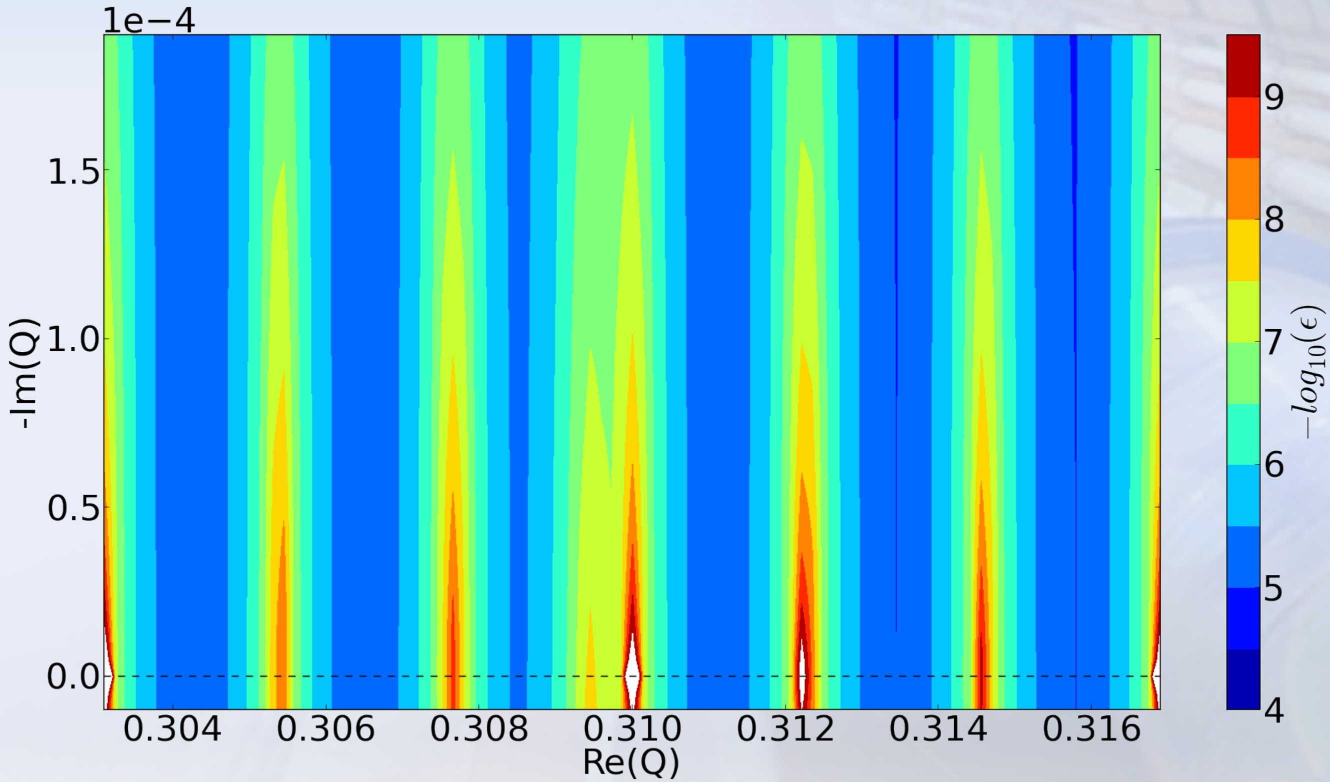
First results

- 8 bunches
- 0.0 chromaticity, no damper



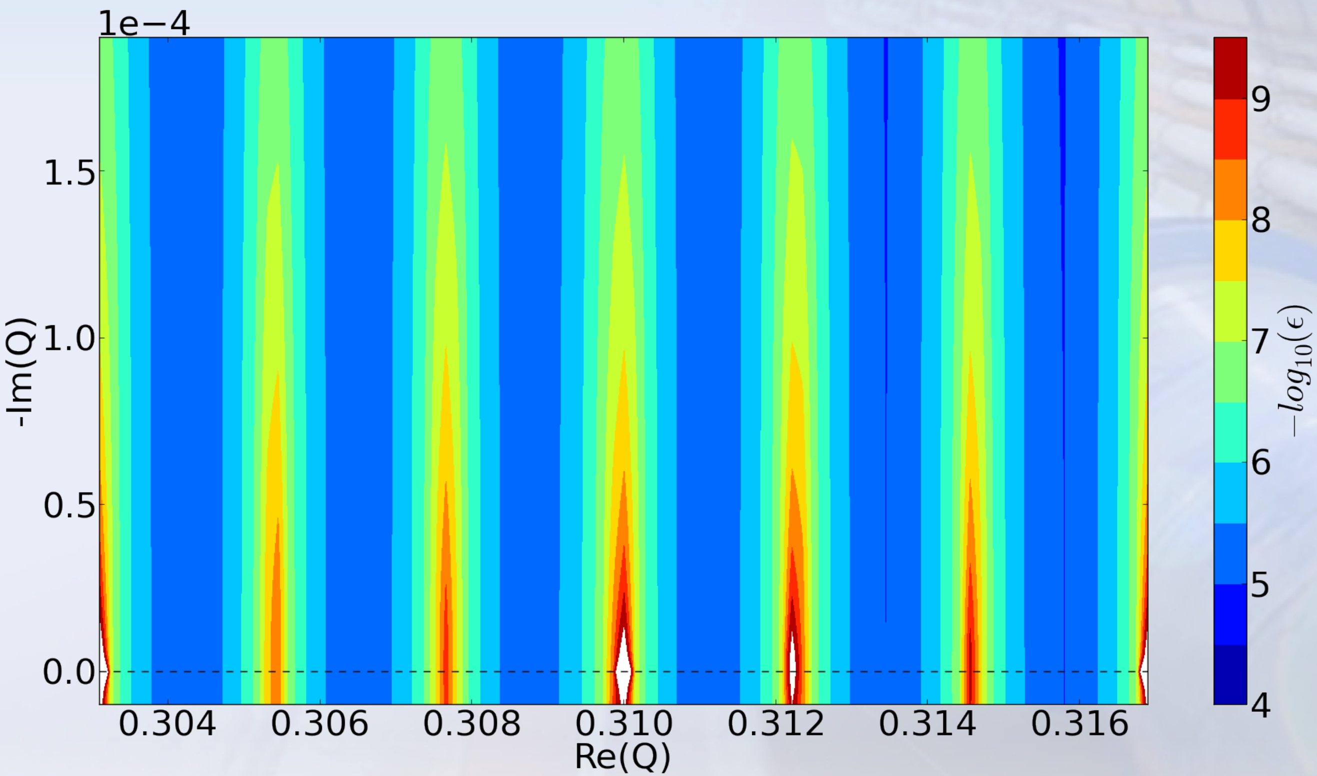
First results

- 4 bunches
- 0.0 chromaticity, 1000 turn damper



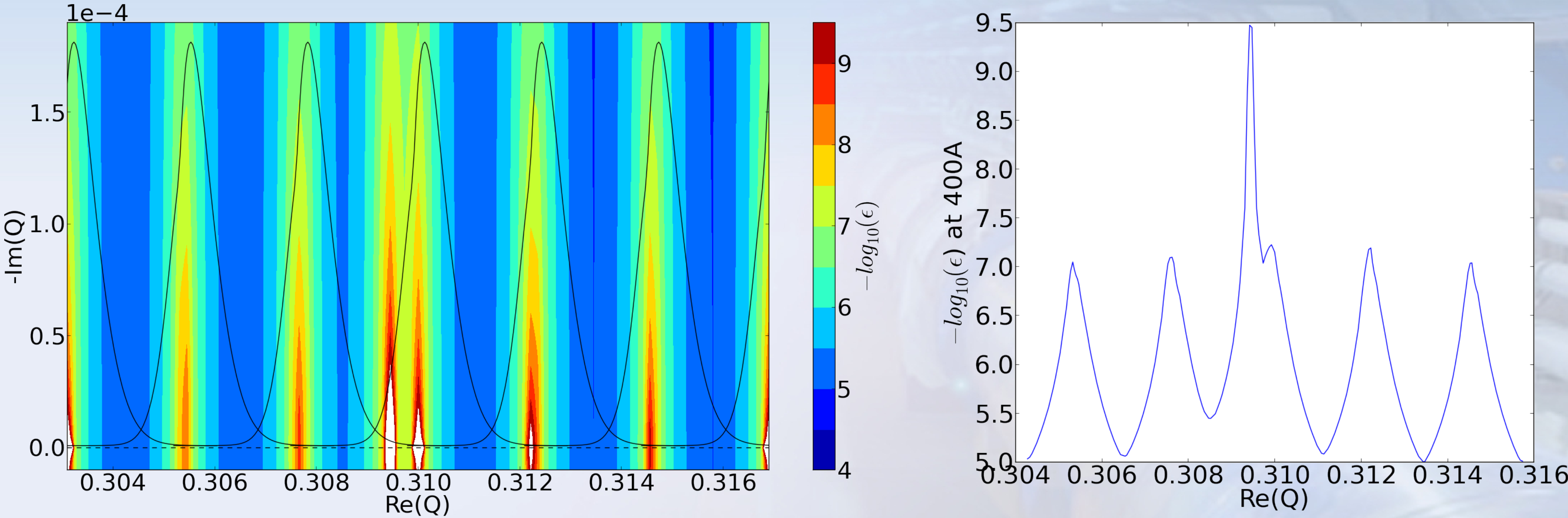
First results

- 4 bunches
- 0.0 chromaticity, 100 turn damper



A naive trial to include transverse non-linearities

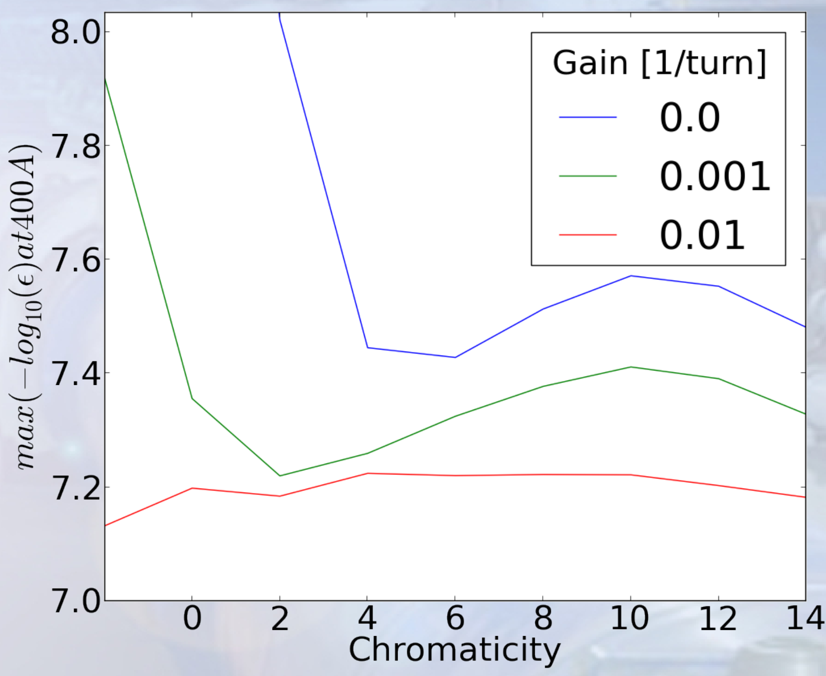
- Small tune shift : each side band can be treated separately



- If all eigenvalues are in the stable area, one can characterize the stability of the beam by the maximum of -log(ε) on the stability diagram

Chromaticity / damper

- Results in accordance with expectations
- Not valid for most cases of interest, where mode coupling is not negligible



Conclusion

- Coupled bunch instabilities cannot be fully treated using the standard eigenvalue approach
  - Transient growth may be expected even in systems with only decaying eigenmodes
  - Behavior depends on initial condition / external excitation
- The pseudo spectrum provides information on the behavior of such non-normal system
- Real life application less obvious than eigenvalues
- Including the effect of transverse non-linearities is also not trivial