From one step to two steps

- If, starting with 1 electron, $p_1(n)$ is the distribution of the number of electrons after 1 step. What is $p_2(n)$, the distribution of the number of electrons after 2 steps ?
- First few expressions:

$$p_2(1) = p_1(1)\, p_1(1)$$
$$p_2(2) = p_1(1)\, p_1(2) + p_1(2)\, (p_1 * p_1)(2)$$
$$p_2(3) = p_1(1)\, p_1(3) + p_1(2)\, (p_1 * p_1)(3) + p_1(3)\, (p_1 * p_1 * p_1)(3)$$

- In general:

$$p_2(n) = \sum_{i=1}^{n} p_1(i)\, p_1^{*i}(n), \quad \textit{where } p^{*i} \textit{ is the } i\textit{-fold convolution of } p$$

The mean after two steps

- The mean (or 1st moment S_1) of p_2 is given by:

$$\bar{p}_2 = \sum_{i=1}^{\infty} i\, p_2(i) = p_1(1)\sum_{i=1}^{\infty} i\, p_1(i) + p_1(2)\sum_{i=1}^{\infty} i\, p_1^{*2}(i) + p_1(3)\sum_{i=1}^{\infty} i\, p_1^{*3}(i) \ldots$$

- Using that the mean is a "cumulant" under convolution:

$$\overline{p_1^{*n}} = n\,\bar{p}_1$$

- the 2-step mean is seen to be the 1-step mean squared:

$$\begin{aligned}\bar{p}_2 &= p_1(1)\,\bar{p}_1 + p_1(2)\,2\,\bar{p}_1 + p_1(3)\,3\,\bar{p}_1 \ldots \\ &= \bar{p}_1 \sum_{i=1}^{\infty} i\, p_1(i) \\ &= \bar{p}_1^{\,2}\end{aligned}$$

The mean after many steps

- The mean of p_3, i.e. the mean after 3 steps, is given by:

$$\bar{p}_3 = \sum_{i=1}^{\infty} i\, p_3(i) = p_2(1)\sum_{i=1}^{\infty} i\, p_1(i) + p_2(2)\sum_{i=1}^{\infty} i\, p_1^{*2}(i) + p_2(3)\sum_{i=1}^{\infty} i\, p_1^{*3}(i)\ldots$$

- Hence, the 3-step mean is the cube of the 1-step mean:

$$\begin{aligned}\bar{p}_3 &= p_2(1)\,\bar{p}_1 + p_2(2)\,2\,\bar{p}_1 + p_2(3)\,3\,\bar{p}_1\ldots \\ &= \bar{p}_1 \sum_{i=1}^{\infty} i\, p_2(i) \\ &= \bar{p}_1\,\bar{p}_2 \\ &= \bar{p}_1^{\;3}\end{aligned}$$

- In general, the mean grows exponentially:

$$\bar{p}_n = \bar{p}_1^{\;n}$$

Check 1: two Legler steps

- Starting from the Legler distribution for $p_1(n)$, binomial with the *total number* of samples as variable; the number of *successes* is 1:

$$p_1(n) = a(1-a)^n,\ with\ a = e^{-\alpha x}$$

- The multiple convolutions are straightforward:

$$p_1^{*2}(n) = a^2(1-a)^n(n+1)$$

$$p_1^{*3}(n) = a^3(1-a)^n(n+1)(n+2)/2!$$

$$p_1^{*4}(n) = a^4(1-a)^n(n+1)(n+2)(n+3)/3!$$

- Summing leads to an unsurprising result: $(a^2 = e^{-\alpha 2x}!)$

$$p_2(0) = a^2$$

$$p_2(1) = a^2(1-a^2)$$

$$p_2(2) = a^2(1-a^2)^2$$

Check 2: two Poisson steps

- Next, the more rounded (shifted) Poisson-distribution

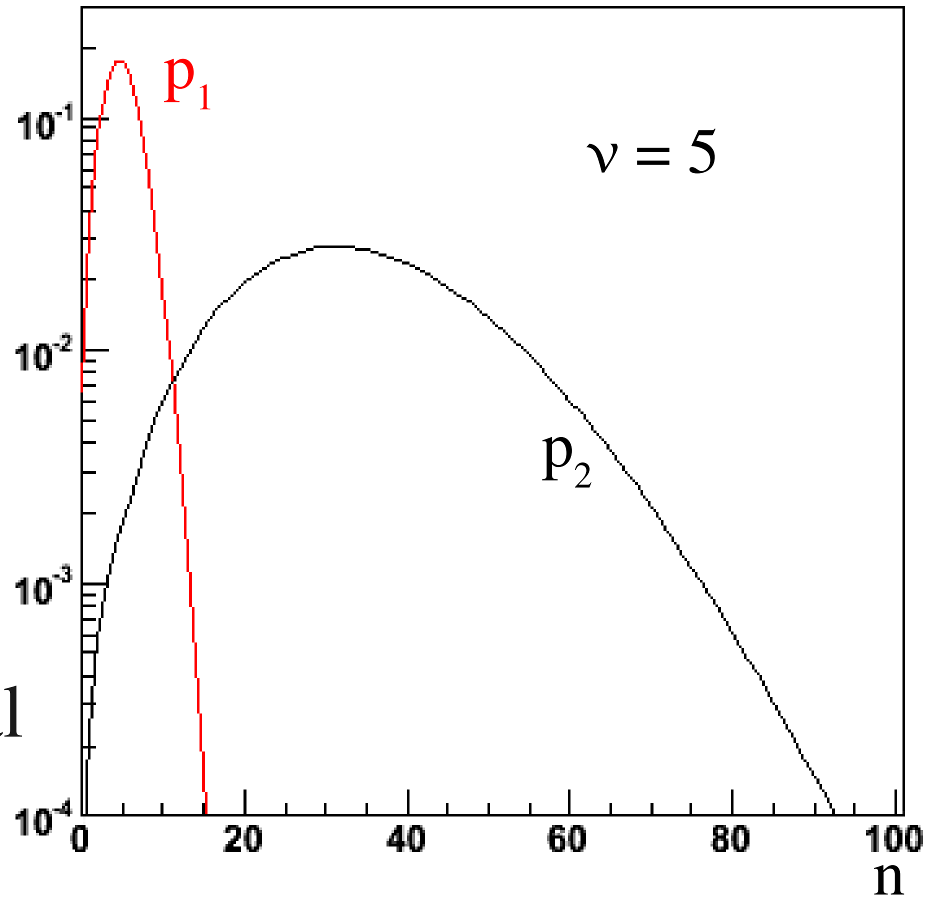
  $$p_1(n,\nu) = e^{-\nu}\nu^{n-1}/(n-1)! \quad (n \geq 1)$$

- Poisson remains Poisson when convoluted, but 2 successive avalanche steps is not Poisson.
- Empirically (numerically):

  $$\mu_1 = \nu + 1 \qquad \mathrm{RMS}_1^2 = \nu$$

  $$\mu_2 = (\nu + 1)^2 \quad \mathrm{RMS}_2^2 = \nu^3 + 3\nu^2 + 2\nu$$

- This too suggests an exponential growth of the mean.



The RMS after two steps

- The RMS is a cumulant derived from the 2nd moment S_2:

$$\mathrm{RMS}^2\left(p_1^{*i}\right) = S_2\left(p_1^{*i}\right) - S_1^2\left(p_1^{*i}\right) = i\left(S_2\left(p_1\right) - S_1^2\left(p_1\right)\right) = i\,\mathrm{RMS}^2\left(p_1\right)$$

- The 2nd moment S_2 after 2 steps is given by:

$$S_2\left(p_2\right) = \sum_{i=1}^{\infty} i^2\, p_2\left(i\right) = \sum_{i=1}^{\infty} p_1\left(i\right) S_2\left(p_1^{*i}\right)$$

- Hence, the 2nd cumulant after 2 steps is:

$$\begin{aligned} S_2\left(p_2\right) - S_1^2\left(p_2\right) &= \sum_{i=1}^{\infty} p_1\left(i\right)\left(S_1^2\left(p_1^{*i}\right) + i\left(S_2\left(p_1\right) - S_1^2\left(p_1\right)\right)\right) - S_1^2\left(p_2\right) \\ &= \sum_{i=1}^{\infty} p_1\left(i\right)\left(i^2\, S_1^2\left(p_1\right) + i\left(S_2\left(p_1\right) - S_1^2\left(p_1\right)\right)\right) - S_1^4\left(p_1\right) \\ &= S_1\left(p_1\right)\left(1 + S_1\left(p_1\right)\right)\left(S_2\left(p_1\right) - S_1^2\left(p_1\right)\right) \end{aligned}$$

Growth of the RMS: verification

- Recall that the growth of the RMS reads:

$$\mathrm{RMS}^2(p_2) = \bar{p}_1(1+\bar{p}_1)\mathrm{RMS}^2(p_1)$$

- This agrees with the Legler growth:

$$\mathrm{RMS}^2(p_2) = \mu^4\left(1-\frac{1}{\mu^2}\right) = \mu^2\left(1-\frac{1}{\mu}\right)\mu^2\left(1+\frac{1}{\mu}\right) = \mu(1+\mu)\mathrm{RMS}^2(p_1)$$

- This also works for a Poisson distribution. Recall:

$$\mu_1 = \nu+1 \qquad \mathrm{RMS}_1^2 = \nu$$

$$\mu_2 = (\nu+1)^2 \quad \mathrm{RMS}_2^2 = \nu^3+3\nu^2+2\nu$$

- Hence:

$$\mathrm{RMS}_2^2 = \nu^3+3\nu^2+2\nu = \mu_1(1+\mu_1)\mathrm{RMS}_1^2$$

The RMS after many steps

- Triple step: a double step followed by a single step:

$$S_2(p_3)=\sum_{i=1}^{\infty} i^2 p_3(i)=\sum_{i=1}^{\infty} p_2(i) S_2(p_1^{*i})$$

- The 2nd cumulant after 3 steps is therefore:

$$\begin{aligned} S_2(p_3)-S_1^2(p_3) &= \sum_{i=1}^{\infty} p_2(i)\left(S_1^2(p_1^{*i})+i\left(S_2(p_1)-S_1^2(p_1)\right)\right)-S_1^2(p_3) \\ &= S_1^2(p_1)\left(1+S_1(p_1)+S_1^2(p_1)\right)\left(S_2(p_1)-S_1^2(p_1)\right) \end{aligned}$$

- The pattern becomes clear:

$$\mathrm{RMS}^2(p_2)=\bar{p}_1(1+\bar{p}_1)\mathrm{RMS}^2(p_1)$$

$$\mathrm{RMS}^2(p_3)=\bar{p}_1^{\,2}(1+\bar{p}_1+\bar{p}_1^{\,2})\mathrm{RMS}^2(p_1)$$

$$\mathrm{RMS}^2(p_4)=\bar{p}_1^{\,3}(1+\bar{p}_1+\bar{p}_1^{\,2}+\bar{p}_1^{\,3})\mathrm{RMS}^2(p_1)$$

...

Relative width, a measure of roundness

- The Polya distribution, a popular parametrisation of the gain, is a Γ-distribution

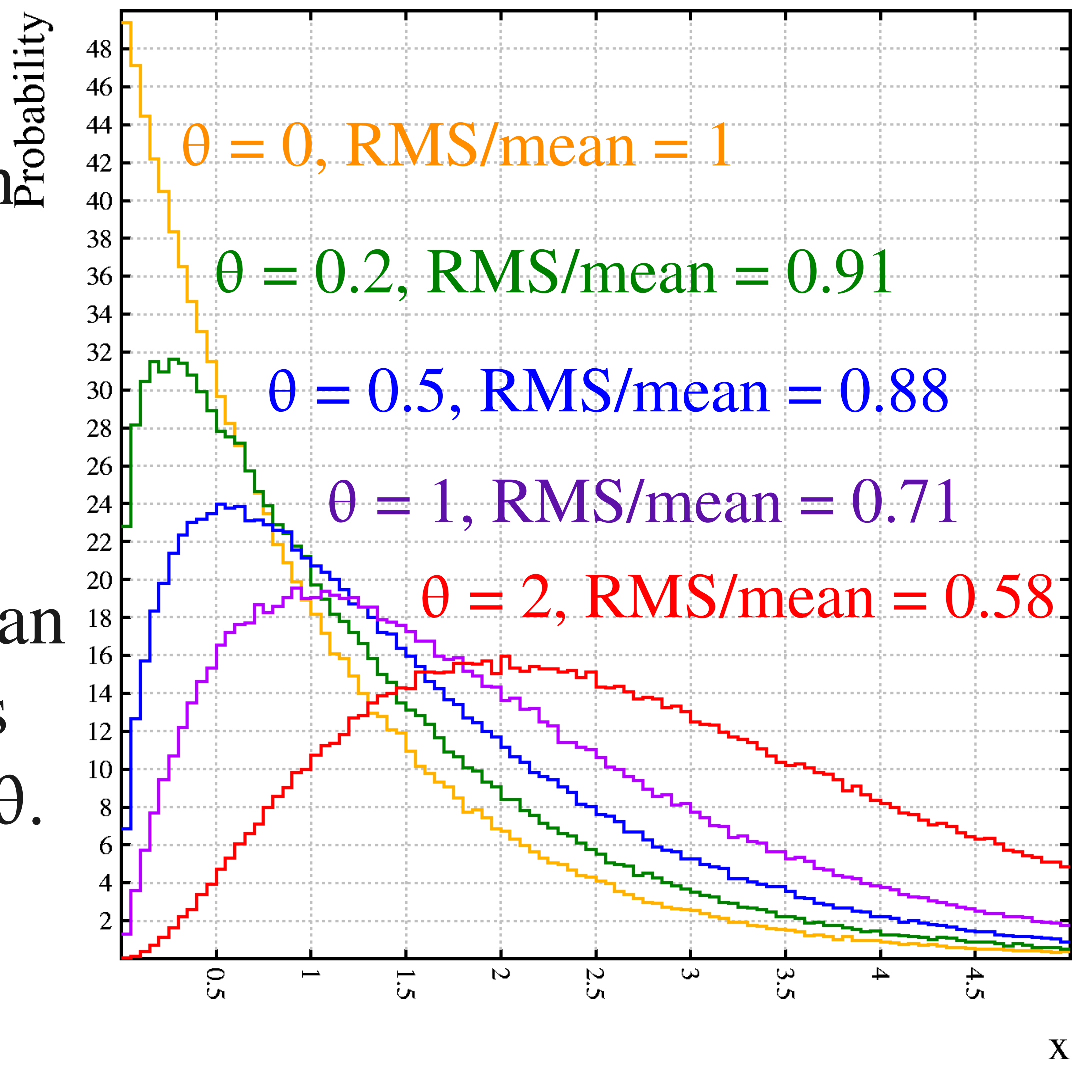
$$f(x,\theta) = x^{\theta} e^{-x}/\Gamma(\theta+1)$$

- Its lowest moments are:

$$\bar{f} = \mathrm{RMS}^2(f) = \theta+1$$

- The relative width is 1 for an exponential (θ=0) and falls towards 0 with increasing θ.

$$\frac{\mathrm{RMS}(f)}{\bar{f}} = \frac{1}{\sqrt{\theta+1}}$$



Evolution of the relative width

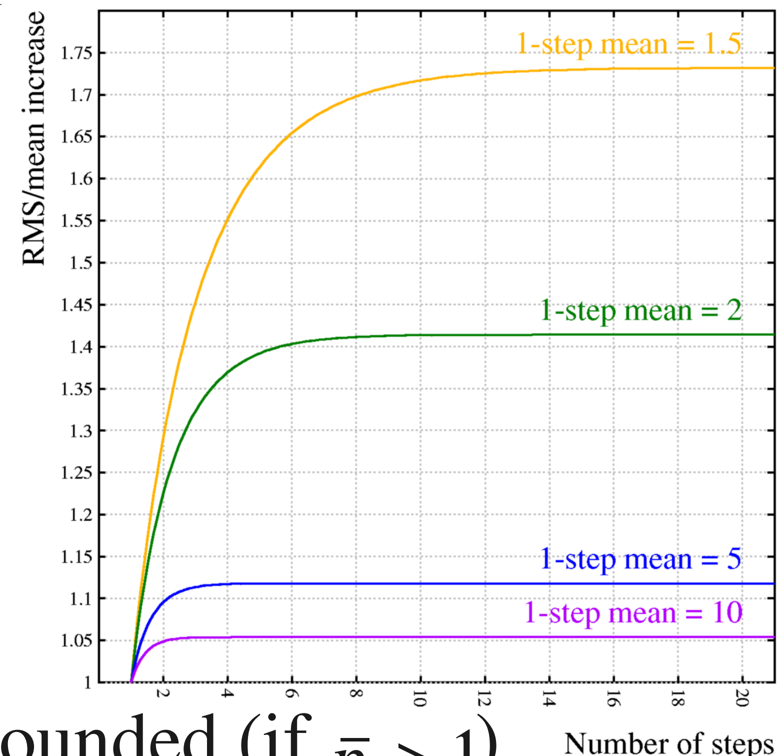
- We found earlier that:

$$\bar{p}_n = \bar{p}_1^{\,n}, \quad \mathrm{RMS}(p_n) = \mathrm{RMS}(p_1)\sqrt{\bar{p}_1^{\,n-1}\sum_{i=0}^{n-1}\bar{p}_1^{\,i}}$$

- The relative width evolves as:

$$\frac{\mathrm{RMS}(p_n)}{\bar{p}_n} = \frac{\mathrm{RMS}(p_1)}{\bar{p}_1}\sqrt{\sum_{i=0}^{n-1}\frac{1}{\bar{p}_1^{\,n-i-1}}}$$

$$= \frac{\mathrm{RMS}(p_1)}{\bar{p}_1}\sqrt{\frac{1-1/\bar{p}_1^{\,n}}{1-1/\bar{p}_1}}$$



- The relative width *always increases*.
- The growth of the relative width is bounded (if $\bar{p}_1 > 1$).
- Once $\bar{p}$ is large, the relative width barely grows further.

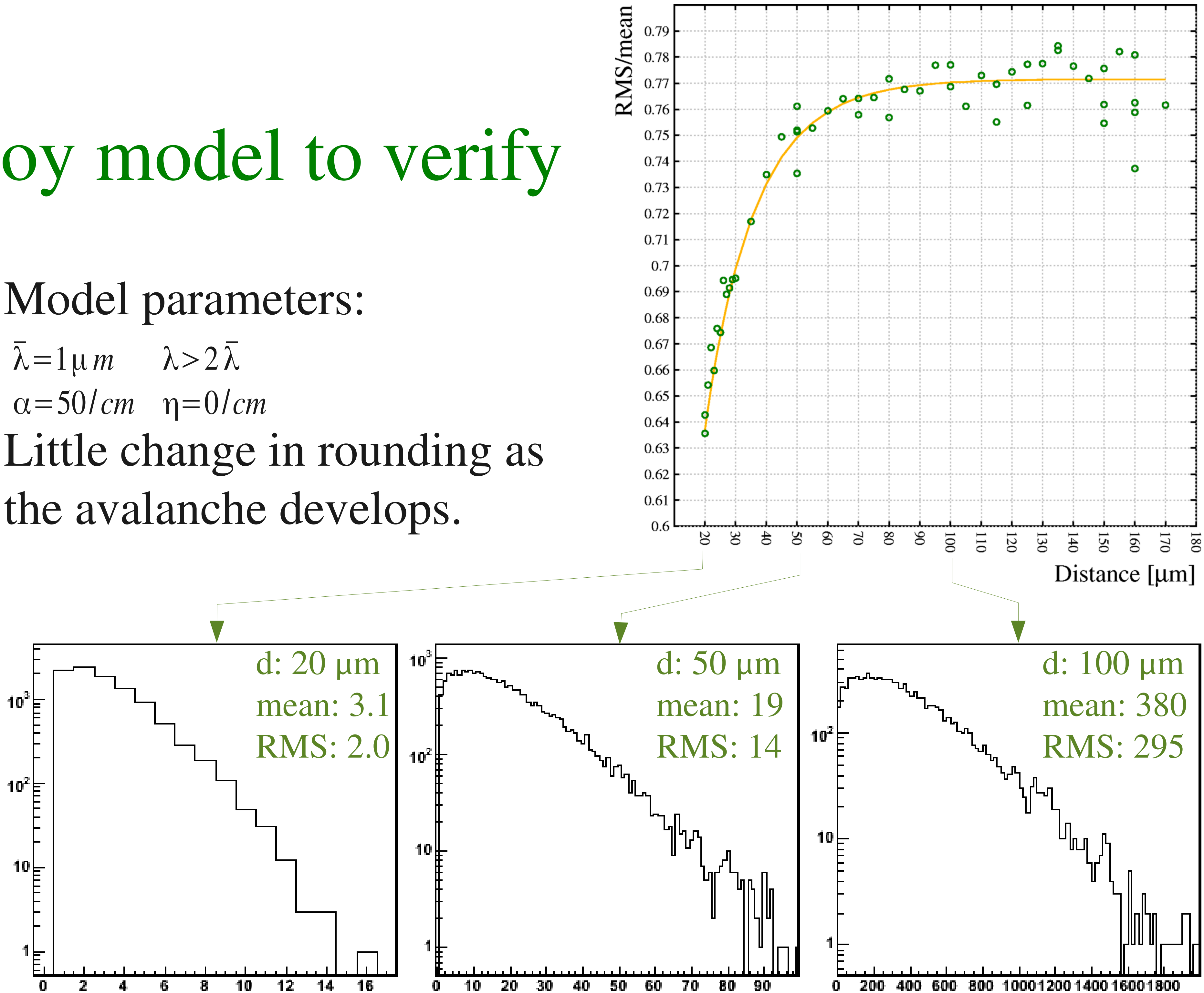
Toy model to verify

- Model parameters:

  $\bar{\lambda}=1\mu m \quad \lambda>2\bar{\lambda}$

  $\alpha=50/cm \quad \eta=0/cm$

- Little change in rounding as the avalanche develops.



Verification with Magboltz

- Model parameters:
  - Ar 80 % CO_2 20 %
  - *E*: 20 kV/cm
  - Gap: 20 µm – 900 µm
  - Gain: 1.08 – 120
- Evolution of the relative width is as expected.
- Note: the RMS also precisely follows the Legler estimate.