Ghosts in Keldysh-Schwinger formalism

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Excited QCD, 2 – 8 Feb 2014, Sarajevo, Bosnia & Herzegovina
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Motivation

- Computations in covariant gauges are usually much simpler than those in physical ones like the Coulomb gauge.

- Covariant gauges require ghosts to compensate unphysical degrees of freedom.

How to introduce ghosts in the Keldysh-Schwinger formalism?

What is the Green’s function of free ghosts?
Keldysh–Schwinger formalism

Description of non-equilibrium many-body systems

Contour Green function of gauge field

\[ i \mathcal{D}^{\mu\nu}_{ab}(x, y) = \langle \tilde{T} A^\mu_a(x) A^\nu_b(y) \rangle \]

\[ \langle \ldots \rangle = \text{Tr}[\hat{\rho}(t_0)\ldots] \]

\[ \tilde{T} - \text{ordering along the contour} \]

\[ \tilde{T} A(x) B(y) = \Theta(x_0, y_0) A(x) B(y) \pm \Theta(y_0, x_0) B(y) A(x) \]

\( c_1 \rightarrow c_2 \quad t_0 \rightarrow -\infty \quad t \rightarrow +\infty \quad t_{\text{max}} \)
**Keldysh–Schwinger formalism**

Contour Green’s function includes 4 Green’s functions with real time arguments:

\[
(D_{ab}^{\mu \nu})^> (x, y) = \left\langle A_\mu^a (x) A_\nu^b (y) \right\rangle
\]

\[
(D_{ab}^{\mu \nu})^< (x, y) = \left\langle A_\nu^b (y) A_\mu^a (x) \right\rangle
\]

\[
(D_{ab}^{\mu \nu})^c (x, y) = \left\langle T^c A_\mu^a (x) A_\nu^b (y) \right\rangle
\]

\[
(D_{ab}^{\mu \nu})^a (x, y) = \left\langle T^a A_\mu^a (x) A_\nu^b (y) \right\rangle
\]

- **Chronological time ordering**
- **Anti-chronological time ordering**
Retarded, advanced & symmetric Green’s functions

\[
\mathcal{D}^+(x, y) = \Theta(x_0 - y_0)(\mathcal{D}^>(x, y) - \mathcal{D}^<(x, y))
\]

\[
\mathcal{D}^-(x, y) = \Theta(y_0 - x_0)(\mathcal{D}^<(x, y) - \mathcal{D}^>(x, y))
\]

\[
\mathcal{D}^{sym}(x, y) = \mathcal{D}^>(x, y) + \mathcal{D}^<(x, y)
\]
Meaning of the functions

\( D^{<\to} (x, y) \)
- phase-space density
- mass-shell constraint
- real particles

\( D^{\pm} (x, y) \)
- retarded & advanced propagator
- no mass-shell constraint
- virtual particles
Full Green’s function of gluon field

Contour-ordered Green’s function has perturbative expansion similar to that of time-ordered Green’s function.

Dyson – Schwinger equation

\[ \mathcal{D} = D - D \Pi D \]

1-st order

Full contour Green’s function = Free contour Green’s function + Contour polarization tensor
From general covariant to Feynman gauge

Equation of motion of the free contour function in general covariant gauge

\[
\left[ \partial_x^2 g^{\mu\nu} - \left( 1 - \frac{1}{\alpha} \right) \partial_x^\mu \partial_x^\nu \right] D_{\nu\rho}^{ab} (x, y) = g^\mu_\rho \delta^{ab} \delta^{(4)}_C (x, y)
\]

\[
\mathcal{L}_{gf} = - \frac{1}{2\alpha} (\partial^\mu A^a_\mu)^2
\]

\[
\delta^{(4)}_C (x, y) = \begin{cases} 
\delta^{(4)} (x - y) & \text{for } x_0, y_0 \text{ on the upper branch} \\
0 & \text{for } x_0, y_0 \text{ on different branches} \\
-\delta^{(4)} (x - y) & \text{for } x_0, y_0 \text{ on the lower branch}
\end{cases}
\]

\[
D^{\mu\nu}_{ab} (x, y) = D^{\mu\nu}_{ab} (x - y) \quad \text{homogeneity, translational invariance}
\]

\[
D^> (p) \propto \delta (p^2) \left( g^{\mu\nu} - (1 - \alpha) \frac{p^\mu p^\nu}{p^2} \right)
\]

Feynman gauge \( \alpha = 1 \)

mass-shell condition

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Free Green’s functions of gluon field

Equations of motion of the real-time argument Green’s functions

\[ p^2 D^{>,<}(p) = 0 \]
\[ p^2 D^{c,a}(p) = \mp \delta^{ab} g_{\mu\nu} \]

\[
D^{>}(p) = \frac{i\pi}{E_p} g_{\mu\nu} \delta^{ab} \left[ \delta(E_p - p_0)[n_g(p) - 1] + \delta(E_p + p_0)n_g(-p) \right]
\]

\[
D^{<}(p) = \frac{i\pi}{E_p} g_{\mu\nu} \delta^{ab} \left[ \delta(E_p - p_0)n_g(p) + \delta(E_p + p_0)[n_g(-p) - 1] \right]
\]

\[
D^{c}(p) = -g_{\mu\nu} \delta^{ab} \left[ \frac{1}{p^2 + i0^+} - \frac{i\pi}{E_p} \left( \delta(E_p - p_0)n_g(p) + \delta(E_p + p_0)n_g(-p) \right) \right]
\]

\[
D^{a}(p) = g_{\mu\nu} \delta^{ab} \left[ \frac{1}{p^2 - i0^+} + \frac{i\pi}{E_p} \left( \delta(E_p - p_0)n_g(p) + \delta(E_p + p_0)n_g(-p) \right) \right]
\]

\[ n_g(p) \] - gluon distribution function
Polarization tensor in QCD

How to find the ghosts Green’s functions?

One can solve the equation of motion of the scalar field.
But what is the distribution function of ghosts?
How to get Green’s function of free ghosts?

Ghost sector should be determined by the gauge symmetry of the theory!

\[ A^a_\mu \rightarrow (A^a_\mu)^U = A^a_\mu + f^{abc} \omega^b A^c_\mu - \frac{1}{g} \partial_\mu \omega^a \]

Slavnov-Taylor identities
Generating functional

\[ W_0[J, \chi, \chi^*] = N_0 \int_{BC} DA \, Dc \, Dc^* \, e^{i \int c^4 x \, L_{\text{eff}}(x)} \]

boundary conditions:
the fields are fixed in \( t = -\infty \pm i0^+ \)

Lagrangian:

\[ L_{\text{eff}}(x) = -\frac{1}{4} F_a^{\mu \nu} F_a^{\mu \nu} + \bar{\psi}(i \gamma_\mu D_\mu - m)\psi - \frac{1}{2\alpha} (\partial_\mu A_\mu^a)^2 \]
\[ - c_a^*(\partial_\mu \partial_\mu \delta_{ab} - g \partial_\mu f^{abc} A_\mu^c) c_b + J_\mu^a A_\mu^a + \chi_a^* c_a + \chi_a c_a^* \]

\[ W[J, \chi, \chi^*] = N \int DA' \, Dc' \, Dc'^* \, DA'' \, Dc'' \, Dc''\]
\[ \times \rho[A', c', c'^*| A'', c'', c''\] W_0[J, \chi, \chi^*] \]

density matrix

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Generating functional

\[ W[J, \chi, \chi^*] = N \int DA' Dc' Dc^* DA'' Dc'' Dc^{*''} \]
\[ \times \rho[A', c', c^*|A'', c'', c^{*''}] W_0[J, \chi, \chi^*] \]

The full Green's function can be generated through

\[ iD^{ab}_{\mu\nu}(x, y) = (-i)^2 \frac{\delta^2}{\delta J^a_\mu(x) \delta J^b_\nu(y)} W[J, \chi, \chi^*] \bigg|_{J=\chi=\chi^*=0} \]

density matrix \( \rho[A', c', c^*|A'', c'', c^{*''}] \) is not specified

the explicit form of the functional and the Green's function cannot be found

The functional provides various relations among Green's functions.

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General Slavnov-Taylor identity

\[ W[J, \chi, \chi^*] = N \int_{BC} \mathcal{D}A \Delta[A] e^{i \int_c d^4 x \mathcal{L}(x)} \]

analog of the Fadeev-Popov determinant

\[ \Delta[A] \equiv \int_{BC} \mathcal{D}c \mathcal{D}c^* e^{-i \int_c d^4 x \left( -f^{abc} \partial_\mu \delta_{ab} - g f^{abc} A_\mu^c c_b + \chi^*_a c_a + \chi c_a \right)} \]

The invariance of \( W[J, \chi, \chi^*] \) under the transformations

\[ A_\mu^a \rightarrow \left( A_\mu^a \right)^U = A_\mu^a + f^{abc} \omega^b A_\mu^c - \frac{1}{g} \partial_\mu \omega^a \]

leads to

\[ \left\{ i \partial_\mu^\mu \frac{\delta}{\delta J_d^\mu(y)} - \int_C d^4 x J_a^\mu(x) \left( \partial_\mu^{(x)} \delta_{ab} + igf^{abc} \frac{\delta}{\delta J_c^\mu(x)} \right) M_{bd}^{-1} \left[ \frac{1}{i} \frac{\delta}{\delta J} \right] |x, y\rangle \right\} W[J, \chi, \chi^*] = 0 \]
Slavnov-Taylor identity for gluon Green’s function

\[
\frac{\delta}{\delta J^\nu_c(z)} \left\{ i \partial_\mu^{(y)} \frac{\delta}{\delta J_\mu^a(y)} - \int_C d^4 x J_\mu^a(x) \left( \partial_\mu^{(x)} \delta^{ab} + ig f^{abc} \frac{\delta}{\delta J_\mu^c(x)} \right) M^{-1}_{bd} \left[ \frac{1}{i} \frac{\delta}{\delta J} \right]_{x, y} \right\} W[J, \chi, \chi^*] = 0
\]

\[
J = \chi = \chi^* = 0
\]

\[
- p^\mu D^{ab}_{\mu \nu}(p) = p_\nu \Delta_{ab}(p)
\]

free ghosts Green’s function

The longitudinal component of the gluon Green’s function is free.
Ghost functions

$$-p^\mu D_{\mu\nu}^{ab}(p) = p_\nu \Delta_{ab}^{p}(p)$$

$$\Delta^<(p) = -\frac{i\pi}{E_p} \delta^{ab} \left[ \delta(E_p - p_0)[n_g(p) + 1] + \delta(E_p + p_0)n_g(-p) \right]$$

$$\Delta^>(p) = -\frac{i\pi}{E_p} \delta^{ab} \left[ \delta(E_p - p_0)n_g(p) + \delta(E_p + p_0)[n_g(-p) + 1] \right]$$

$$\Delta^c(p) = \delta^{ab} \left[ \frac{1}{p^2 + i0^+} - \frac{i\pi}{E_p} \left( \delta(E_p - p_0)n_g(p) + \delta(E_p + p_0)n_g(-p) \right) \right]$$

$$\Delta^a(p) = -\delta^{ab} \left[ \frac{1}{p^2 - i0^+} + \frac{i\pi}{E_p} \left( \delta(E_p - p_0)n_g(p) + \delta(E_p + p_0)n_g(-p) \right) \right]$$

$n_g(p)$ - gluon distribution function
Contributions to the contour polarization tensor

$$
\Pi_{ab}^{\mu \nu}(x, y) = \mp ig^2 N_c \delta_{ab} \text{Tr}[\gamma^{\mu} S(x, y) \gamma^{\nu} S(y, x)]
$$

quark-loop contribution to contour polarization tensor
From contour to retarded polarization tensor

\[ \Pi^{\text{contour}}(x, y) \rightarrow \Pi^{<,>}(x, y) \rightarrow \Pi^{\pm}(x, y) \]

**Position of** \( x_0 \) **and** \( y_0 \) **on the contour**

**Identity**

\[ \Pi_{\mu\nu}^{\pm}(x, y) = \pm \Theta(\pm x_0 \mp y_0)(\Pi_{\mu\nu}^{>}(x, y) - \Pi_{\mu\nu}^{<}(x, y)) \]
Fermion-loop contribution to retarded polarization tensor

\[
(\Pi^+(k))_{ab}^{\mu\nu} = -\frac{ig^2}{2} N_c \delta_{ab} \int \frac{d^4 p}{(2\pi)^4} \times \\
\times \text{Tr}[\gamma^\mu S^+(p + k)\gamma^\nu S^{\text{sym}}(p) + \gamma^\mu S^{\text{sym}}(p)\gamma^\nu S^-(p - k)]
\]

Free quark Green’s functions:

\[
S^{\pm}(p) = \frac{p^\mu \gamma_\mu}{p^2 \pm ip_0 0^+} \quad S^{\text{sym}}(p) = S^>(p) + S^<(p)
\]

\[
S^>(p) = \frac{i\pi}{E_p} \hat{p} \left[ \delta(E_p - p_0)[n_q(p) - 1] + \delta(E_p + p_0)\bar{n}_q(-p) \right]
\]

\[
S^<(p) = \frac{i\pi}{E_p} \hat{p} \left[ \delta(E_p - p_0)n_q(p) + \delta(E_p + p_0)[\bar{n}_q(-p) - 1] \right]
\]

For gluon and ghosts loop and tadpole there are analogical formulas
Wavelength of a quasi-particle is much bigger than inter-particle distance in the plasma:

\[ \lambda \gg d \]

\[ k^\mu \ll p^\mu \]
Polarization tensor

\[
\Pi_{ab}^{\mu\nu}(k) = g^2 \delta_{ab} \int \frac{d^3 p}{(2\pi)^3} \frac{f(p)}{E_p} \frac{k^2 p^\mu p^\nu - [p^\mu k^\nu + k^\mu p^\nu - g^{\mu\nu} (k \cdot p)](k \cdot p)}{(k \cdot p + i0^+)^2}
\]

distribution function

\[f(p) \equiv 2N_c n_g(p) + n_q(p) + \bar{n}_q(p)\]

(vacuum effect is subtracted)

- symmetric \( \Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k) \)
- transversal \( k_\mu \Pi^{\mu\nu}(k) = 0 \)

Ghosts work properly!

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Conclusions

- The generating functional of QCD in the Keldysh-Schwinger formalism was constructed.
- The general Slavnov-Taylor identity was derived.
- The ghost Green’s function was expressed through the gluon one.
- The computed polarization tensor in the hard loop approximation is automatically transverse.
- QCD calculations in Keldysh-Schwinger formalism are possible in the Feynman gauge.