Vortices and chiral symmetry breaking

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What do we know about the mechanism of chiral symmetry breaking in SU(2) lattice QCD?

in cooperation with
Urs Heller, Roman Höllwieser, Thomas Schweigler
J. Greensite, and Š. Olejník

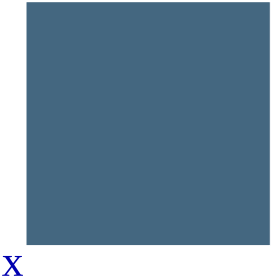
Outline

- Reminder of lattice QCD
- Notions of confinement
- Center vortex dominance
- Chiral symmetry breaking and Dirac spectrum
- Center vortices and topological charge
- Center vortices and Dirac eigenmodes
- Dirac spectra for spherical vortices and instantons
- SummaryClassical

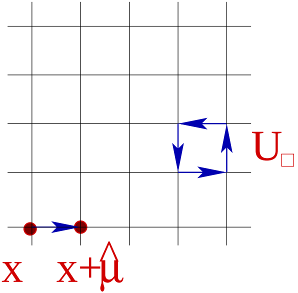
R. Höllwieser *et al.*, PRD78, 054508 (2008) [arXiv:0805.1846]
R. Höllwieser *et al.*, JHEP 1106, 052 (2011) [arXiv:1103.2669]
T. Schweigler *et al.*, PRD87, 054504 (2013) [arXiv:1212.3737]
R. Höllwieser *et al.*, PRD88, 114505 (2013) [arXiv:1304.1277]

SU(2) Gauge theory on a lattice

Continuum



Lattice

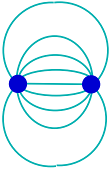


$$A^a_\mu(x)\tfrac{\tau_a}{2} \in \mathrm{su}(2), \qquad U_\mu(x) = \mathrm{e}^{\mathrm{i}ga\bar{A}^a_\mu(x)\frac{\tau_a}{2}} \in \mathrm{SU}(2)$$

$$S = \tfrac{1}{4}\int \mathrm{d}^4x (F_{\mu\nu,a})^2, \quad S = \beta\sum_{\Box}(1 - \frac{1}{2}\,\mathrm{Re\,Tr}\,U_\Box), \qquad \beta = \frac{4}{g^2}$$

Field distributions for point charges

QED
QCD } Maxwell equations
perturbative vacuum }



Coulomb field
unconfined charges

{ Maxwell equations
Non-perturbative vacuum

What is condensed in the vacuum ? Magnetic flux

Vortices and chiral symmetry breaking

- Vortices are quantised magnetic fluxes
- Vortices explain confinement
- Is there a relation to chiral symmetry breaking?
- Nambu (1960): explained low pion mass
  pion is a collective excitation of broken chiral symmetry
- Lagrangian of massless QCD in continuum: chiral symmetric
- $\langle\bar{\psi}\psi\rangle^{\overline{MS}}(2\ \mathrm{GeV}) = -(250 \pm\ \mathrm{MeV})^3$

Notions of confinement

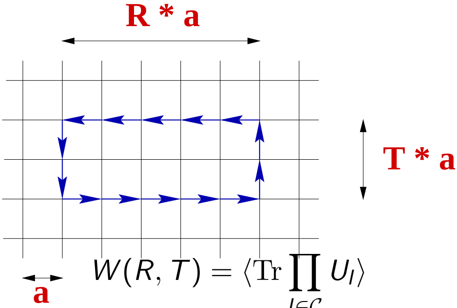
- Confinement means that the spectrum consists only of color singlet states.
- Confinement means that no color-nonsinglet asymptotic states exist.
- In some cases, for theories with a center symmetry, confinement means that Wilson loops have an area law;
  the center symmetry is unbroken.

For theories with a center symmetry we have a true order parameter for confinement, the expectation value of a Polyakov loop, $\langle L \rangle$,

$$L(\vec{x}) = \prod_{t=1}^{N_t} U_4(\vec{x}, t) \ , \qquad \text{world-line of static quark}$$

Under center transformations: $L(\vec{x}) \to zL(\vec{x})$, with $z \in Z_N$.

Wilson loops



$$W(R,T) = \langle \mathrm{Tr} \prod_{l \in \mathcal{C}} U_l \rangle$$

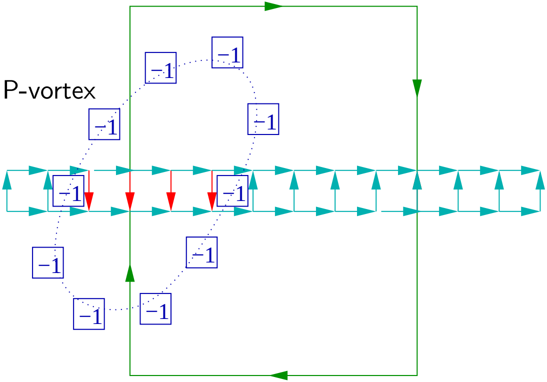
- Wilson loop $\Longleftrightarrow$ Expectation value of a static quark antiquark pair

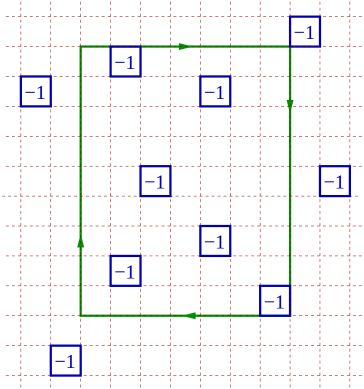
$$\begin{aligned} W(R,T) &= \exp\left(-V(R)T - \alpha(R+T)\right) \\ &= \exp\left(-\sigma RT - \alpha(R+T)\right) \end{aligned}$$

- String tension from area law

$$\sigma = \lim_{R\to\infty} \frac{1}{R} V(R) = -\lim_{R,T\to\infty} \frac{1}{RT} \ln W(R,T)$$

Area law for center projected Wilson loops





denote f the probability that a plaquette has the value -1

$$\langle W(A)\rangle = [(-1)f + (+1)(1-f)]^A = \exp[\underbrace{\ln(1-2f)}_{-\sigma} A] = \exp[-\sigma \underbrace{R \times T}_{A}],$$

$$\sigma \equiv -\ln(1-2f) \approx 2f$$

Center projection

A method to detect vortices.
Fix to direct maximal center gauge, *i.e.*, to Landau gauge in the adjoint repr. Then, center projection for $SU(2)$ consists in

$$U_\mu(x) \to \tilde{U}_\mu(x) = Z_\mu(x) = \text{sign}\,\text{Tr}\Big(U_\mu(x)\Big) \ ,$$

and vortex removal

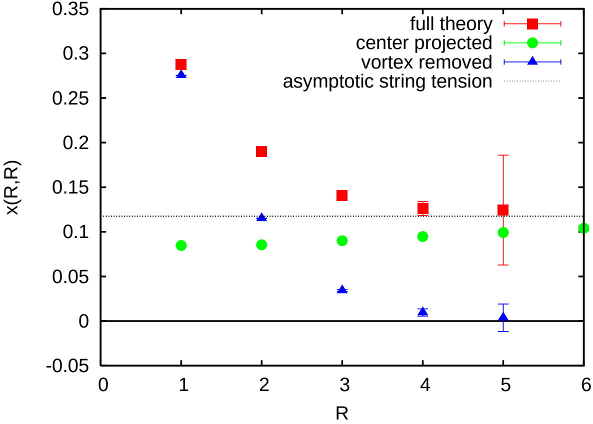
$$U'_\mu(x) = Z_\mu(x) U_\mu(x) \ .$$

P-vortices occupy world sheets that are dual to negative plaquettes in the center-projected gauge fields.

Center dominance means that long distance (nonperturbative) “physics” is reproduced by the center-projected fields.

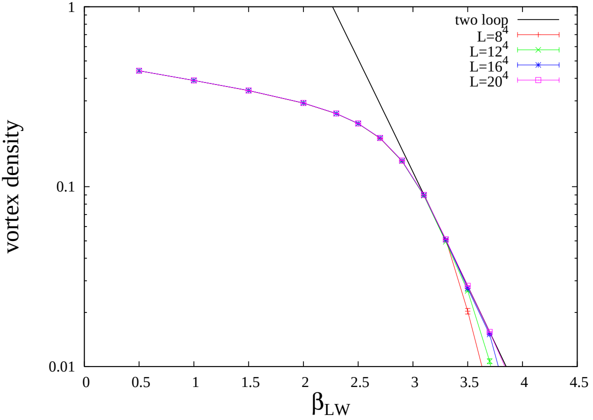
The vortex-removed gauge fields retain short-distance (perturbative) physics only.

Center vortex dominance



Creutz ratios for full, center-projected, and vortex-removed gauge fields for $\beta_{LW} = 3.3$.

Center vortex dominance



P-vortex surface density vs. β_{LW}

"Two-loop" line is scaling prediction with $\sqrt{\rho_v/6\Lambda^2} = 50$.

Scaling shows the vortex density is a physical quantity, with a well defined continuum limit.

Chiral symmetry breaking and Dirac spectrum

A strong enough attractive force is expected to induce spontaneous chiral symmetry breaking.
A force strong enough to confine is certainly expected to do so.

According to Banks-Casher, chiral symmetry breaking is necessarily associated with a finite density of near-zero modes of the chiral-invariant Dirac operator

$$\langle\bar{\psi}\psi\rangle = \pi\rho(0^+) \ ,$$

where $\rho(\lambda)$ is the density of Dirac operator eigenvalues.

Banks-Casher relation

Chiral symmetry breaking $\Longrightarrow$

$\Longrightarrow$ Low-lying eigenmodes of Dirac operator

Dirac equation: $D[A]\,\psi_n = \mathrm{i}\lambda_n\psi_n,$
$\{\gamma_5, \gamma_\mu\} = 0, \quad D[A]\,\gamma_5\psi_n = -\mathrm{i}\lambda_n\gamma_5\psi_n$
Non-zero eigenvalues appear in imaginary pairs $\pm\mathrm{i}\lambda_n$.

$$\langle\bar{\psi}\psi\rangle = -\lim_{m\to 0}\lim_{V\to\infty}\left\langle\frac{1}{V}\sum_n\frac{1}{m+\mathrm{i}\lambda_n}\right\rangle =$$

$$= -\lim_{m\to 0}\lim_{V\to\infty}\left\langle\frac{1}{V}\int\mathrm{d}\lambda\,\rho_V(\lambda)\frac{1}{2}\left(\frac{1}{m+\mathrm{i}\lambda}+\frac{1}{m-\mathrm{i}\lambda}\right)\right\rangle$$

$$-\lim_{m\to 0}\frac{m}{m^2+\lambda^2} = \lim_{m\to 0}\frac{\mathrm{d}}{\mathrm{d}\lambda}\arctan\frac{m}{\lambda} \quad\longrightarrow\quad \pi\delta(0)$$

Chiral condensate $\Longrightarrow$ Density of Near-Zero-modes

$$\langle\bar{\psi}\psi\rangle = \frac{\pi\rho_V(0)}{V}$$

➜ *Banks, Casher, 1980*

Atiyah-Singer index theorem

- zero-modes of fermionic matrix: $D[A]\psi(x) = 0$
- ψ has definite chirality:

$$\psi_{\substack{R\\L}} = \frac{1}{2}(1 \pm \gamma_5)\psi, \quad \Rightarrow \quad \gamma_5 \psi_{\substack{R\\L}} = \pm \psi_{\substack{R\\L}}$$

- Index theorem (wilson, overlap fermions):

n_-, n_+: number of left-/right-handed zeromodes

$$\mathrm{ind} D[A] = n_- - n_+ = Q[A]$$

- (Asqtad) staggered fermions:

$$\mathrm{ind}\ D[A] = 2Q[A] \text{ (SU(2), double degeneracy)}$$

- Adjoint overlap fermions:

$$\mathrm{ind}\ D[A] = 2NQ[A] = 4Q[A] \text{ (real representation)}$$

➜ *Neuberger, Fukaya (1999)*

Topological charge Q

- QCD-vacua characterised by winding number n_w
- scalar (gauge) function: $g(x) = \mathrm{e}^{-\mathrm{i}\vec{\alpha}(x)\vec{\sigma}} \in$ SU(2)$\simeq$ S^3
- R$^3 \to$ S^3: $n_w = -\frac{1}{4\pi^2}\int_{\mathrm{S}^3} \mathrm{d}^3x \mathrm{Sp}(\partial_i g g^\dagger \partial_j g g^\dagger \partial_k g g^\dagger)$
- vector field: $\mathrm{i}\partial_\mu g g^\dagger = \mathcal{A}_\mu = \frac{\vec{\sigma}}{2}\vec{A}_\mu$
- $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - \varepsilon_{ijk} A^j_\mu A^k_\nu = 0$
- Topological charge: $Q = \frac{1}{32\pi^2}\int \mathrm{d}^4x\, F^i_{\mu\nu}\, \tilde{F}^i_{\mu\nu} = \frac{1}{4\pi^2}\vec{E}\vec{B}$
- Lattice: $F_{\mu\nu} = \frac{1}{2\mathrm{i}}(U_{\mu\nu} - U^\dagger_{\mu\nu})$
- admissibility condition: $\mathrm{tr}(\mathbb{1} - U_{\mu\nu}) < 0.0011$
  uniquness of topological charge on the lattice: ➜ Lüscher (1998)

Center vortices and topological charge

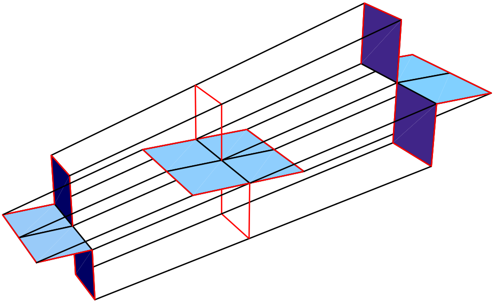
It is believed that in a non-Abelian gauge theory spontaneous chiral symmetry breaking is intimately linked to gauge configurations having topological charge. So one may ask whether center vortices can give rise to topological charge.

Recall that the topological charge density is defined as

$$q(x) = \frac{1}{16\pi^2}\mathrm{Tr}\left(F_{\mu\nu}\tilde{F}_{\mu\nu}\right) = \frac{1}{4\pi^2}\vec{E}\cdot\vec{B}\,, \qquad \tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}\,.$$

So we need flux in all four directions. A vortex has flux perpendicular to its world sheet. Hence to generate topological charge we need two orthogonal vortices intersecting, or one vortex “writhing,” *i.e.*, twisting around itself.

Generating topological charge



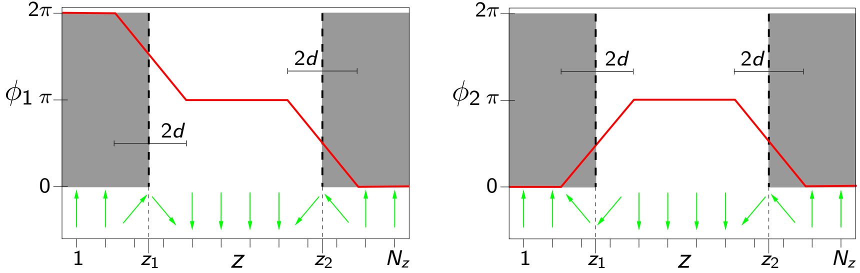
Intersections and writhing points contribute to the topological charge of a P-vortex surface

- intersections $Q = \pm\frac{1}{2}$
- writhing points $Q = \pm\frac{1}{4}$

H. Reinhardt, NPB628 (2002) 133 [hep-th/0112215], hep-th/0204194

Classical vortices and Dirac eigenmodes

We can create a pair of plane vortices, or a vortex-antivortex pair, in the xy plane by setting the time like links in one time slice t_0 to $U_4(t_0, z) = \exp\{i\sigma_3\phi(z)\}$ with ϕ as



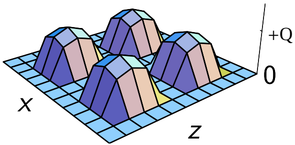
All other links are set to unity, $U_\mu = \mathbb{1}$. Due to periodic boundary conditions, plane vortices come in parallel or antiparallel pairs.

The corresponding P-vortices are located at z_1 and z_2.

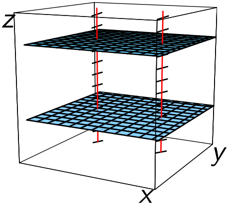
Intersecting plane vortices

Intersecting two orthogonal pairs of plane vortices we can generate topology. A xy vortex generates a chromo-electric field, E_z, and a zt vortex a chromo-magnetic field, B_z. Each intersection point contributes $Q = \pm 1/2$ to the total topological charge.

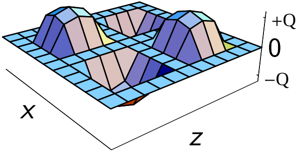
Parallel Vortices



Geometry



Antiparallel Vortices



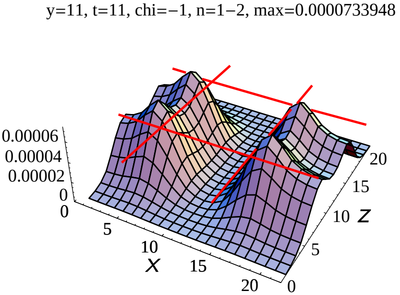
So we can get $Q = 2$ with parallel intersecting vortices and $Q = 0$ with antiparallel intersecting vortices.

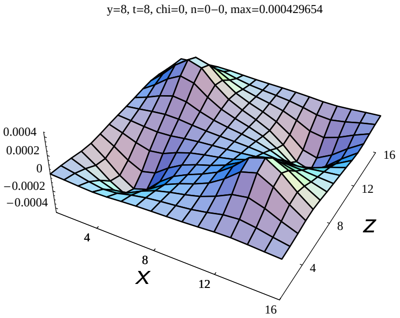
Intersecting plane vortices

We define the scalar density as $\rho = \psi^\dagger \psi$ and the chiral density as $\rho_5 = \psi^\dagger \gamma_5 \psi$, with ψ an overlap Dirac operator eigenmode.

Left: scalar density of the two zero modes for the $Q = 2$ intersecting vortex pairs on a 22^4 lattice.

Right: chiral density of the near-zero modes of the $Q = 0$ intersecting vortex pairs on a 16^4 lattice.





Planar vortices are “special”

The strictly planar (zy) vortices introduced, with the nontrivial gauge links $U_4(t_0, z) = \exp\{i\sigma_3\phi(z)\}$, have all links in a single U(1) subgroup. They are thus like 2-d U(1) vortices, replicated in the two orthogonal directions (x and y).

2-d U(1) vortices have topological charge and thus exact zero modes.

The replication in the orthogonal directions leaves this unchanged.

And indeed, we find two zero modes for a parallel planar vortex pair, but no zero modes, with lifted near-zero modes instead, for an antiparallel planar vortex pair.

Colorful spherical vortex

A nonorientable spherical vortex, with radius R and thickness Δ, is given by

$$U_\mu(x) = \begin{cases} \exp\left(\mathrm{i}\alpha(|\vec{r}-\vec{r}_0|)\vec{n}\cdot\vec{\sigma}\right) & t = t_i, \mu = 4 \\ \mathbb{1} & \text{elsewhere} \end{cases},$$

with $\vec{n} = (\vec{r}-\vec{r}_0)/|\vec{r}-\vec{r}_0|$, and $\vec{r}$ the spatial coordinates of site x.
The profile function α is either

$$\alpha_+(r) = \begin{cases} 0 \\ \frac{\pi}{2}\left(1 - \frac{r-R}{\frac{\Delta}{2}}\right) \\ \pi \end{cases}, \alpha_-(r) = \begin{cases} \pi & r < R - \frac{\Delta}{2} \\ \frac{\pi}{2}\left(1 + \frac{r-R}{\frac{\Delta}{2}}\right) & R - \frac{\Delta}{2} < r < R + \frac{\Delta}{2} \\ 0 & R + \frac{\Delta}{2} < r \end{cases}$$

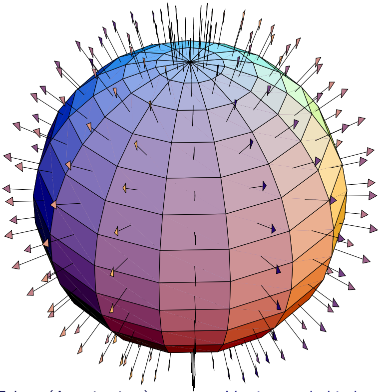
All other links are set to unity, $U_\mu = \mathbb{1}$.

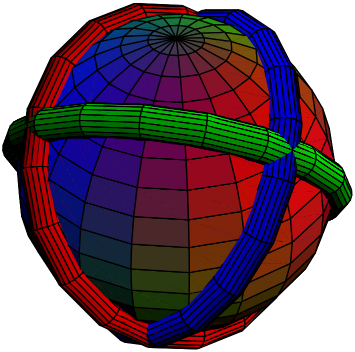
Colorful spherical vortex

The corresponding P-vortex is the sphere at radius R.

The color structure of the spherical vortex, a hedgehog configuration, is illustrated in the left plot.

The right plot illustrates the monopole lines after Abelian projection in the maximal Abelian gauge.



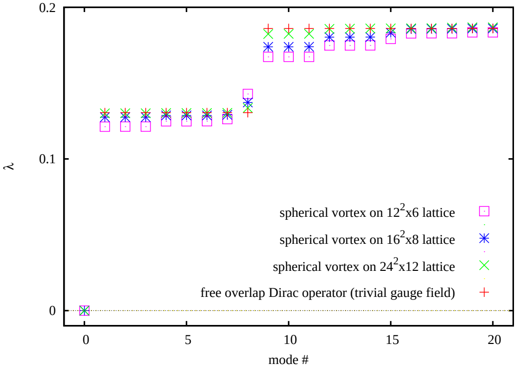


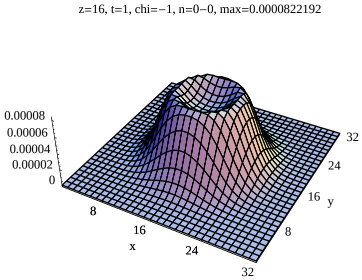
Colorful spherical vortex

Assigning to the space-like boundary constant $U_4(\mathbf{r}) = \pm 1$,
gives a map $\mathbf{r} \in S^3 \to SU(2) \ni U_4(\mathbf{r})$ characterized by a winding number

$$N = \frac{1}{24\pi^2} \int d^3x \, \epsilon_{ijk} \, \mathrm{Tr}[(\partial_i U_4 U_4^\dagger)(\partial_j U_4 U_4^\dagger)(\partial_k U_4 U_4^\dagger)],$$

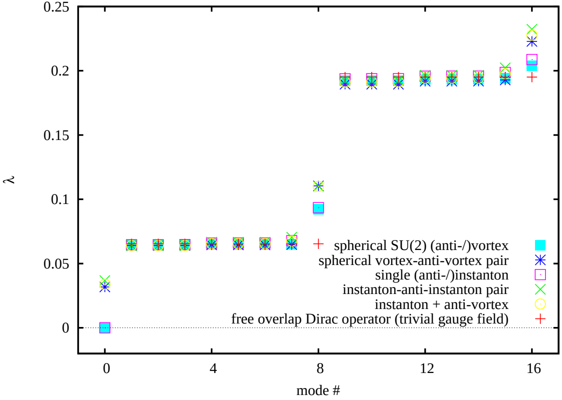
that contributes to the topology and leads to an exact zero mode.





Dirac spectra, spherical vortices and instantons

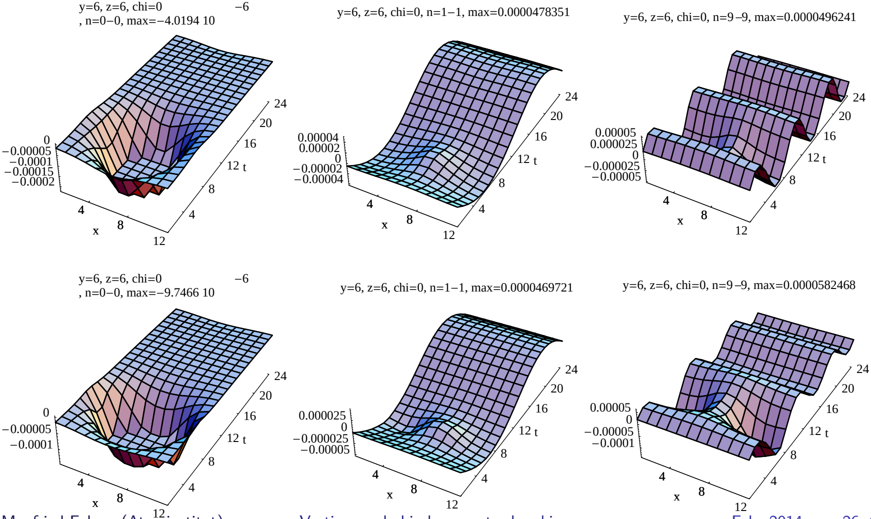
The overlap Dirac eigenvalues, and even the eigenmodes, in the background of spherical vortices are very similar to those with instantons.



With objects of opposite topological charge, the would-be zero modes interact and become near-zero modes.

Dirac spectra, spherical vortices and instantons

Chiral density for the zero mode, first and ninth modes in an instanton background (top row) and a spherical vortex background (bottom row).



Dirac spectra, spherical vortices and instantons

The similarity of the behavior of the Dirac spectrum in the background of instantons and of spherical vortices suggests that almost classical vortices, similarly to an instanton liquid model, can produce a finite spectral density near zero and thus, by the Banks-Casher argument, lead to the spontaneous breaking of chiral symmetry.

Summary

- Confirmed center dominance (saturation of string tension) with improved gauge action
- Confirmed that center vortices also induce chiral symmetry breaking
- So center vortices dominate long-distance, non-perturbative physics
- Classical vortices, planar and spherical, can contribute to the topological charge, leading to zero modes and near-zero modes
- A center vortex model of QCD can thus lead to both confinement and spontaneous chiral symmetry breaking

Thank you for your attention!

Questions?

Notions of confinement

There is another loop operator, $B(C)$, introduced by 't Hooft (1978), with C a loop on the dual lattice, intended to be "dual" to the Wilson loop operator, $W(C')$, with the property

$$B(C)W(C') = \exp(2\pi i/N)W(C')B(C) \ ,$$

if C and C' are topologically linked. In a confined phase, $B(C)$ has a perimeter law falloff, $\exp[-\mu P(C)]$, but an area law falloff otherwise.

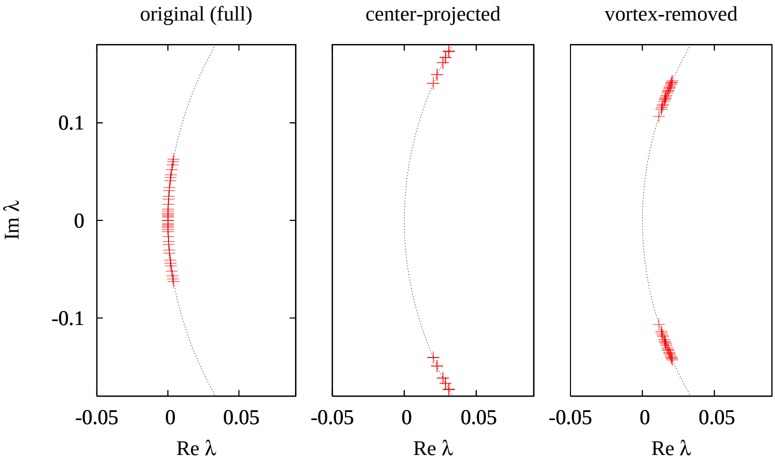
$B(C)$ can be interpreted as creation operator of a "physical" center vortex, leading to the center vortex picture of confinement.

Physical center vortices, with an expected thickness of order 1 fm, are rather difficult to locate in a gauge configuration.

Much easier to define and find are "thin" vortices in center-projected gauge fields, called "P-vortices".

It is assumed that P-vortices track the physical center vortices.

A puzzle

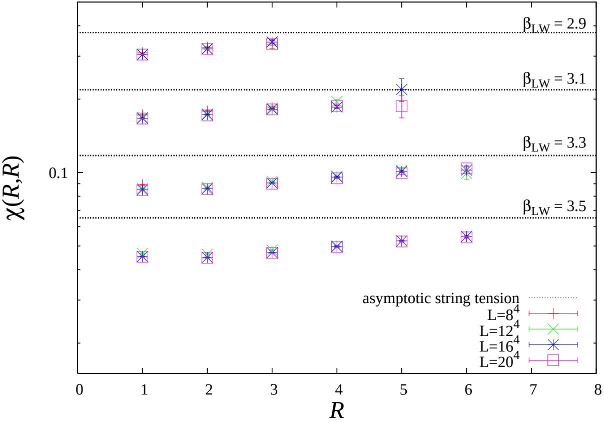


On full and vortex-removed configurations we observe the expected, but on center-projected configs, we find a gap and hence no χSB.

Center vortex dominance

Well known with Wilson action. We test with Lüscher-Weisz action.

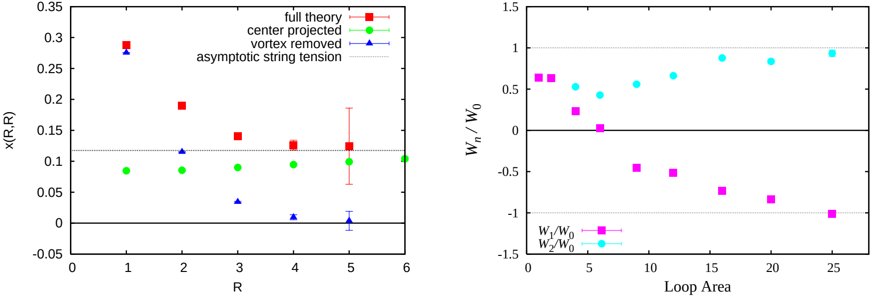
Creutz ratios $\chi(R,R)$ approach string tension σ as $R \to \infty$.



Lattice spacing $a \approx$ 0.27, 0.21, 0.15, 0.11 fm for $\beta_{LW} =$ 2.9, 3.1, 3.3, 3.5.

Find that center-projected Creutz ratios approach full string tension.

Center vortex dominance



Left: Creutz ratios for full, center-projected, and vortex-removed gauge fields for $\beta_{LW} = 3.3$.

Right: Wilson loop pierced by n P-vortices W_n.
Expect $W_n \to (-1)^n W_0$ as area is increased.

Cancellations lead to area-law of confinement.

Chiral symmetry breaking and Dirac spectrum

Since center-projected gauge fields induce confinement, one expects them to induce chiral symmetry breaking.

Vortex-removed, non-confining gauge fields are expected to have a gap and no χSB.

Overlap fermions, with "exact chiral symmetry" should be a good probe.

$$aD_{ov} = M\left[1 + \gamma_5 \varepsilon\left(\gamma_5 D_W(-M)\right)\right] \ ,$$

satisfying the Ginsparg-Wilson relation

$$\{\gamma_5, D_{ov}\} = \frac{a}{M} D_{ov} \gamma_5 D_{ov} \ ,$$

or

$$\left\{\gamma_5, D_{ov}^{-1}\right\} = \frac{a}{M}\gamma_5 \ .$$

This is the mildest breaking of continuum chiral symmetry on the lattice.

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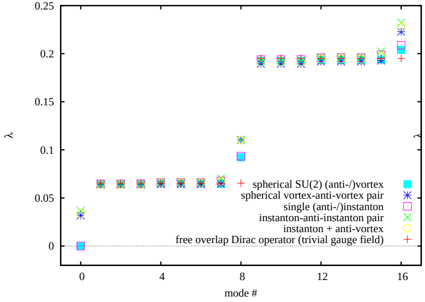
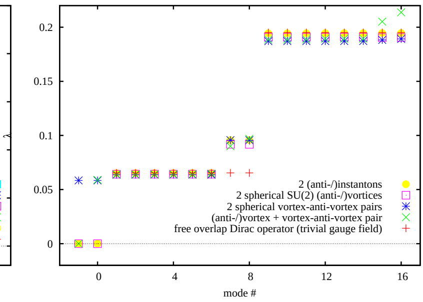
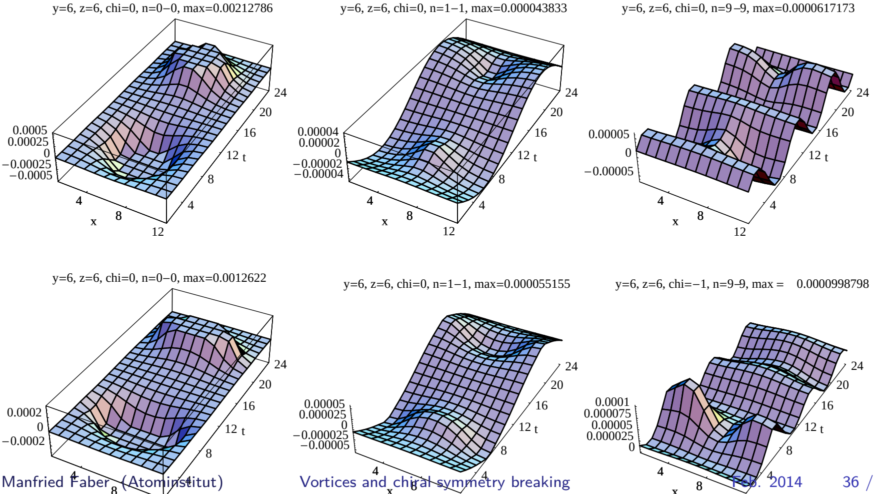




With objects of opposite topological charge, the would-be zero modes interact and become near-zero modes.

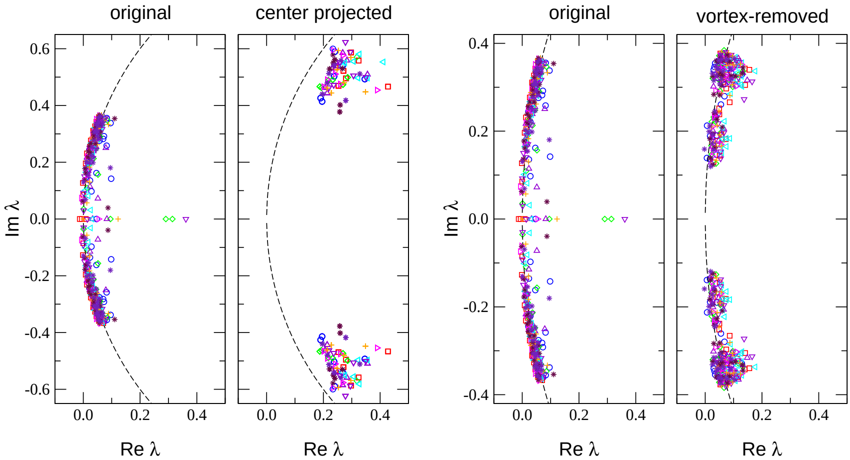
Dirac spectra, spherical vortices and instantons

Chiral density for the first (near-zero mode), second and ninth modes in an instanton–anti-instanton (top row) and a spherical vortex-antivortex background (bottom row).



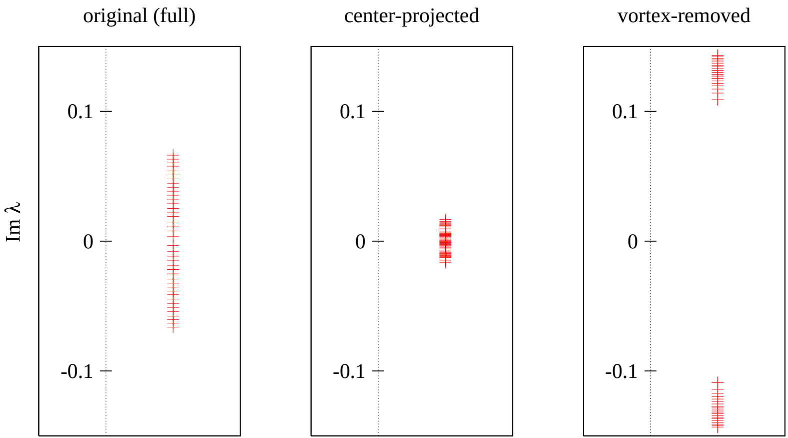
A puzzle

Confirm Gattnar *et al.*, NPB716 (2005) 105 [hep-lat/0412032], who used a "chirally improved" Dirac operator.



A puzzle

Compare to improved staggered (asqtad) fermions:



Here find expected behavior on center-projected configs, too.

And its resolution

The chiral symmetry of overlap fermions is not quite the usual one

$$\delta\psi = i\alpha\gamma_5 \left(1 - \frac{a}{M} D_{ov}\right)\psi\ , \quad \delta\bar{\psi} = i\alpha\bar{\psi}\gamma_5\ .$$

It is field dependent for finite a.

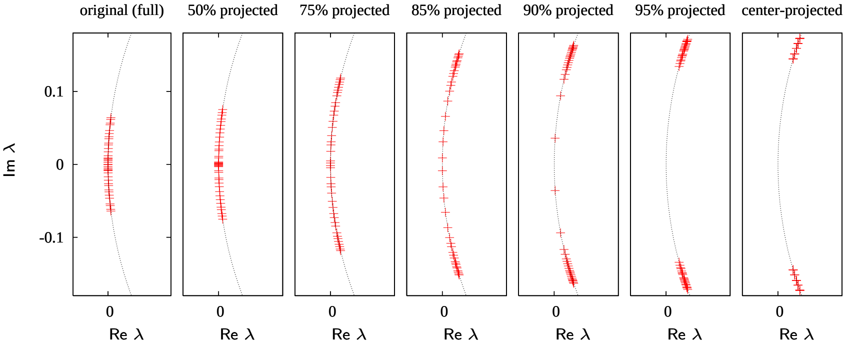
For center-projected configs $\tilde{U}_\mu(x) = \pm 1$.
So the configs are very rough, and easily $D_{ov}(\tilde{U}) = \mathcal{O}(1/a)$, *i.e.*, $aD_{ov}(\tilde{U}) = \mathcal{O}(1)$.
Then the notion of "exact chiral symmetry" is lost.

Note that staggered fermions have a field-independent $U(1) \times U(1)$ chiral symmetry. So center-projected configs induce no gap, signaling χSB, as expected.

And its resolution

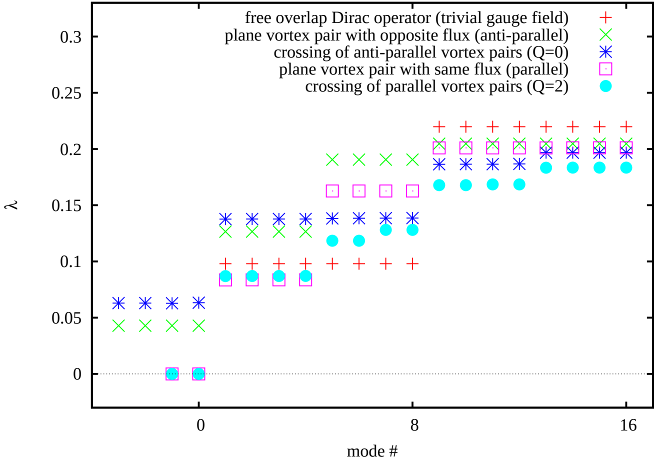
Interpolating, on the $SU(2)$ manifold, between $U_\mu(x)$ and $\tilde{U}\mu(x)$, the configs can be made smoother.

For sufficiently smooth interpolation, that gap closes.



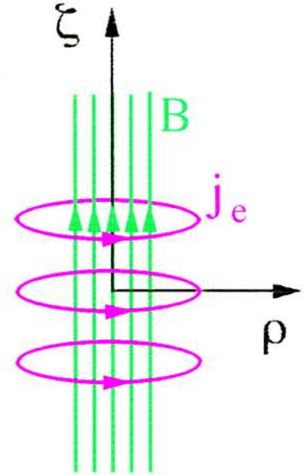
Intersecting plane vortices

We compare a few low-lying overlap eigenvalues in plane vortex configurations with the free case, always using antiperiodic boundary conditions in the time direction.



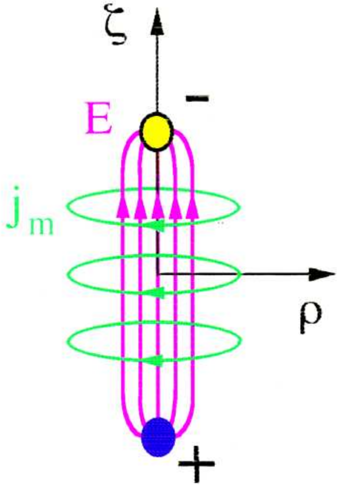
Confinement due to Magnetic Monopoles?

type II superconductor



magnetic fluxoid quantisation

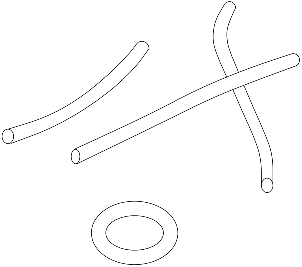
dual superconductor

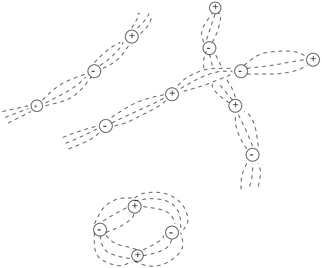


electric fluxoid quantisation

Monopoles versus Vortices

The "spaghetti vacuum" appears, under abelian projection,



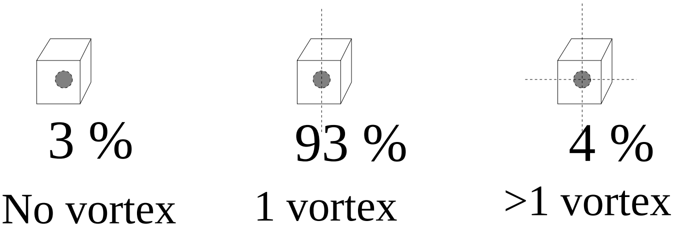


as a "monopole vacuum"

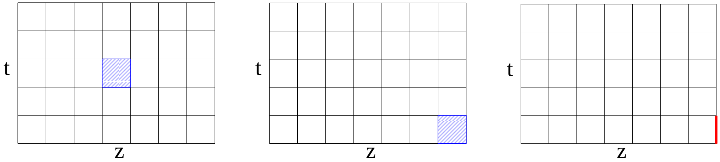
Monopoles and Vortices

Piercing of monopole cubes by vortices

➜ *Greensite et al. (1997)*



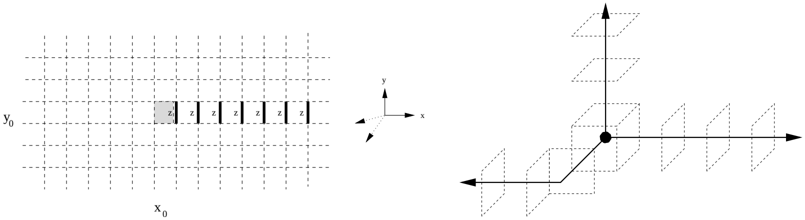
't Hooft loop



How to create a 't Hooft loop $\partial\tilde{\Sigma}$ in the (x, y) plane:
(left) in each (z, t) plane intersecting $\tilde{\Sigma}$, multiply by a center element one plaquette with coordinates (z_0, t_0);
(middle) equivalently, choose (z_0, t_0) in the corner;
(right) equivalently, multiply by a center element the link U_t at the boundary.
Thus, a 't Hooft loop of maximal size is equivalent to twisted boundary conditions for the Polyakov loop.

$$\langle \tilde{W}(\partial\tilde{\Sigma})\rangle \equiv \frac{\int \mathcal{D}U \exp(-\beta \sum_{U_P\in\mathcal{P}(\tilde{\Sigma})}(1-\frac{1}{N}\mathrm{ReTr}\ \zeta U_P) - \beta\sum_{U_P\notin\mathcal{P}(\tilde{\Sigma})}(1-\frac{1}{N}\mathrm{ReTr}\ U_P))}{\int \mathcal{D}U \exp(-\beta\sum_{U_P}(1-\frac{1}{N}\mathrm{ReTr}\ U_P))} \ .$$

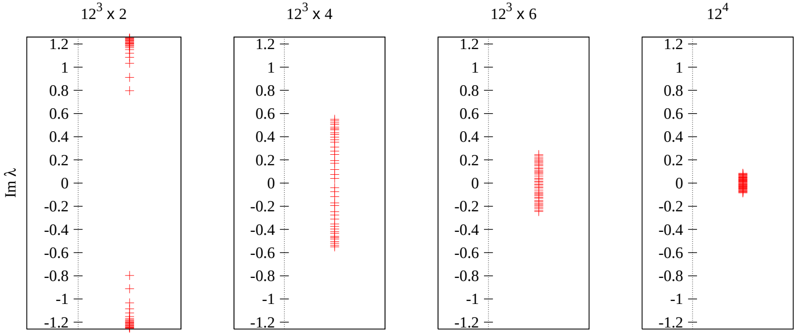
Center vortex sketch



Left: Creation of a thin center vortex: $x - y$ plaquettes (shaded) at sites $(x_0, y_0, \vec{x}_\perp)$ form the center vortex (on the dual lattice).

Right: Three center vortices (solid lines), for $SU(3)$, piercing plaquettes on the original lattice (dashed contours) diverging from a monopole (solid circle) on the dual lattice.

Dirac spectrum at finite temperature



Finite temperature and center projection:
Asqtad Dirac-operator eigenvalues on $12^3 \times N_t$ center-projected lattices with $N_t = 2, 4, 6, 12$ at $\beta_{LW} = 3.5$ – above $\beta_c(N_t = 6)$.
On center-projected lattices chiral restoration appears to occur at higher temperatures than deconfinement.

Vortices and Dirac eigenmode densities

On center-projected configs, Dirac eigenmodes can be expected to be correlated with center vortices.
Does this hold for red full configs?

Note: center vortices generate topological charge at intersection and "writhing" points (Engelhardt and Reinhardt (2000))

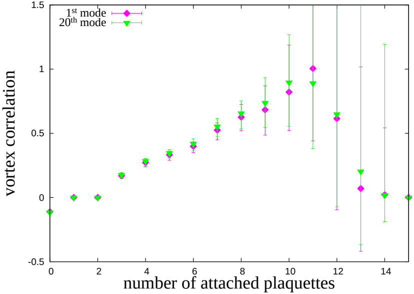
Define correlator (see Kovalenko *et al.*, PLB648 (2007) 383 [hep-lat/0512036])

$$C_\lambda(N_\mathrm{v}) = \frac{\sum_{p_i} \sum_{x\in H}(V\rho_\lambda(x) - 1)}{\sum_{p_i}\sum_{x\in H} 1} \; ,$$

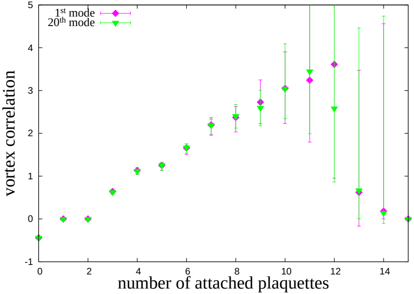
where $\rho_\lambda(x) = |\psi_\lambda(x)|^2$, p_i points on the dual lattice with N_v plaquettes on the vortex surface, and H the hypercube on the original lattice dual to the point p_i.

Vortex correlation: staggered fermions

The vortex correlation $C_\lambda(N_v)$ as function of the number of vortex plaquettes N_v at a given site.



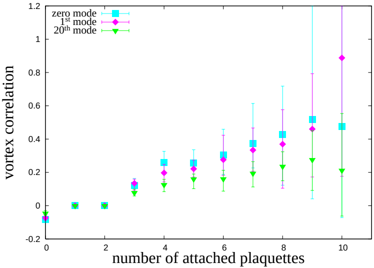
Full configs



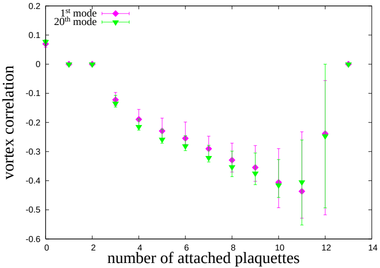
Center-projected configs

The eigenmode density is larger on sites with larger N_v, and more pronounced on center-projected configs, where center vortices are thin and extremely localized.

Vortex correlation: overlap fermions



Full configs



Center-projected configs

For full configs, vortex correlation $C_\lambda(N_v)$ for overlap fermions is similar to staggered fermions.

For center-projected – and too rough for overlap fermions – we see even an anti-correlation.

Dimensionality of Dirac eigenmodes densities

How localized are the low-lying eigenmodes?

Often used to quantify localization of eigenmodes is the inverse participation ratio (IPR)

$$I = N \sum_x \rho_\lambda^2(x) \ .$$

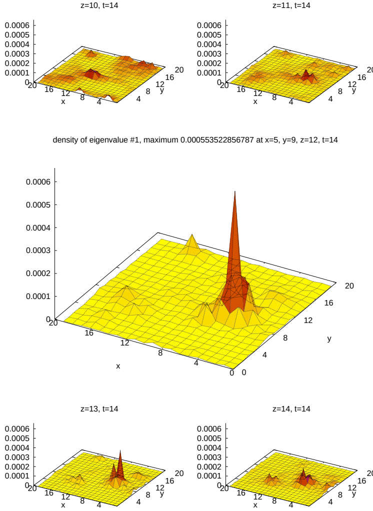
If an eigenmode has support on a submanifold of dimension d, with thickness in the orthogonal $4-d$ direction a fixed number of lattice spacings, then $I \propto 1/a^{4-d}$.

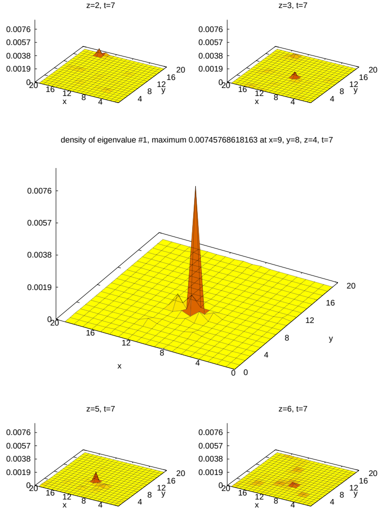
The results are controversial, ranging from $d=0$ (pointlike) to $d=3$.

Scaling of the number of “disconnected” support structures can change the scaling behavior of I.

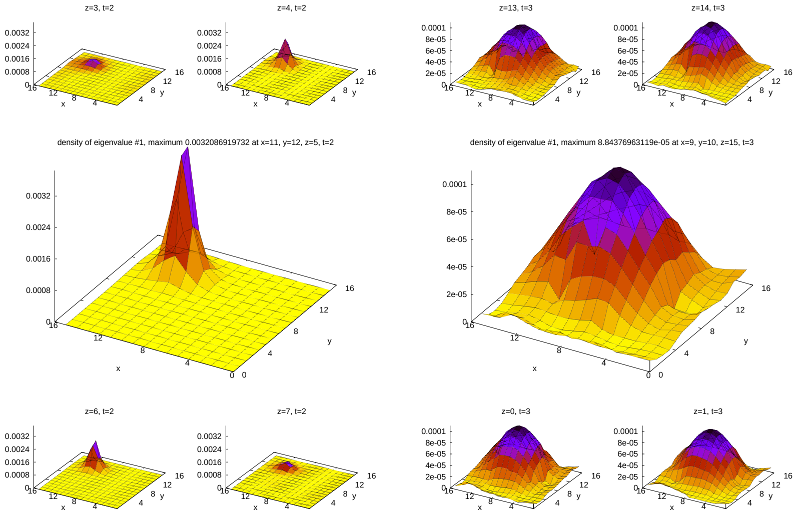
We just inspect by looking at $\rho_\lambda(x)$:

Dimensionality: staggered fermions





Dimensionality: overlap fermions



Topological charge Q

- QCD-vacua characterised by winding number n_w
- scalar (gauge) function: $g(x) = \mathrm{e}^{-\mathrm{i}\vec{\alpha}(x)\vec{\sigma}} \in$ SU(2)$\simeq$ S^3
- R$^3 \to$ S^3: $n_w = -\frac{1}{4\pi^2}\int_{\mathrm{S}^3} \mathrm{d}^3x \mathrm{Sp}(\partial_i g g^\dagger \partial_j g g^\dagger \partial_k g g^\dagger)$
- Topological charge: $Q = \frac{1}{32\pi^2}\int \mathrm{d}^4x\, F^i_{\mu\nu}\, \tilde{F}^i_{\mu\nu} = \frac{1}{4\pi^2}\vec{E}\vec{B}$
- Lattice: $F_{\mu\nu} \Rightarrow \frac{1}{2\mathrm{i}}(U_{\mu\nu} - U^\dagger_{\mu\nu})$
- admissibility condition: $\mathrm{tr}(\mathbb{1} - U_{\mu\nu}) < 0.0011$
  uniquness of topological charge on the lattice

➜ *Lüscher (1998)*

Topological charge Q

- QCD-vacua characterised by winding number n_w
- scalar (gauge) function: $g(x) = \mathrm{e}^{-\mathrm{i}\vec{\alpha}(x)\vec{\sigma}} \in$ SU(2)$\simeq$ S^3
- vector field: $\mathrm{i}\partial_\mu g g^\dagger = \mathcal{A}_\mu = \frac{\vec{\sigma}}{2}\vec{A}_\mu$
- perturbative vacuum: $\mathcal{A}'_\mu(x) \equiv 0, n'_w = 0$
  $\mathcal{A}_\mu(x) = g(\mathcal{A}'_\mu - \mathrm{i}\partial_\mu)g^\dagger = -\mathrm{i}g\partial_\mu g^\dagger = \mathrm{i}\partial_\mu g g^\dagger$
- R$^3 \to$ S^3: $n_w = -\frac{1}{4\pi^2}\int_{\mathrm{S}^3} \mathrm{d}^3x \mathrm{Sp}(\partial_i g g^\dagger \partial_j g g^\dagger \partial_k g g^\dagger)$
- $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - \varepsilon_{ijk} A^j_\mu A^k_\nu = 0$
- vacuum to vacuum transition: $Q = n_w - n'_w$
- Topological charge: $Q = \frac{1}{32\pi^2}\int \mathrm{d}^4x\, F^i_{\mu\nu}\, {}^*F^i_{\mu\nu}$
- Lattice: $F_{\mu\nu} = \frac{1}{2\mathrm{i}}(U_{\mu\nu} - U^\dagger_{\mu\nu})$
- admissibility condition: $\mathrm{tr}(\mathbb{1} - U_{\mu\nu}) < 0.03$
  uniquness of topological charge on the lattice: ➜ Lüscher (1998)

Atiyah-Singer index theorem

- zero-modes of fermionic matrix: $D[A]\psi(x) = 0$
- ψ has definite chirality:

$$\psi_{\substack{R\\L}} = \frac{1}{2}(1 \pm \gamma_5)\psi, \quad \Rightarrow \quad \gamma_5 \psi_{\substack{R\\L}} = \pm \psi_{\substack{R\\L}}$$

- Index theorem (wilson, overlap fermions):

$$n_-, n_+ \text{: number of left-/right-handed zeromodes}$$

$$\mathrm{ind} D[A] = n_- - n_+ = Q[A]$$

- (Asqtad) staggered fermions:

$$\mathrm{ind}\ D[A] = 2Q[A] \text{ (SU(2), double degeneracy)}$$

- Adjoint overlap fermions:

$$\mathrm{ind}\ D[A] = 2NQ[A] = 4Q[A] \text{ (real representation)}$$

➜ *Neuberger, Fukaya (1999)*

Questions

- Do zero-modes locate topological charge distributions?

  analyse $\rho(x) = \psi^\dagger(x)\psi(x), \quad \rho_5(x) = \psi^\dagger(x)\gamma_5\psi(x)$

- Are there configurations with fractional topological charge?

  analyse $Q(x) = \frac{1}{32\pi^2} F^i_{\mu\nu}(x) \, {}^*F^i_{\mu\nu}(x)$

- Can we identify fractional topological charge with adjoint fermions? (Pisarsky, Greensite)

Topological excitations

- instantons: vacuum-vacuum-transitions
- merons, calorons, dyons: parts of instantons
- vortices: singular gauge transformations,
  world sheets of magnetic flux
- magnetic monopoles: projections of magnetic flux
  or singularities of the gauge field

➜ *Ilgenfritz, Martemyanov, Müller-Preussker, Shcheredin, Veselov, Phys.Rev.2002*

➜ *Ilgenfritz, Müller-Preussker, Peschka, Phys.Rev.2005*

➜ *Ilgenfritz, Martemyanov, Müller-Preussker, Veselov, Phys.Atom.Nucl.2005*

➜ *Ilgenfritz, Martemyanov, M. Müller-Preussker, Veselov, Phys.Rev.2006*

➜ *Bornyakov, Ilgenfritz, Martemyanov, Morozov, M. Müller-Preussker, Veselov, Phys.Rev.2007*

➜ *Bornyakov, Ilgenfritz, Martemyanov, Morozov, Müller-Preussker, Veselov, Phys.Rev.2008*

➜ *Bornyakov, Ilgenfritz, Martemyanov, M. Müller-Preussker, Phys.Rev.2009*

Center Vortices

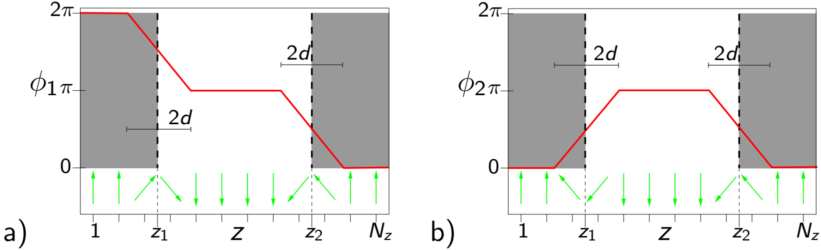
➜ *'t Hooft 1979, Nielsen, Ambjorn, Olesen, Cornwall, 1979*
Mack, 1980; Feynman, 1981

- QCD vacuum is a condensate of closed magnetic flux-lines, they have topology of tubes (3D) or surfaces (4D),
- magnetic flux corresponds to the center of the group,
- Vortex model may explain ...
  - **Confinement** $\rightarrow$ piercing of Wilson loop $\equiv$ crossing of static electric flux tube and moving closed magnetic flux
  - **Topological charge**: vortices carry topological charge at intersection points and writhing points
  - **Spontaneous chiral symmetry breaking**: also center-projected configurations show SCSB

➜ *Höllwieser, M.F., Greensite, Heller, Olejnik 2008*

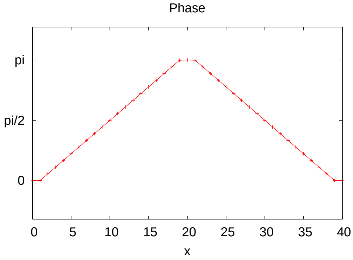
Definition of Plain Vortices

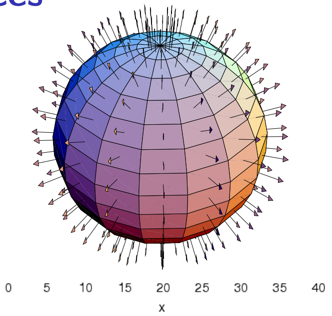
xy-vortices: along z-direction we vary t-links,
in a U(1) subgroup of SU(2), e.g. σ_3,
$U_\mu = \exp\{\mathrm{i}\phi(z)\sigma_3\}$.



a) ϕ_1 of a parallel and b) ϕ_2 of an anti-parallel vortex pair

Thick Spherical SU(2)-vortices





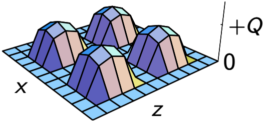
$$U_\mu(x^\nu) = \begin{cases} \exp\{\mathrm{i}\alpha(r)\frac{\vec{r}}{r}\vec{\sigma}\}, & t = 1, \mu = 4 \\ 1 & \text{else} \end{cases}$$

$$\alpha_\pm(r) = \begin{cases} \begin{cases} \pi \\ 0 \end{cases} & r < R - \frac{\Delta}{2} \\ \frac{\pi}{2}\left(1 \mp \frac{r-R}{\frac{\Delta}{2}}\right) & R - \frac{\Delta}{2} < r < R + \frac{\Delta}{2} \\ \begin{cases} 0 \\ \pi \end{cases} & R + \frac{\Delta}{2} < r \end{cases}$$

Topological charge of intersecting vortices

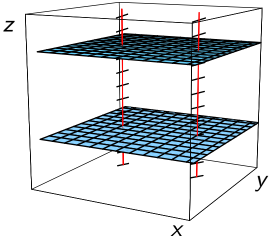
xy-vortices: along *z*-direction we vary *t*-links,
zt-vortices: along *x*-direction the *y*-links vary
⇒ x-z-intersection plane

Parallel Vortices

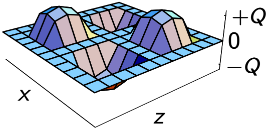


Q=2

Geometry



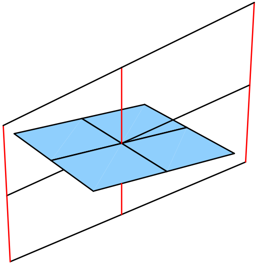
Anti-parallel Vortices



Q=0

Contributions to topological charge Q

vortex intersection

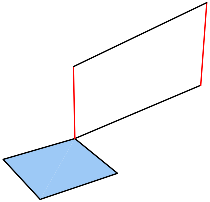


$4 \cdot 4 = 16$ contributions

$Q = \pm\frac{1}{2}$

1 contribution

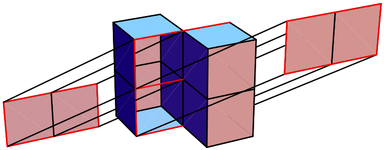
$Q = \pm\frac{1}{32}$

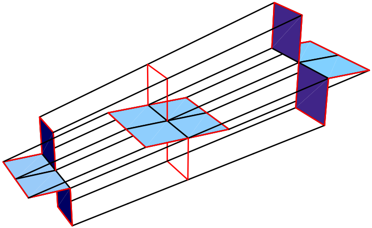


vortex surfaces need orientation

→ *Engelhardt, Reinhardt (2000)*

Intersections and Writhing points





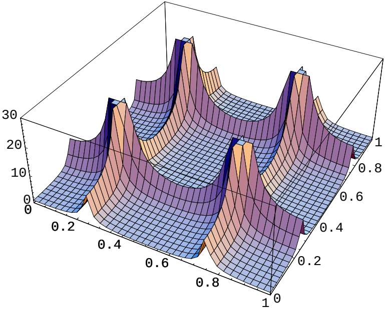
contributions to topological charge

- intersections $Q = \pm\frac{1}{2}$
- writhing points $Q = \pm\frac{1}{4}$

Analytical results

➜ *Reinhardt, Schroeder, Tok and Zhukovsky (2002)*

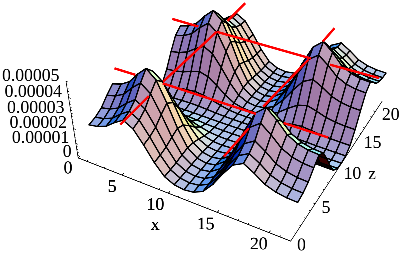
Analytical calculations by the Tübingen group.
Zero-modes peak at intersections of vortices.



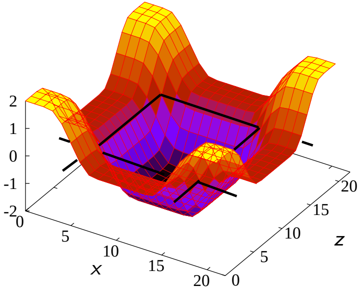
Probability density of zero-mode in the background of four intersecting vortices.

Zero-modes for Q=2

y=11, t=11, chi=−1, n=1−2, max=0.0000498734

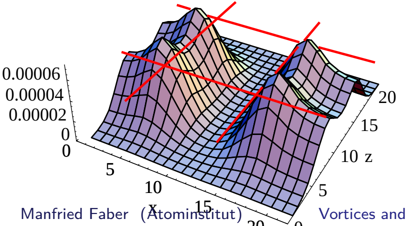


periodic boundary conditions



$\text{Sp } L_y + \text{Sp } L_t$

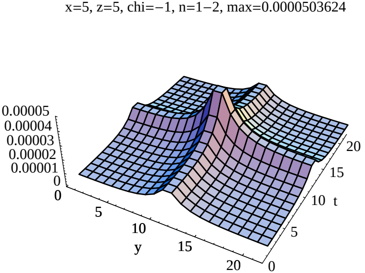
y=11, t=11, chi=−1, n=1−2, max=0.0000733948



antiperiodic boundary cond.

intersections at x and $z = 5.5$ and 16.5

Perpendicular to intersections



zero-modes at maximum, perpendicular to intersections

Conclusions:

- Vortices contribute with fractions of topological charge
- Zero-modes are related to vortex intersections
- Zero mode positions depend on values of Polyakov loops

Localisation

Scalar density: $\rho_i(x) = \sum_{c,d} |\psi_i(x)_{cd}|^2$
c and d, color und Dirac indices

$\sum_{x=1}^{N} \rho_i(x) = 1$

Chiral densities $\rho_+(x)$ and $\rho_-(x)$: $\rho_{i\pm}(x) = \sum_{c,d} \psi_i(x)^*_{cd} \frac{1-\gamma_5^{c,d'}}{2} \psi_i(x)_{cd'}$

Inverse participation ratio (IPR)

➜ *Ilgenfritz:2007, Polikarpov:2005, Aubin:2004, Bernard:2005*

$I = \frac{1}{\text{fraction of contributing sites}}$

$I = N \sum_{x=1}^{N} \rho_i^2(x), \qquad \frac{1}{N} \leq I \leq N$

locate fractional topological charge contributions with I

Scalar densities for adjoint overlap fermions

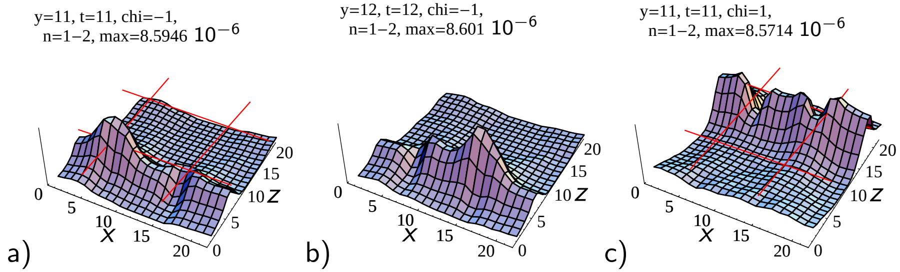
Adjoint fermions: $\mathrm{ind} D[A] = 2NQ[A] = 4Q[A]$

hope to separate various contributions

Q=0 configuration with periodic boundary conditions:

no zero-modes for fundamental fermions

2+2 zero-modes for adjoint fermions

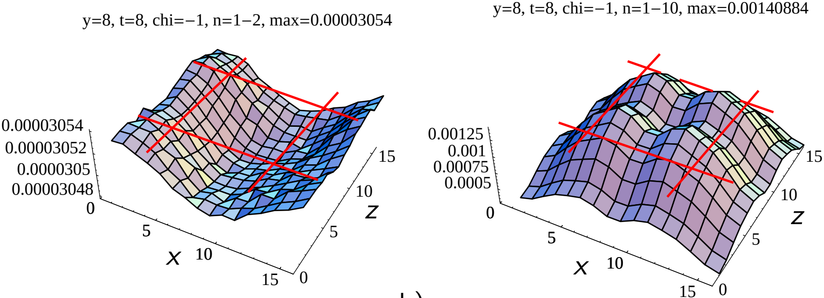


negative positive

chirality

Adjoint overlap fermions

Q=2 configuration with periodic boundary conditions:
Scalar density plots
2 positive and 10 negative adjoint zero-modes



a) 2 non-topological zero-modes

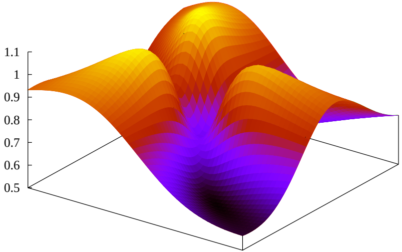
b) 10 negative chirality

Search for IPR maxima

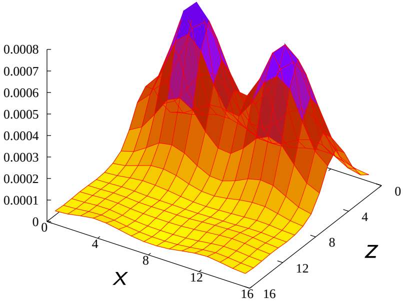
Q=2 configuration with antiperiodic boundary conditions:

parameter space of linear combinations of the eight topological zero-modes,

systematic and random (20 000) search of maxima



a)

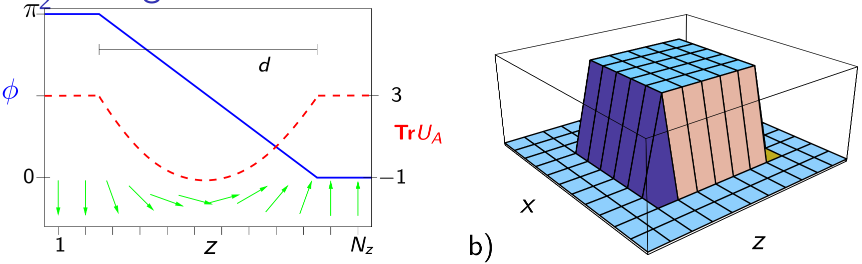


b)

a) Plane through the highest three IPR maxima:
The peaks are very broad and easy to identify

b) Density with maximal IPR still peaks at two vortex intersections

$Q = \frac{1}{2}$-configuration



a) b)

Thin and thick vortices:

xy-vortices: along *z*-direction we vary *t*-links,

zt-vortices: along *x*-direction the *y*-links vary

Adjoint fermions are blind against thin vortices:

$$(\mathbf{Tr}\,U)(\mathbf{Tr}\,U)^\dagger = (\mathbf{Tr}\,U)^2 = 1 + \mathbf{Tr}\,U_A$$

a) Link profile

b) Topological charge density in the intersection plane,

"thick-thick" vortex intersection

$Q = \frac{1}{2}$-configuration

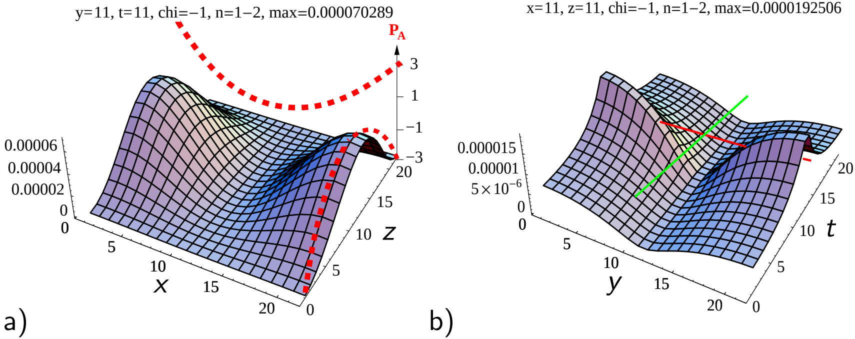
Adjoint fermions: $\mathrm{ind}D[A] = 2NQ[A] = 4Q[A]$

xy-vortices: along z-direction we vary t-links,

zt-vortices: along x-direction the y-links vary

antiperiodic boundary conditions

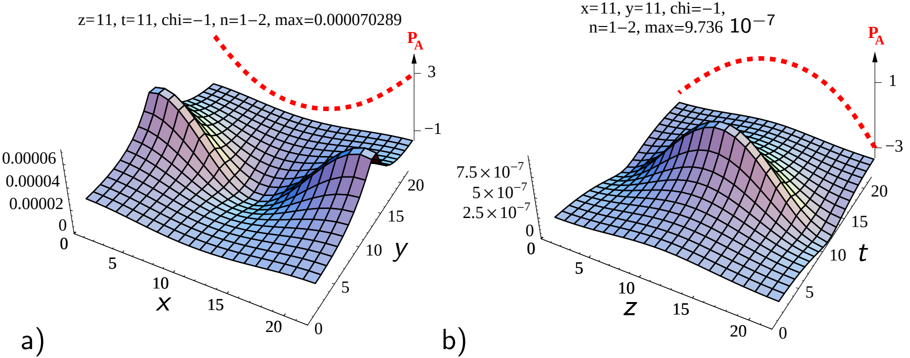
two adjoint overlap zero-modes



Zero-modes avoid negative adjoint Polyakov (Wilson) lines

$Q = \frac{1}{2}$-configuration

xy-vortices: along *z*-direction we vary *t*-links,
zt-vortices: along *x*-direction the *y*-links vary
antiperiodic boundary conditions



a) in *x*-direction profile of adjoint *y*-Wilson lines,
b) in *z*-direction profile of adjoint *t*-Wilson lines

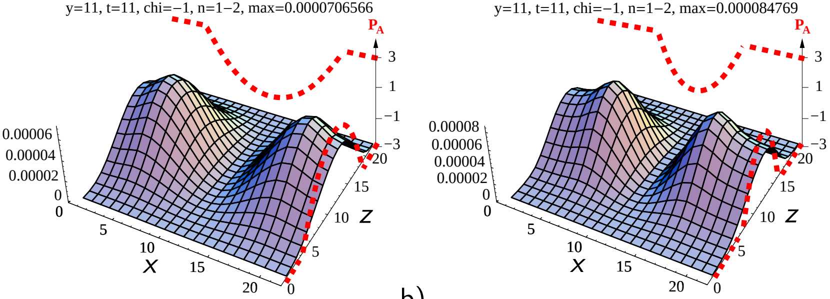
$Q = \frac{1}{2}$-configuration

various vortex profiles

xy-vortices: along *z*-direction we vary *t*-links,

zt-vortices: along *x*-direction the *y*-links vary

antiperiodic boundary conditions



a) $d_x = d_z = 16$

b) $d_x = d_z = 12$

Zero mode densities follow values of adjoint Wilson lines

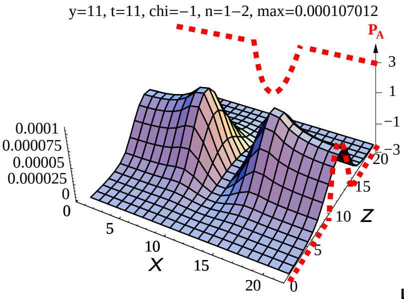
$Q = \frac{1}{2}$-configuration

various vortex profiles

xy-vortices: along z-direction we vary t-links,

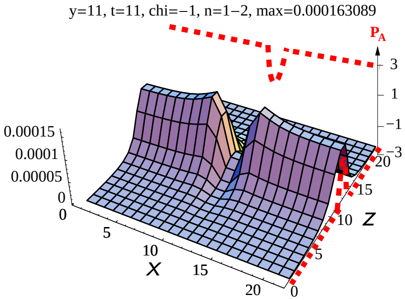
zt-vortices: along x-direction the y-links vary

antiperiodic boundary conditions



a)

$d_x = d_z = 8$



b)

$d_x = d_z = 4$

Zero mode densities follow values of adjoint Wilson lines

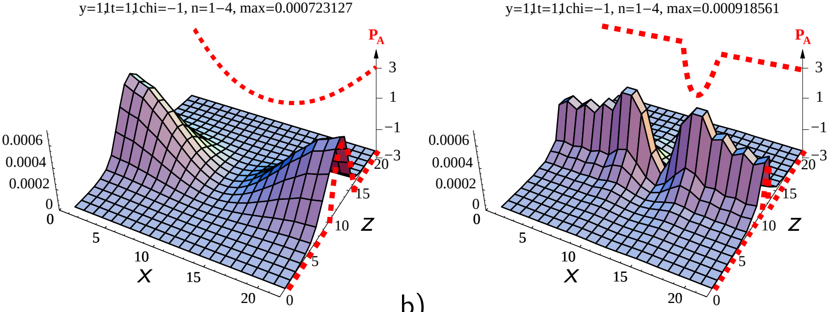
$Q = \frac{1}{2}$-configuration

various vortex profiles

xy-vortices: along *z*-direction we vary *t*-links,

zt-vortices: along *x*-direction the *y*-links vary

antiperiodic boundary conditions

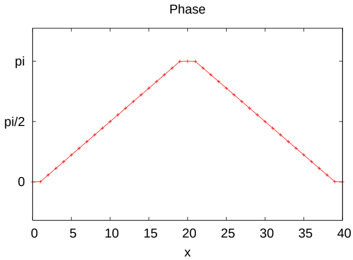


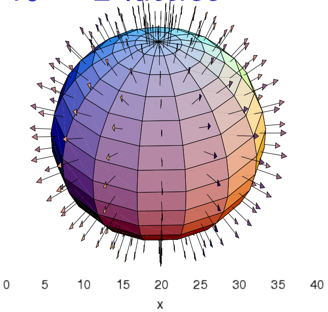
a) $d_x = 20,\ d_z = 8$

b) $d_x = 8,\ d_z = 2$

Zero mode densities follow values of adjoint Wilson lines and cover the whole lattice extent

Thick Spherical SU(2)-vortices, $40^3 * 2$-lattice





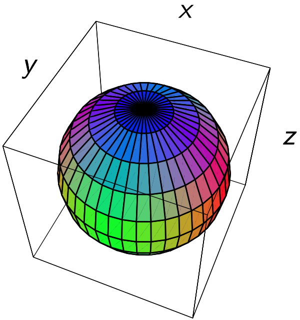
$$U_\mu(x^\nu) = \begin{cases} \exp\{\mathrm{i}\alpha(r)\frac{\vec{r}}{r}\vec{\sigma}\}, & t = 1, \mu = 4 \\ 1 & \text{else} \end{cases}$$

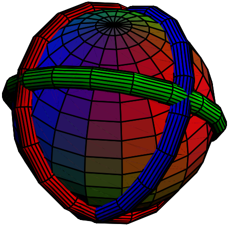
$$L(\vec{r}) = \exp\{\mathrm{i}\alpha(r)\frac{\vec{r}}{r}\vec{\sigma}\} \quad \text{time-like Wilson lines}$$

admissibility condition is fullfilled: $1 - \frac{1}{2}\mathrm{tr}\, U_\square \leq 0.015$

Non-orientable spherical vortices

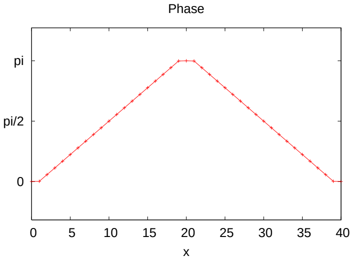
Non-orientable vortex surface . . . leads to monopole lines after **abelian projection**

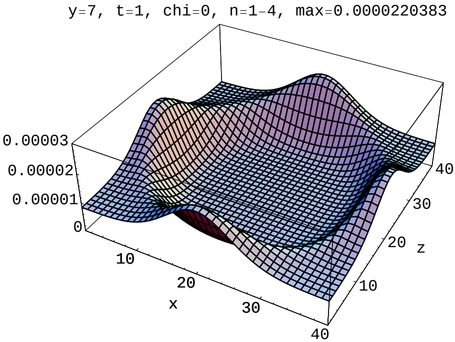




Thick spherical SU(2)-vortex, $40^3 * 2$-lattice

"Positive" Spherical Vortex



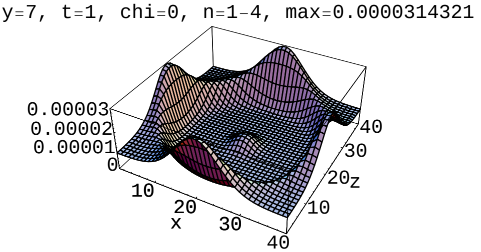


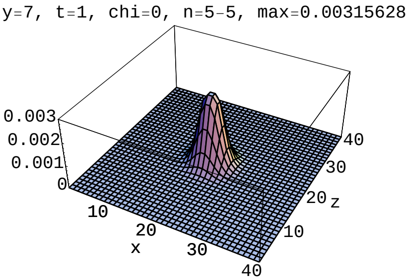
- Index of overlap operator:

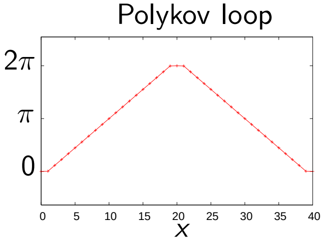
$$n_+ = 3, n_- = 4 \quad \Longrightarrow \quad \mathrm{ind} D[A] = n_- - n_+ = 1$$

Two thick spherical SU(2)-vortices, $40^3 * 2$-lattice

One vortex at $t = 1$ and another at $t = 2$







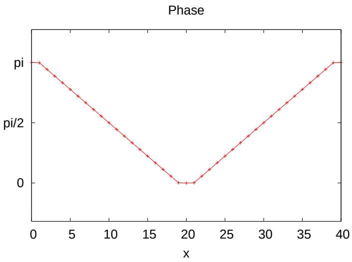
Index of overlap operator:

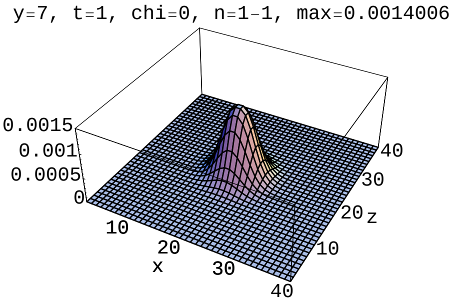
$$n_+ = 3, n_- = 5$$

$$\mathrm{ind} D[A] = n_- - n_+ = 2$$

Thick spherical SU(2)-vortex, $40^3 * 2$-lattice

"Negative" Spherical Vortex



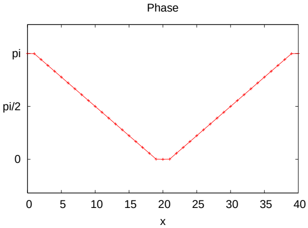


- Index of overlap operator:

$$n_+ = 1, n_- = 0 \quad \Longrightarrow \quad \mathrm{ind}\, D[A] = n_- - n_+ = -1$$

Two thick spherical SU(2)-vortices, $40^3 * 2$-lattice

One vortex at $t = 1$ and another at $t = 2$



Index of overlap operator:

$$n_+ = 0, n_- = 0 \quad \Rightarrow \quad \mathrm{ind}D[A] = n_- - n_+ = 0$$

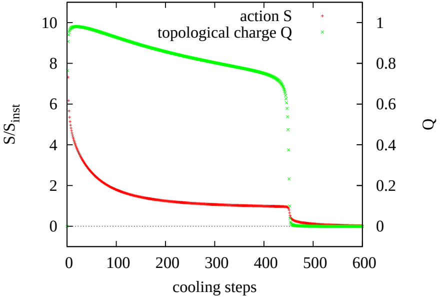
For all examples of spherical vortices:

- only time-like links non-trivial $\quad \Rightarrow \quad$ no magnetic field
  $$\Rightarrow B = 0 \quad \Rightarrow \quad Q[A] = \frac{1}{32\pi^2} \int \mathrm{d}^4x \, F^i_{\mu\nu} \, {}^*F^i_{\mu\nu} = 0!$$
- admissibility condition is fullfilled: $\quad 1 - \frac{1}{2}\mathrm{tr} U_\Box \leq 0.015$

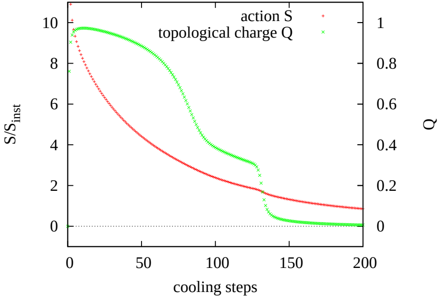
NO save determination of $\mathrm{ind}D[A]$ with $Q[A]$!

Cooling histories of spherical vortex

- Index of the overlap operator: ind $D = 1$
- Topological charge before cooling: $\vec{B} = 0 \;\Rightarrow\; Q[A] = 0$
- Topological charge with cooling: $Q[A] \approx 1$



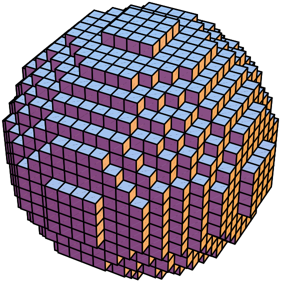
$40^3 \times 4$-lattice



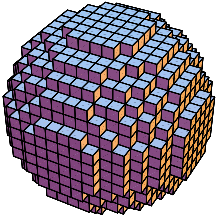
$40^3 \times 2$-lattice

What is the object appearing after 120 cooling steps?

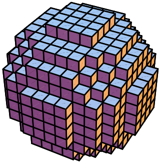
Cooling a spherical vortex on a $40^3 \times 2$ lattice



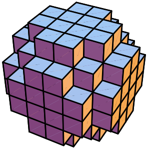
0 cooling steps



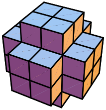
20 cooling steps



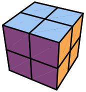
40 cooling steps



60 cooling steps

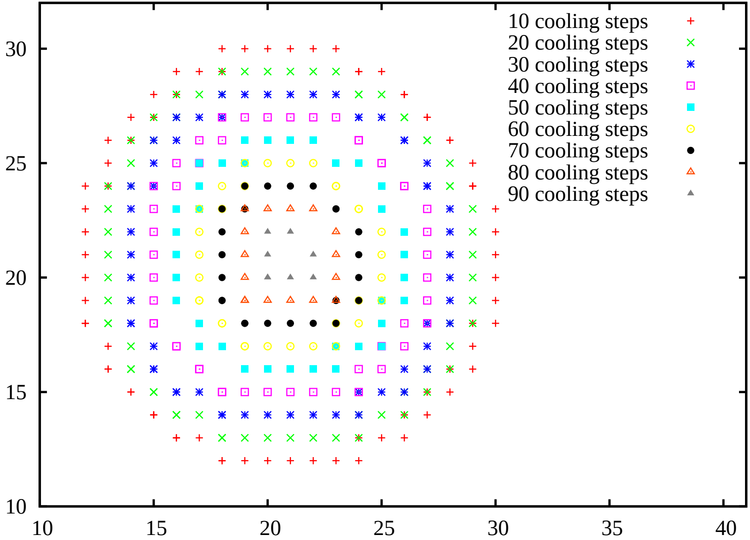


70 cooling steps



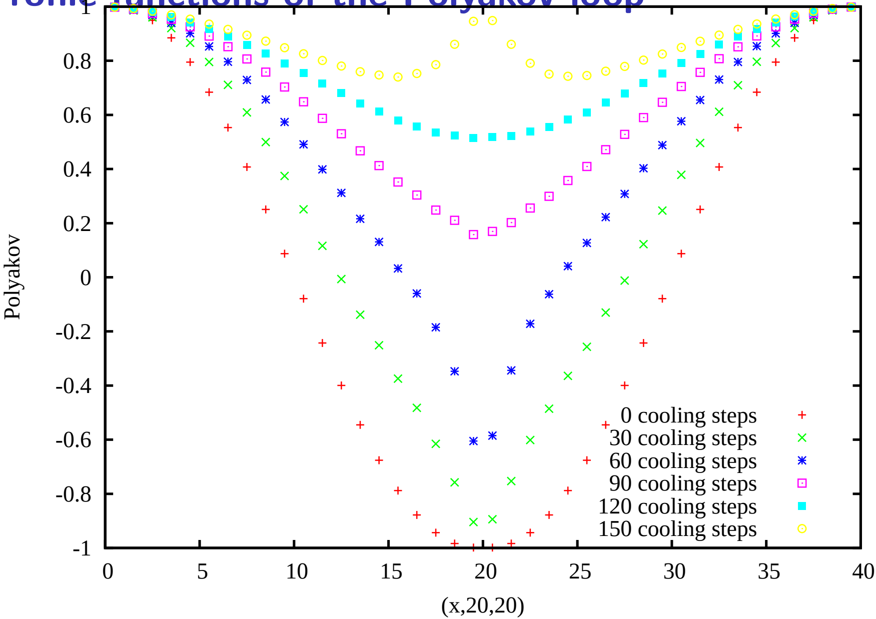
78 cooling steps

Monopole loops during cooling

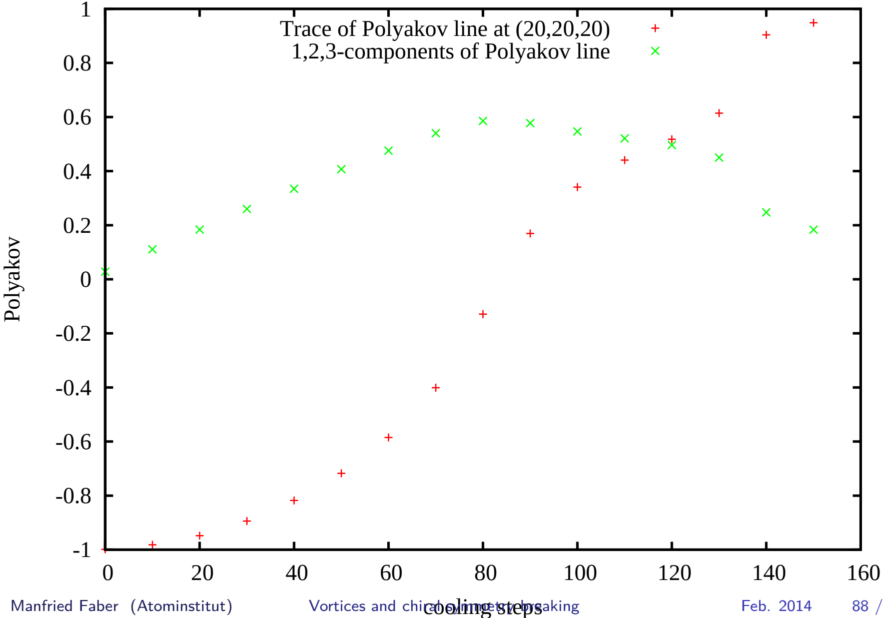


After 100 cooling steps all Abelian monopole loops are gone

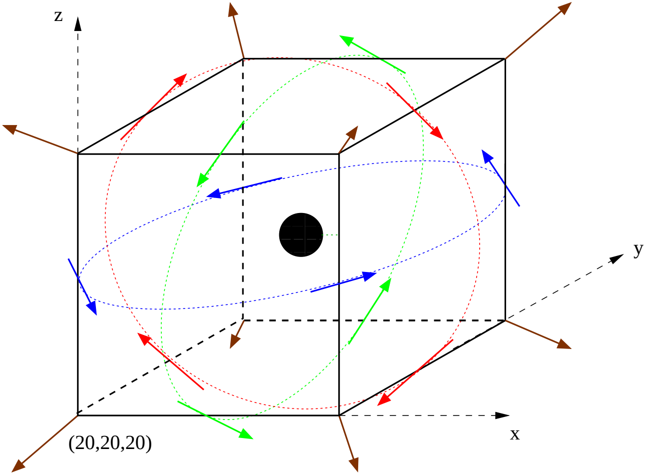
Profile functions of the Polyakov loop



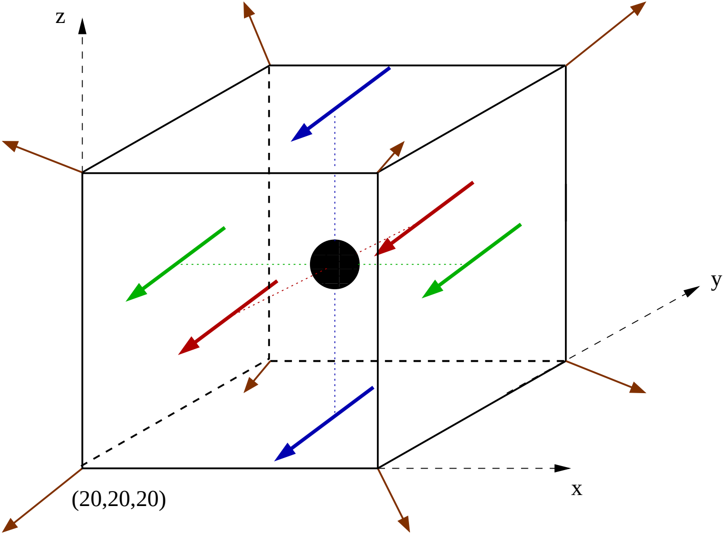
Polyakov loop components during cooling



Central cube after 120 cooling steps



Color flux through central plaquettes



Central cube is a Dirac monopole in non-Abelian representation with a gauge field fading away at large distances

Overlap, Asqtad & Adjoint Results

negative spherical vortex

	fundamental:				adjoint:			
lattice:	ovl:	apbc:	stag:	apbc:	ovl:	apbc:	stag:	apbc:
8^4:	3+ 4-	0+ 1-	6+ 8-	0+ 2-	4+ 6-	0+ 2-	8+ 12-	0+ 4-
12^4:	3+ 4-	0+ 1-	6+ 8-	0+ 2-	4+ 6-	0+ 2-	8+ 12-	0+ 4-
16^4:	3+ 4-	0+ 1-	6+ 8-	0+ 2-	4+ 8-	0+ 4-	8+ 16-	0+ 8-
20^4:	3+ 4-	0+ 1-	6+ 8-	0+ 2-	4+ 8-	0+ 4-	8+ 16-	0+ 8-
$40^3 \times 2$:	3+ 4-	0+ 1-	6+ 8-	0+ 2-	4+ 8-	0+ 4-	8+ 16-	0+ 8-
cooled:	3+ 4-	0+ 1-	6+ 8-	0+ 2-	4+ 6-	0+ 2-	8+ 12-	0+ 4-
$40^3 \times 4$:	3+ 4-	0+ 1-	6+ 8-	0+ 2-	4+ 8-	0+ 4-	8+ 16-	0+ 8-

positive spherical vortex

	fundamental:				adjoint:			
lattice:	ovl:	apbc:	stag:	apbc:	ovl:	apbc:	stag:	apbc:
8^4:	1+ 0-	4+ 3-	2+ 0-	8+ 6-	6+ 4-	2+ 0-	12+ 8-	4+ 0-
12^4:	1+ 0-	4+ 3-	2+ 0-	8+ 6-	6+ 4-	2+ 0-	12+ 8-	4+ 0-
16^4:	1+ 0-	4+ 3-	2+ 0-	8+ 6-	8+ 4-	4+ 0-	16+ 8-	8+ 0-
20^4:	1+ 0-	4+ 3-	2+ 0-	8+ 6-	8+ 4-	4+ 0-	16+ 8-	8+ 0-
$40^3 \times 2$:	1+ 0-	4+ 3-	2+ 0-	8+ 6-	8+ 4-	4+ 0-	16+ 8-	8+ 0-
cooled:	1+ 0-	4+ 3-	2+ 0-	8+ 6-	6+ 4-	2+ 0-	12+ 8-	4+ 0-
$40^3 \times 4$:	1+ 0-	4+ 3-	2+ 0-	8+ 6-	8+ 4-	4+ 0-	16+ 8-	8+ 0-

Answers

- Vortices contribute with fractions to topological charge
- Number of zero-modes is related to vortex intersections
- Zero mode positions depend on values of Wilson lines
- Adjoint zero-modes can't be attributed to $Q = \frac{1}{2}$ configurations
- Even for "admissible" fields: $\mathrm{ind}\, D[A] \neq Q[A]$
- Dirac monopoles differ from Abelian projected monopoles
- Dirac monopoles fading away at large distances have $Q = \frac{1}{2}$

Chiral Symmetry Breaking

- parity acting on a Dirac fermion is called **chiral symmetry**
- two chiralities of quarkfield ψ

$$\psi_{\substack{L\\R}} = \frac{1}{2}(1 \pm \gamma_5)\psi, \quad \bar{\psi}_{\substack{L\\R}} = \psi^\dagger_{\substack{L\\R}}\gamma_0 = \bar{\psi}\frac{1}{2}(1 \mp \gamma_5)$$

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3, \quad \gamma_5^2 = 1, \quad \gamma_5^\dagger = \gamma_5, \quad \{\gamma_\mu, \gamma_5\} = 0$$

- chiral projection $P_{\substack{L\\R}} = \frac{1}{2}(1 \mp \gamma_5)$ on QCD-Lagrangian:
  - kinetic term $\bar{\psi}\gamma^\mu D_\mu \psi$ gives

$$\bar{\psi}\gamma_\mu\psi = \bar{\psi}_R\gamma_\mu\psi_R + \bar{\psi}_L\gamma_\mu\psi_L$$

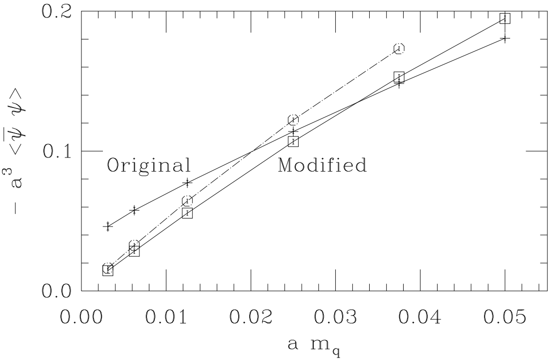
  - whereas mass term $m\bar{\psi}\psi$ gives interaction

$$\bar{\psi}\psi = \bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R$$

    and breaks chiral symmetry explicitly
- spontanous symmetry breaking gives rise to Goldstone-boson: pion

Vortex removal restores chiral symmetry

➜ *De Forcrand and D'Elia (1999)*



Chiral condensate in quenched lattice configurations before ("Original") and after ("Modified") vortex removal.

Banks-Casher relation

Chiral symmetry breaking $\Longrightarrow$

$\Longrightarrow$ Low-lying eigenmodes of Dirac operator

$$\bar{\psi}\psi = -\lim_{m\to 0}\lim_{V\to\infty}\left\langle \frac{1}{V}\sum_n \frac{1}{\mathrm{i}\lambda_n + m}\right\rangle$$

Non-zero eigenvalues appear in pairs $\pm\mathrm{i}\lambda_n$

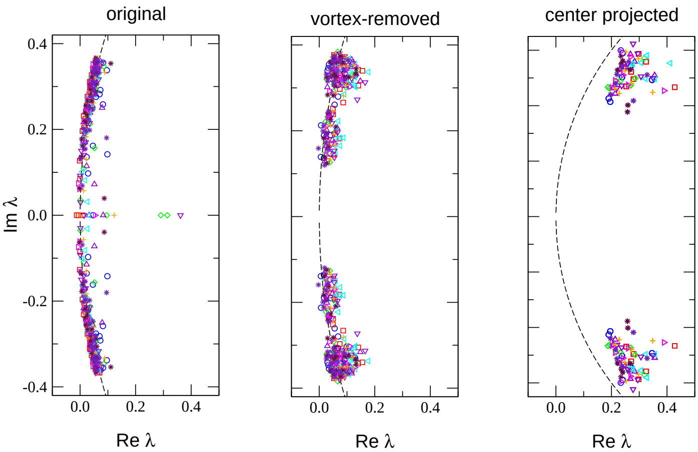
$$\lim_{m\to 0}\frac{2m}{\lambda_n^2 + m^2} \quad \longrightarrow \quad \pi\delta(0)$$

Chiral condensate $\Longrightarrow$ Density of Near-Zero-modes.

$$\bar{\psi}\psi = \frac{\pi\rho(0)}{V}$$

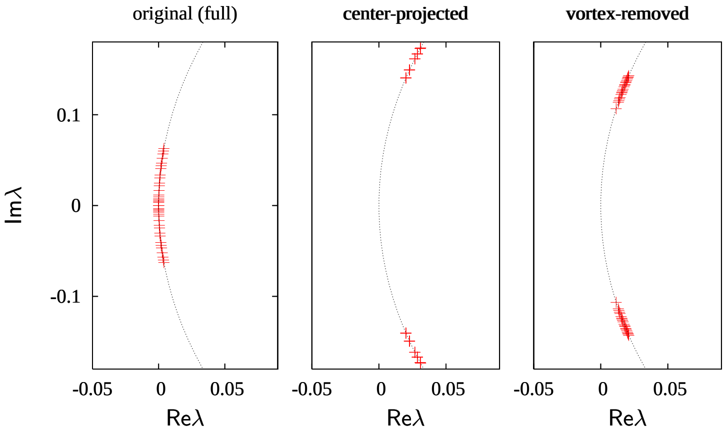
➜ *Banks, Casher, 1980*

Chiral Improved Fermions



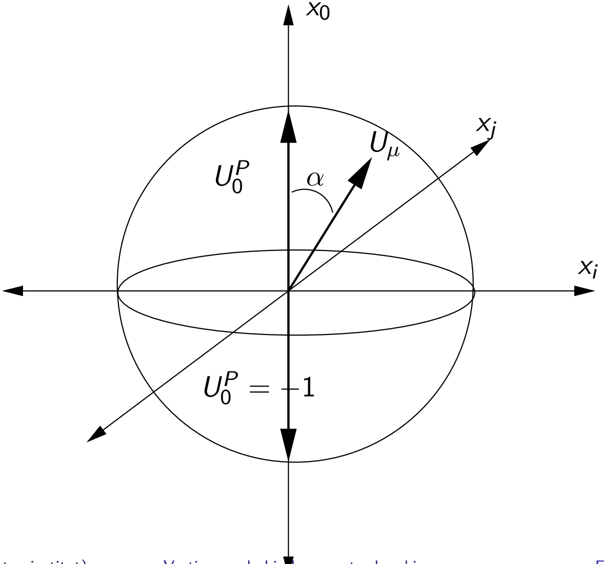
➜ *From J. Gattnar et al., Nucl. Phys. B716 (2005)105.*

Eigenvalues of the Overlap Dirac operator on the Ginsparg-Wilson circle

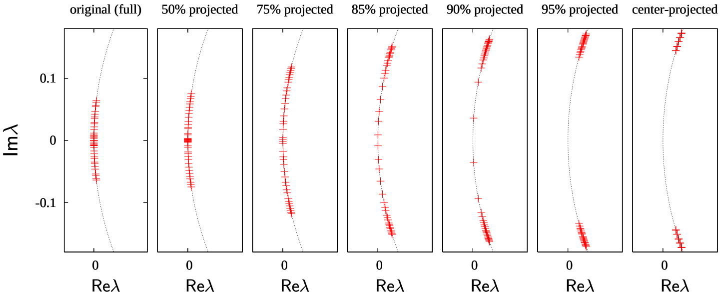


Interpolated gauge fields

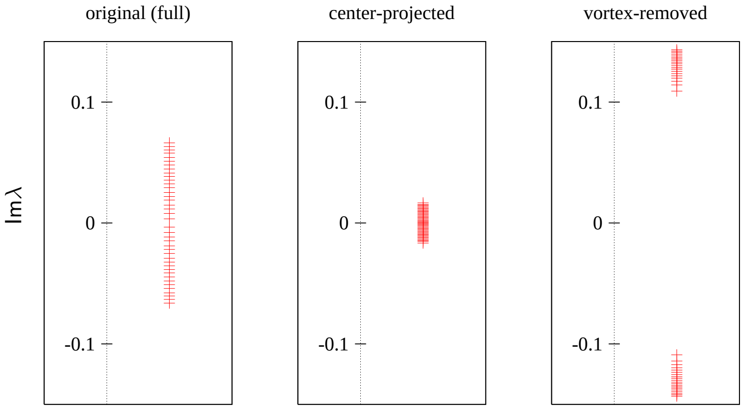
Interpolation between full U_μ and center-projected $U_0^P = \pm 1$.



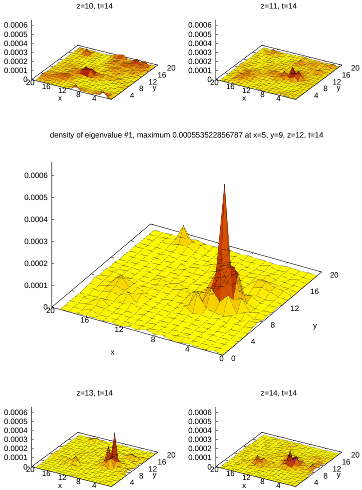
Overlap eigenvalues on interpolated gauge fields



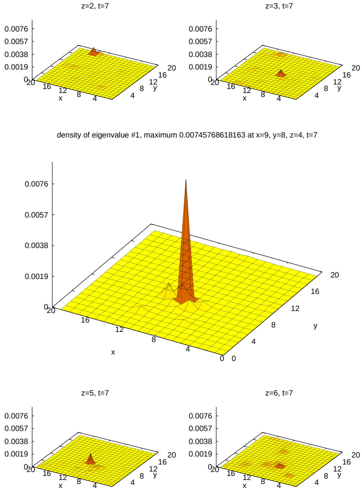
Asqtad Staggered Fermions



Asqtad Eigenmode Density Peaks

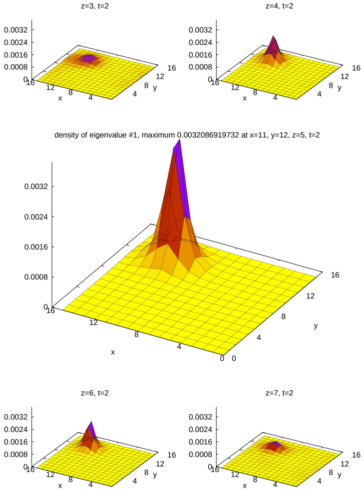


full configuration

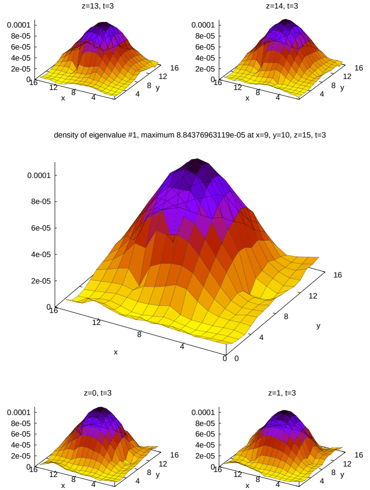


center-projected

Overlap Eigenmode Density Peaks



full configuration



center-projected

Correlation between vortices and Dirac modes

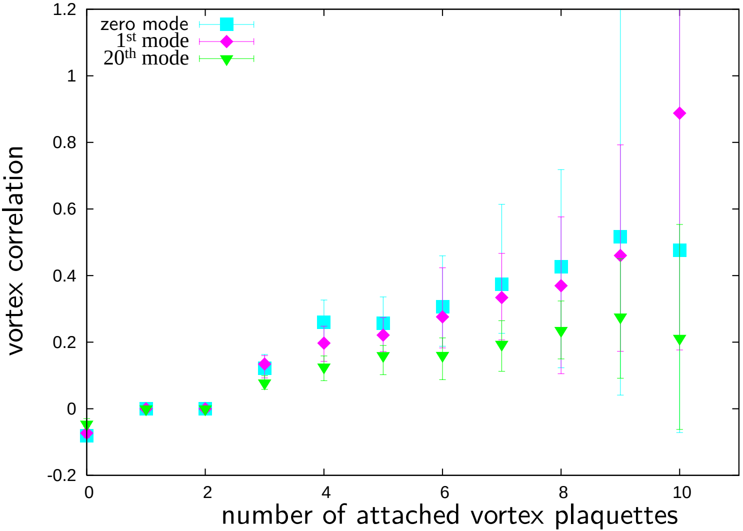
Correlator

$$C_\lambda = \frac{\sum_{P_i} \sum_{x \in H} (V\rho_\lambda(x) - \langle V\rho_\lambda(x) \rangle)}{\sum_{P_i} \sum_{x \in H} 1}$$

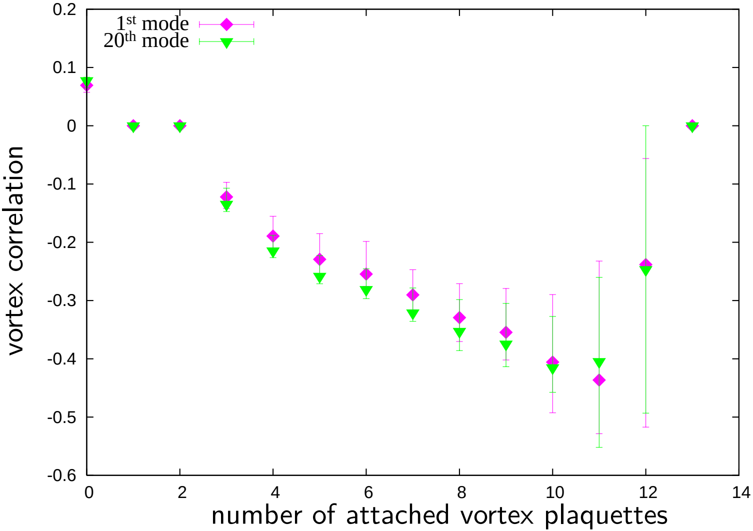
➜ *Kovalenko, Morozov, Polikarpov and Zakharov 2005*

- (vortex) points P_i on the dual lattice
- scalar eigenmode density $\rho_\lambda(x)$, averaged over the vertices x of the 4d hypercube H, dual to P_i
- strongly depends on the number of the vortex plaquettes, attached to a point P_i

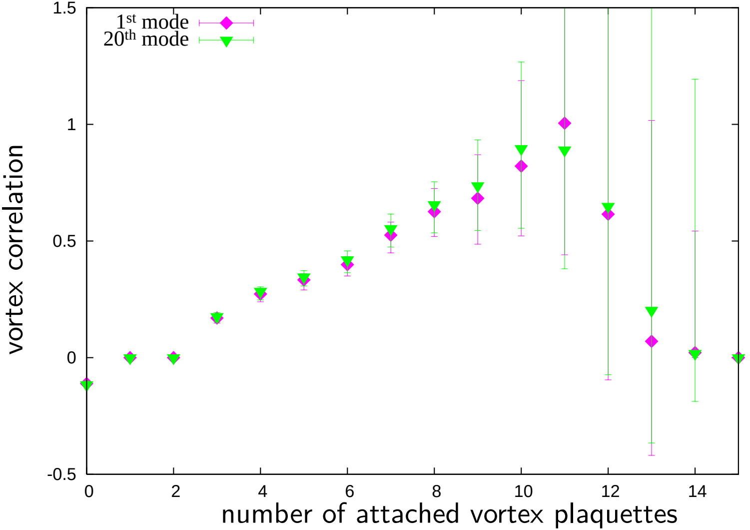
Vortex correlation for full overlap modes



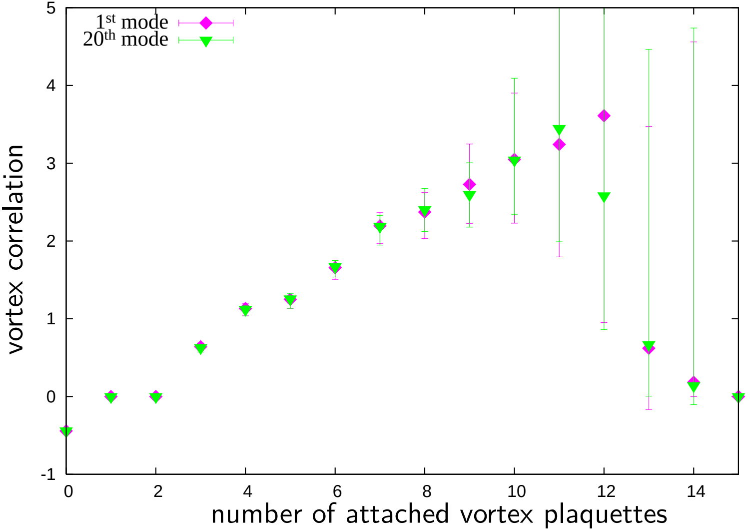
Vortex correlation for projected overlap modes



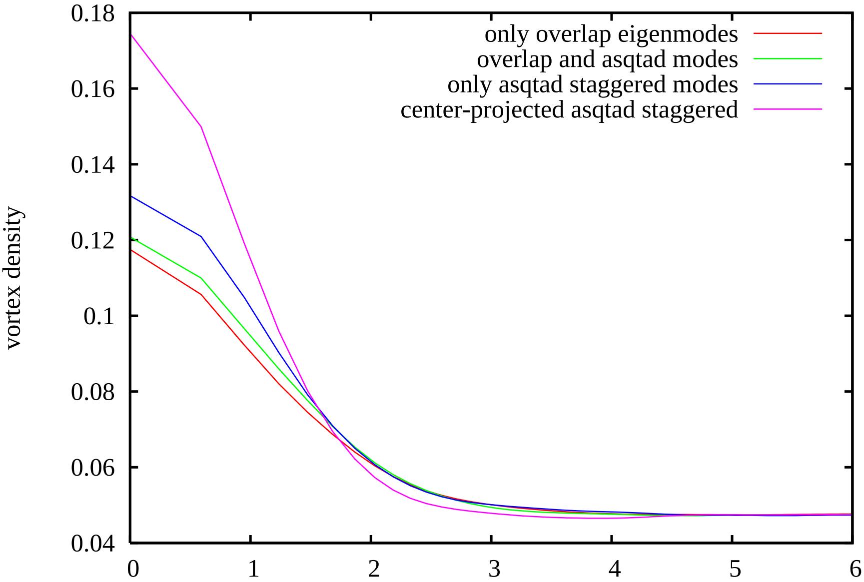
Vortex correlation for full asqtad staggered modes



Vortex correlation for projected staggered modes

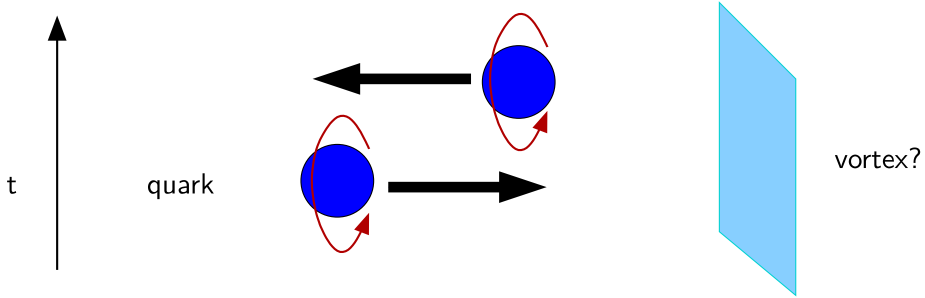


Vortex density at eigenmode peaks



Mechanism of chiral symmetry breaking?

QCD Dynamics breaks Chiral Symmetry



Strong indications that Center Vortices explain Chiral Symmetry Breaking.

Conclusions

- Vortices contribute to with fractions to topological charge
- Number of zero-modes is related to vortex intersections
- Zero mode positions depend on values of Wilson lines
- Adjoint zero-modes can't be attributed to $Q = \frac{1}{2}$ configurations
- Even for "admissible" fields $\mathrm{ind} D[A] \neq Q[A]$
- Dirac monopoles differ from Abelian projected monopoles
- Dirac monopoles fading away at large distances have
  $Q = \frac{1}{2}$
- Strong indications that Center Vortices explain Chiral Symmetry Breaking

Phenomena in QCD

Perturbative Phenomena

- Asymptotic freedom
- Antiscreening

Nonperturbative Phenomena

- Confinement
- Chiral symmetry breaking
- Topological Charge and Axial Anomaly
- Phase Structure of QCD

Is there a common picture? Center Vortices?

Center Vortices

30 years of vortices

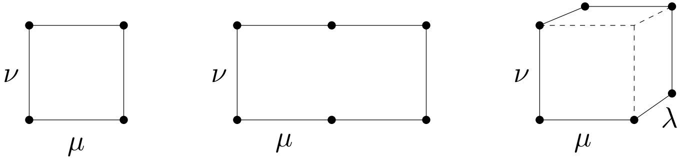
➜ *'t Hooft 1979, Nielsen, Ambjorn, Olesen, Cornwall, 1979*
Mack, 1980; Feynman, 1981

- QCD vacuum is a condensate of closed magnetic flux-lines, they have topology of tubes (3D) or surfaces (4D),
- magnetic flux is quantised,
  it corresponds to the center of the group,
  quarks feel Aharanov-Bohm effect
- Vortex model may explain ...
  - **Confinement** $\rightarrow$ piercing of Wilson loop $\equiv$ crossing of static electric flux tube and moving closed magnetic flux
  - **QCD phase transition**: by type of 4D surfaces
  - **Topological charge**: vortices carry topological charge at intersection points and writhing points
  - **Spontaneous chiral symmetry breaking ?**

Lattice QCD action

tadpole improved version of the one-loop continuum limit improved SU(2) action of Lüscher and Weisz

➜ *Lüscher and Weisz 1985, Poulis 1997*



standard plaquette, 2×1 rectangle and $1 \times 1 \times 1$ parallelogram

$$S = \beta \sum_{pl} S_{pl} - \frac{\beta}{20 u_0^2}[1 + 0.2227\alpha_s] \sum_{rt} S_{rt} - 0.02224 \frac{\beta}{u_0^2} \alpha_s \sum_{pg} S_{pg} .$$

$$u_0 = < \frac{1}{N} \mathrm{Re\ Tr} U_{pl} >^{1/4}, \qquad \alpha_s = -4 \frac{\ln u_0}{1.72597}$$

Center Vortices

How to Identify Center Vortices?

➜ *Del Debbio, Faber, Greensite, Olejnik (1996–1998)*

- Fix thermalized SU(2) lattice configurations to maximal center (adj. Landau) gauge by maximizing the expression:

$$\sum_{x,\mu} \left| \mathrm{Tr}[U_\mu(x)] \right|^2 \qquad \text{or} \qquad \sum_{x,\mu} \mathrm{Tr}[U_\mu^A(x)]$$

+overrelaxation

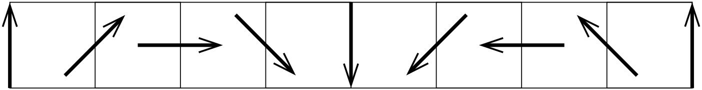
- Make center projection by replacing:

$$U_\mu(x) \to Z_\mu(x) \equiv \text{sign}\ \mathrm{Tr}[U_\mu(x)]$$

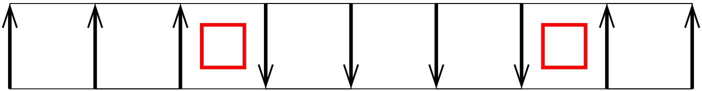
P-Vortex

A plaquette is pierced by a P-vortex, if the product of its center projected links gives -1.

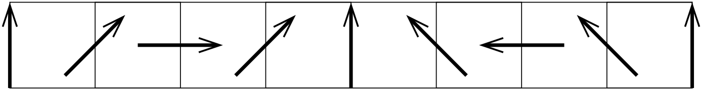
center vortex in one dimension



center-projected (P-vortex plaquettes)

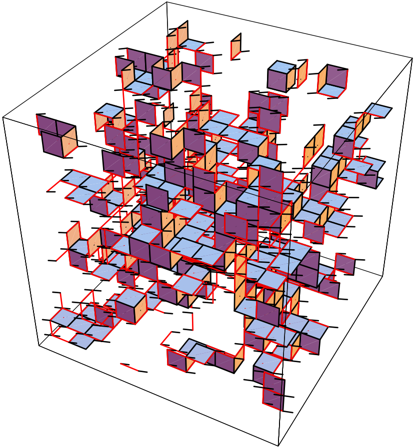


vortex removed configuration



Structure of P-Vortices

In 4D they form closed 2D-surfaces in Dual Space,
Random Structure



3-dimensional cut through the dual of a 12^4-lattice.

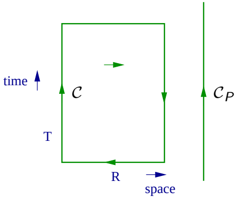
Wilson loops

A signature for confinement

$$\textit{continuum} \quad \cdots \quad W(C) = \langle \mathrm{Tr}\, \mathcal{P} \exp\left(i \oint_C A_\mu(x) dx_\mu\right) \rangle,$$

$$\textit{lattice} \quad \cdots \quad W(R,T) = \langle \mathrm{Trd}\, \mathcal{P} \prod_{l \in \mathcal{C}} U_l \rangle,$$

$$\textit{Polyakov} \quad \cdots \quad P(R) = \langle \mathrm{Trd}\, \mathcal{P} \prod_{l \in \mathcal{C}_P} U_l x \rangle.$$



$$W(\hat{R}, \hat{T}) = \exp\left(-\hat{\sigma}\hat{R}\hat{T} - \hat{\alpha}(\hat{R} + \hat{T}) + \hat{\gamma}\right)$$

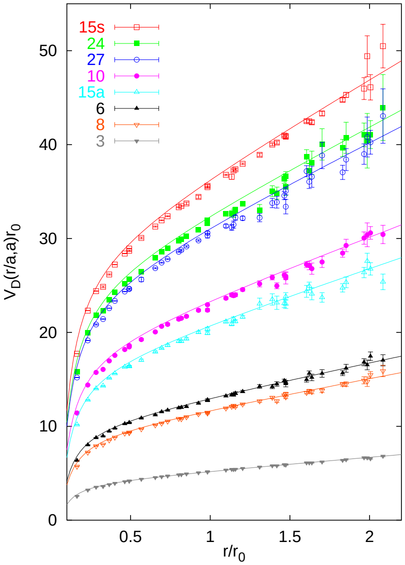
String tension σ

$$\sigma \quad \leftarrow \quad \chi(\hat{R}, \hat{T}) \; = \; -\ln\left(\frac{W(\hat{R}+1, \hat{T}+1) W(\hat{R}, \hat{T})}{W(\hat{R}, \hat{T}-1) W(\hat{R}-1, \hat{T})}\right)$$

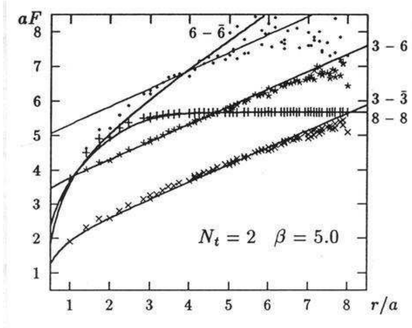
Potential

$$\begin{aligned} V(R)d \quad &= \quad -\lim_{T\to\infty} \frac{1}{T} \ln\, W(R,T) \\ &\to \quad C + \hat{\sigma}\hat{R} - \frac{e}{\hat{R}} \end{aligned}$$

Confinement potentials

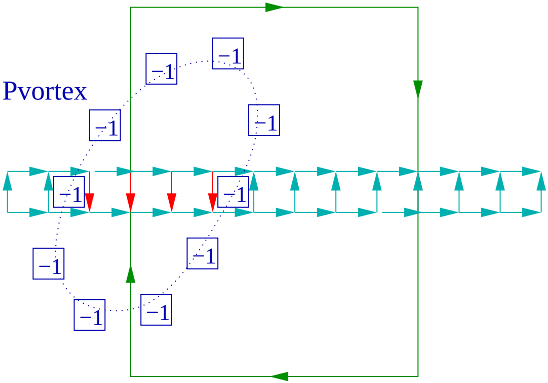


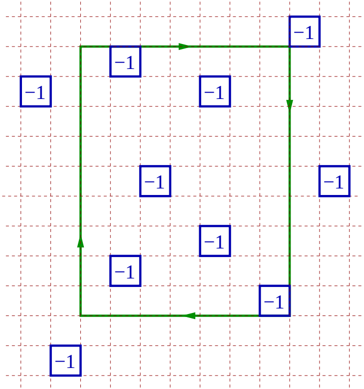
G. Bali, Phys. Rev. D62 (2000)



M. Müller et al., Lattice 1991

Area law for center projected loops



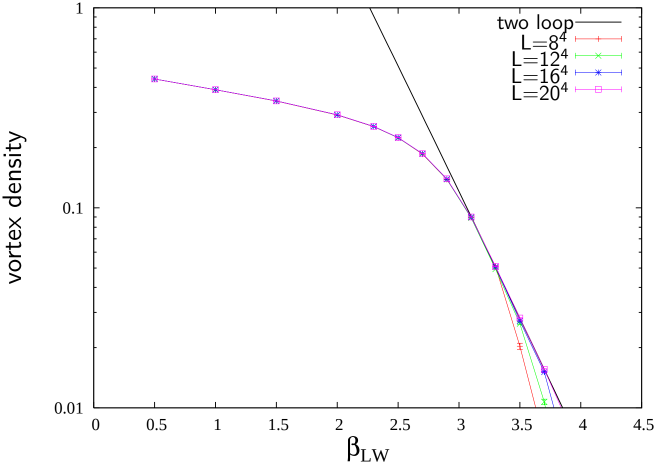


denote f the probability that a plaquette has the value -1

$$\langle W(A)\rangle = [f(-1) + (1-f)\cdot 1]^A = \exp[\underbrace{\ln(1-2f)}_{-\sigma} A], =$$

$$= \exp[-\sigma R \times T], \qquad \sigma \equiv -\ln(1-2f) \approx 2f$$

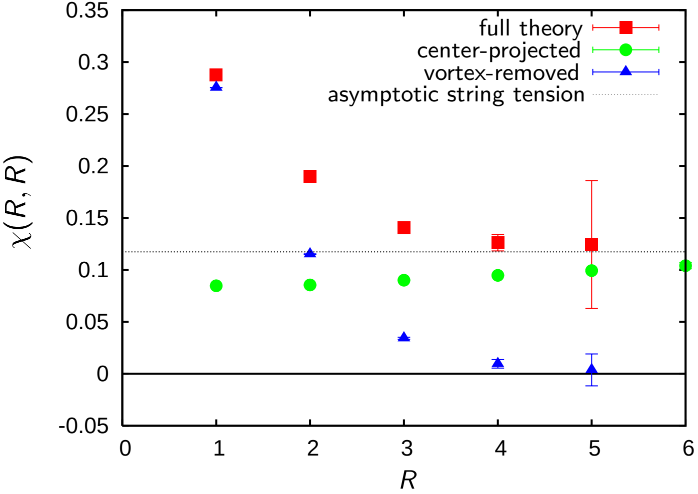
Scaling of vortex density



P-vortex surface density vs. coupling constant β_{LW}. "Two loop" line is the scaling prediction with $\sqrt{\rho/6\Lambda^2} = 50$.

Asymptotic string tension

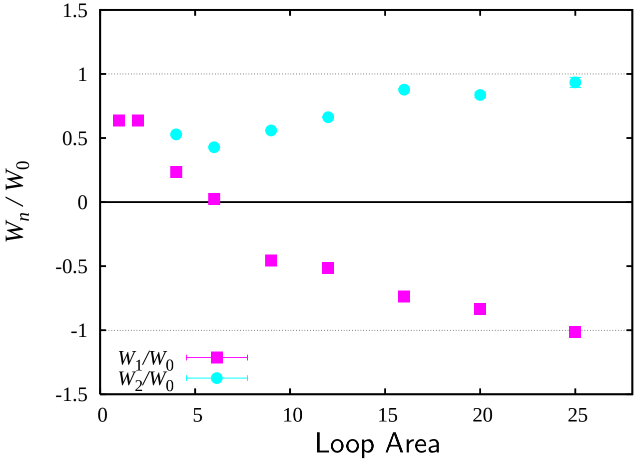
Creutz ratios: $\chi(R,T) = \frac{W(R,T)\,W(R-1,T-1)}{W(R-1,T)\,W(R,T-1)} \to \sigma$



Precocious linearity of center projected Creutz ratios.

String tension sweeps away the $1/r$-potential.

P-vortices locate thick center vortices



$$W_n(C)/W_0(C) \longrightarrow (-1)^n$$

Thick center vortex model

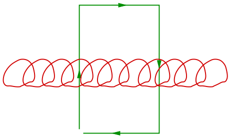
Vortex crossing the Wilson loop

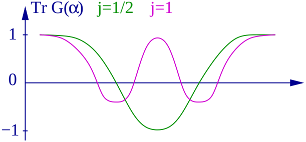
$W(C) = Tr[UU...U] \longrightarrow Tr[UU...G...U]$

$G = exp\{i\alpha_C(x)\vec{n}\vec{T}\}$

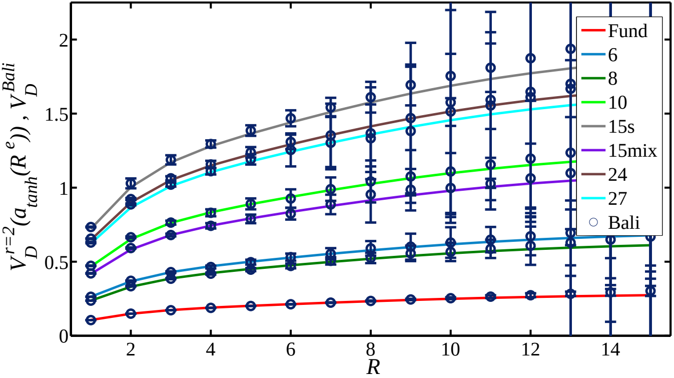
explains Confinement

and Screening of Adjoint Sources



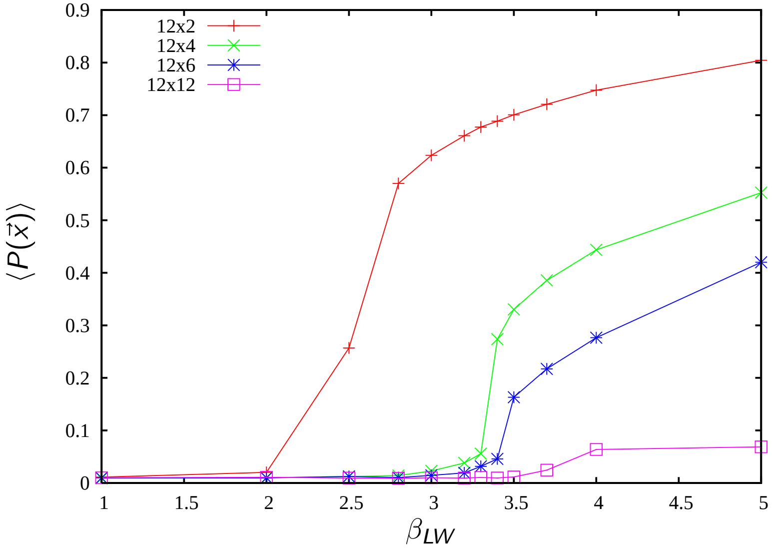


Casimir scaling

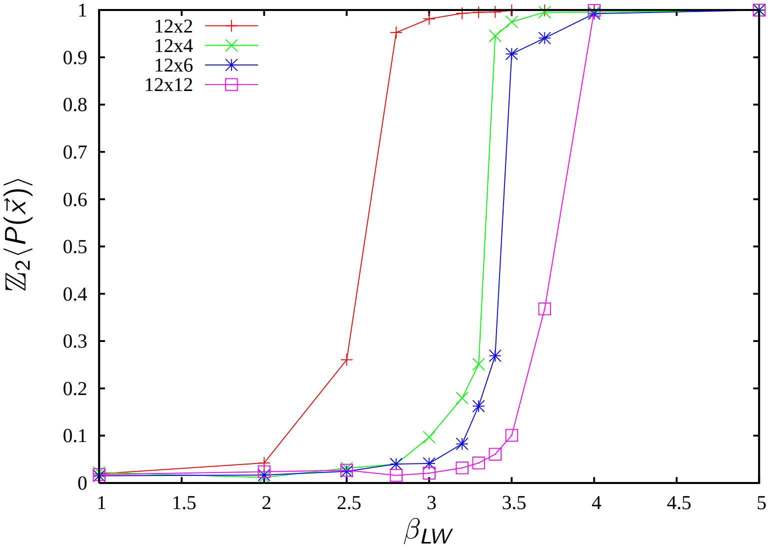


Inter quark potential $V(R)$ induced by center vortices, according to the thick vortex model, for quarks in variousx representations.

Polyakov loops at finite T

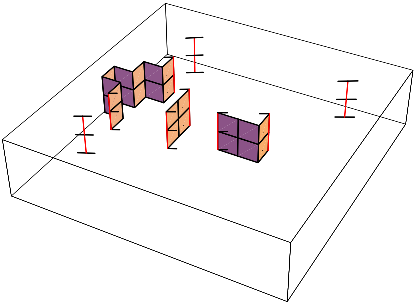


Center-projected Polyakov loops at finite T



P-Vortices at finite temperature

➜ *Engelhardt, Langfeld, Reinhardt, Tennert, 1999*



Two 3-dimensional cuts through the dual of a $2 \cdot 12^3$-lattice.
Two successive z-slices for the x-y-t-subspace are shown.

Pontryagin index Q

singlet axial current contains anomaly

$$\partial^\mu j_{\mu 5} = -\frac{N_f}{16\pi^2} \mathrm{tr}(\mathcal{F}_{\mu\nu}\tilde{\mathcal{F}}_{\mu\nu})$$

P-Vortices: closed surfaces of quantised flux

$$d^2\sigma_{\mu\nu} = \epsilon_{ab} \frac{\partial \bar{x}_\mu}{\partial \sigma_a} \frac{\partial \bar{x}_\nu}{\partial \sigma_b} d^2\sigma$$

$Q =$ Topological winding number
$Q =$ Self intersection number

➜ *Engelhardt, Reinhardt (2000)*

$$Q = -\frac{1}{16} \epsilon_{\mu\nu\alpha\beta} \int_S d^2\sigma_{\alpha\beta} \int_S d^2\sigma'_{\mu\nu} \delta^4(\bar{x}(\sigma) - \bar{x}(\sigma'))$$

one intersection contributes $\pm\frac{1}{2}$
Specify surface orientation !

Exact zero-modes and the Atiyah-Singer index theorem

- Topological charge:

$$Q := \int \mathrm{d}^4 x \partial_\mu j^5_\mu$$

- Axial anomaly:

$$\partial_\mu j^5_\mu = -\frac{N_f}{16\pi^2} \mathrm{tr}(\mathcal{F}_{\mu\nu} \tilde{\mathcal{F}}_{\mu\nu})$$

- Index theorem:

n_-, n_+: number of left-/right-handed zeromodes

$$\mathrm{ind} D[A] = n_- - n_+ = Q[A]$$

Localisation

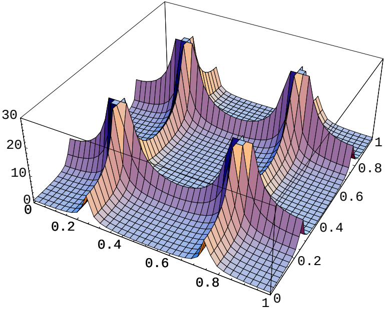
- Scalar density
$$\rho(x) = \sum_{c,d} |\vec{v}(x)_{cd}|^2,$$
c and d, color und Dirac indices

- Chiral densities $\rho_+(x)$ and $\rho_-(x)$
$$\rho_\pm(x) = \sum_{c,d} \vec{v}(x)^*_{cd} \frac{1 - \gamma_5^{c,d'}}{2} \vec{v}(x)_{cd'}$$

Analytical results

➜ *Reinhardt, Schroeder, Tok and Zhukovsky (2002)*

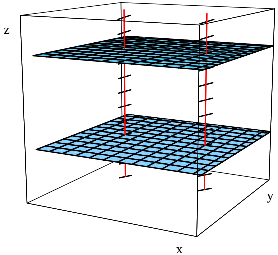
Analytical calculations by the Tübingen group.
Zero-modes peak at intersections of vortices.



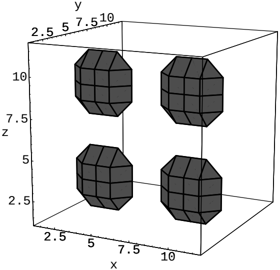
Probability density of zero-mode in the background of four intersecting vortices.

Plane Vortices in U(1) subgroup

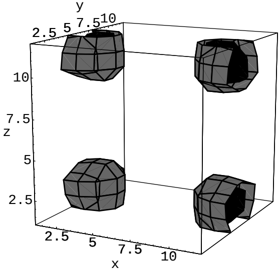
Geometry



Topological charge



Fermionic density



Index of the overlap operator: $\text{ind}\, D = 0$

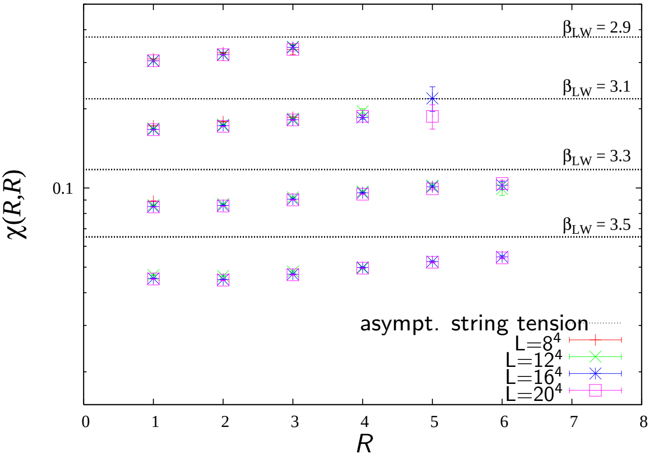
Topological charge: $Q = 0$

Conclusion: index theorem valid

Conclusions

- Confining Disorder $\equiv$ Center Disorder
- P-vortices locate center vortices $W_n/W_0 = (-1)^n$
- Center Dominance: The projected string tension is close to the asymptotic string tension $\sigma\ \chi_{cp}(R,R) \approx \sigma \quad (R \geq 2)$
- Vortex structure changes at the QCD phase transition
- Vortices allow for determination of topological charge
- Vortex removal restores chiral symmetry
- Asqtad staggered fermions show confinement and chiral symmetry breaking also for center-projected configurations
- Strong correlations between Dirac eigenmodes and center vortices
- Dirac eigenmodes show sharp peaks at intersection and writhing points
- Index theorem fulfilled for U(1), but puzzeling for SU(2) vortices

Center Dominance



Center-projected Creutz ratios at $\beta_{LW} = 2.9 - 3.5$ obtained after direct Laplacian center gauge fixing. Horizontal bands indicate the asymptotic string tensions.

The Overlap Dirac operator

- **Overlap operator:** is of Ginsparg-Wilson type

$$D_{ov} = \frac{1}{2}\left[1 + \gamma_5 \epsilon(H_L^+)\right]$$

$$\epsilon \ldots \text{sign-function},$$

$$H_L^+ = \gamma_5 D_{\mathrm{W}}(-m_0)$$

D_{W}: lattice Wilson Dirac operator for $r = 1$
for massless fermions: $m_c < m_0 < 2$

The Staggered Fermion operator

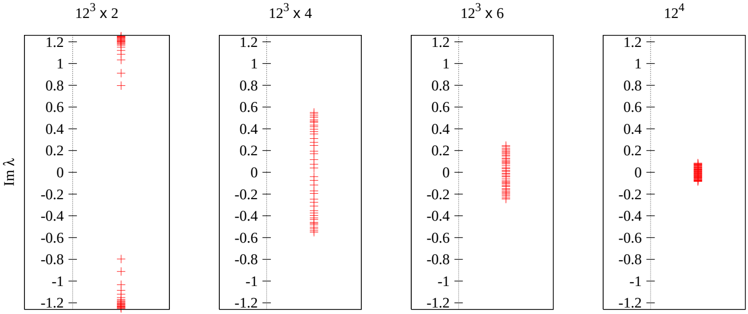
- **Staggered Fermion operator:** is of Ginsparg-Wilson type for fermionfield χ with massless fermions: $m_c < m_0 < 2$

$$D_{sf} = \frac{1}{2a}\sum_{\mu}\eta(x,\mu)P(x,\mu)$$

$$P(x,\mu) = \Big[U(x,\mu)\chi(x+a_\mu) - U^\dagger(x-a_\mu,\mu)\chi(x-a_\mu)\Big]$$

$\eta(x,\mu) = (-1)^{\sum_{\nu(<\mu)} x_\nu}$ are the staggered fermion phases
asqtad formulation includes next-to-nearest neighbour and staple terms, namely the Naik term, 3-, 5- and 7-staple terms and the Lepage term.

Asqtad Staggered Fermions at finite T



Localization and fractal dimension of eigenmodes

The Inverse Participation Ratio (IPR) of a normalized field $\rho_i(x)$ is defined as

$$I = N \sum_{x=0}^{N} \rho_i^2(x).$$

$$\begin{array}{rl} \text{Unlocalized: } \rho(x) = const. & I = 1 \\ \delta\text{-function: } \rho(x) = \delta(x_0) & I = N \\ \text{localized on fraction } f \text{ of sites:} & I = 1/f \end{array}$$

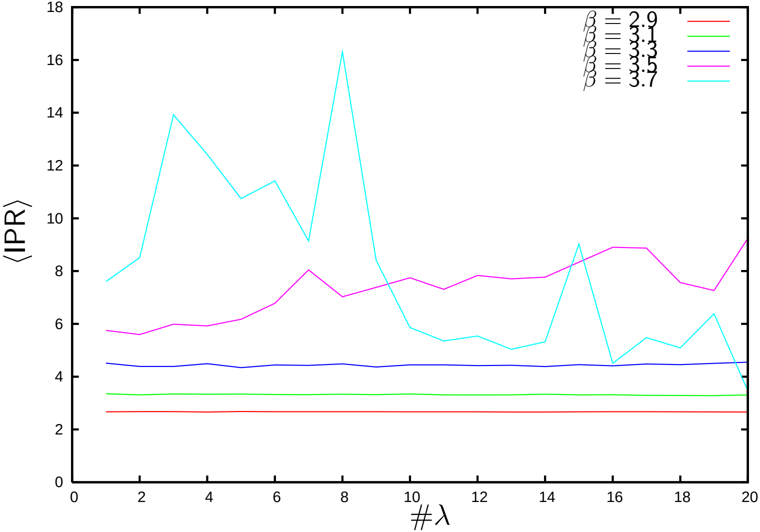
Scaling properties:

$$a \to 0 \text{ at fixed volume: } I \sim a^{4-d}$$
$$L \to \infty \text{ at fixed } a\text{: } I \sim \text{constant.}$$

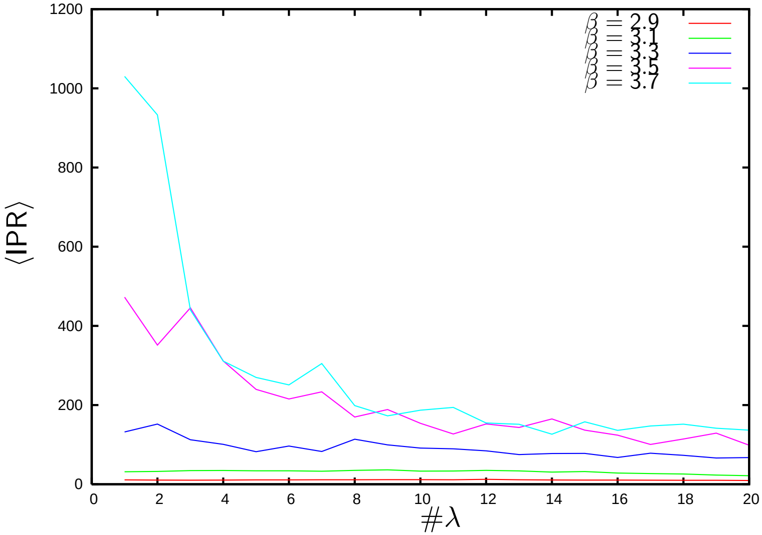
The remaining norm R is defined as

$$R(n) = 1 - \sum_{i=1}^{n} \rho_i(x) \quad \ldots \quad 1 \leq n \leq L^4,$$

Asqtad IPR of full config vs. β



Asqtad IPR for center-projected config vs. β



Asqtad IPR for vortex removed config vs. β