Studying and removing effects of fixed topology

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Outline

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Topology and vacuum

- To avoid infinite action, one needs $\lim_{r\to\infty} A_\mu = G\partial_\mu G^{-1}$, where G is an element of the group of symmetry
- For each G, you have a **classical vacuum** which can be classified by a topological **charge** Q("winding number")

  same Q: one topological sector: no energy wall

  different Q: different topological sector: energy wall

- Tunneling is possible between topological sectors ⇒**Quantum vacuum** is a superposition

$$|\Omega, \theta\rangle = \sum_{n=-\infty}^{n=+\infty} e^{-in\theta}|n\rangle \qquad (1)$$

where θ is a constant angle characterizing the theory : $\theta_{QCD} \approx 0$

Topology in Euclidean QCD

Instantons and topological charge

- We can include the θ-parameter in the action.

$$S^E_{QCD}(\theta) = S^E_{QCD} - iQ\theta = S^E_{QCD} - i\theta \frac{g^2}{32\pi^2} \int d^4x \, F_{\mu\nu} \tilde{F}_{\mu\nu} \qquad (2)$$

Definition: Instantons

Pseudo particles which are solutions of the Euclidean EOM
Carry a topological charge Q=+1

- In a classical vacuum at finite volume, the topological charge is $Q = N_{inst} - N_{anti}$
- Definition:Topological Susceptibility can be defined as:

$$\chi_T(\theta) = \frac{\partial^2 E_0(\theta)}{\partial \theta^2} = \frac{\langle Q^2 \rangle}{V} \text{ if one assumes } \langle Q \rangle = 0. \qquad (3)$$

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1. Introduction
   - Introduction to topology
   - Motivation
2. Working at fixed Topology
   - From real QCD to fixed topological Sector
   - Method to obtain QCD results
3. Results
   - 3 different models
   - QM: Particle on a circle with well potential
   - QFT Model: Schwinger and SU(2)

Topology on lattice

- Generate **field configurations** with Monte-Carlo algorithms
- Each configuration has finite volume and one can assign a topological charge to it (not a unique definition on a lattice)
- To simulate QCD $\Rightarrow$"sum" on field configurations with different topological charge

Problem

Topology freezes for a too small lattice spacing $a \lesssim 0.05 fm$[a]
$\Rightarrow$Find a way to get from one sector a QCD observable

[a] Luscher, Martin JHEP 1008 (2010) 071 arXiv:1006.4518 [hep-lat] CERN-PH-TH-2010-143

Other uses of fixed topology:

- High quality fermions $\rightarrow$ Overlap fermions
- Mixed action, ...

Outline

Fixed Topological sectors
From fixed θ to fixed Q

- Topological sectors: path integral superposition of classical field configuration with the same Q

  $\Rightarrow$No Hamiltonian! (Q non-local in time)

- Partition function at fixed Q

$$Z_Q = \int DAD\psi D\bar{\psi}\delta_{Q,Q[A]}e^{-S_{QCD}^E[A,\bar{\psi},\psi]} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \, Z(\theta)e^{iQ\theta}. \tag{4}$$

  where $Z(\theta)$ is the partition function of $S_{QCD}^E(\theta)$

- Correlator (two-point correlation function) transformation

$$Z_Q C_Q = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \, C(\theta)Z(\theta)e^{iQ\theta}. \tag{5}$$

From real QCD to Fixed Topological Sector

Correlator at fixed topology

Using Saddle point approximation for large volume :

$$Z_Q = \frac{e^{-TE_0(0)}}{\sqrt{2\pi\chi_t V}} e^{-\frac{1}{2}\frac{Q^2}{\chi_t V}} \left(1 + O\left(\frac{1}{\chi_T V}\right)\right) \tag{6}$$

$$M_Q(t) := -\frac{d}{dt} ln(C_Q(t)) \tag{7}$$

- From one topological sector we can extract the mass of real QCD[1][2]

$$\boxed{M_Q = M(0) + \frac{M''(0)}{2\chi_T V}\left(1 - \frac{Q^2}{\chi_T V}\right) + O\left(\frac{1}{(\chi_T V)^2}\right)} \tag{8}$$

[1] R. Brower , S. Chandrasekharan , John W. Negele , U.J. Wiese in Phys.Lett. B560 (2003) 64-74

[2] Aoki, Sinya et al. Phys.Rev. D76 (2007) 054508

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Method

General Method:

1. Compute correlator at fixed topology for different volumes and different Q
2. Fit formula (3 parameters)
3. Extract $M(\theta = 0)$ and χ_T

Conditions:

(C1) $1/\chi_T V \ll 1$ and $|Q|/\chi_T V \ll 1$: Taylor development and saddle point approximation

(C2) $\left| M_H^{(2)}(0) t \right| / \chi_T V \ll 1$: Taylor development

(C3) $m_\pi L \gg 3$: Avoid non topological finite size effect (Lattice simulation)

(C4) $(M_H^* - M_H) t \gg 1$ and $M_H (T - 2t) \gg 1$: Avoid exited state, and periodicity contribution.

Improvement

Motivations of improvement:

1. Increase precision on results
2. Perhaps increase range of validity (C1 and C2)

Improved Formula

$$C_{Q,V}(t) = \alpha(0)\exp\Bigg(-M_H(0)t - \frac{1}{\mathcal{E}_2 V}\frac{x_2}{2} - \frac{1}{(\mathcal{E}_2 V)^2}\left(\frac{x_4 - 2(\mathcal{E}_4/\mathcal{E}_2)x_2 - 2x_2^2}{8} - \frac{x_2}{2}Q^2\right)$$
$$- \frac{1}{(\mathcal{E}_2 V)^3}\Bigg(\frac{16(\mathcal{E}_4/\mathcal{E}_2)^2 x_2 + x_6 - 3(\mathcal{E}_6/\mathcal{E}_2)x_2 - 8(\mathcal{E}_4/\mathcal{E}_2)x_4 - 12x_2x_4}{48}$$
$$+ \frac{18(\mathcal{E}_4/\mathcal{E}_2)x_2^2 + 8x_2^3}{48} - \frac{x_4 - 3(\mathcal{E}_4/\mathcal{E}_2)x_2 - 2x_2^2}{4}Q^2\Bigg)\Bigg)$$
$$+ \mathcal{O}\left(\frac{1}{(\mathcal{E}_2 V)^4}, \frac{1}{(\mathcal{E}_2 V)^4}Q^2, \frac{1}{(\mathcal{E}_2 V)^4}Q^4\right). \tag{9}$$

Cost of improvement: 11 parameters for the correlator and $M_Q(t)$ is a function of time!

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3 different models

Exploratory studies on 3 different models (Aim is full QCD)

1. A QM Model: A particle on a circle with a square well potential:

   Interest: Solvable with Mathematica. Allowed to compare formula with exact results, test conditions.

   Method: Compute exact energy differences (M^{eff}) at fixed Q, using formula to extract M^{eff} at fixed θ. Compare with the exact results.

2. Schwinger Model with Wilson fermions: QED at 2 dimension:

   Interest: Share property with QCD (as confinement) and have fermions. Cheap to compute.

   Method: Compute fixed top. masses. Extracted the mass at fixed θ and infinite volume from formula. Compare with classical lattice result obtained at unfixed top.

3. SU(2) Model:

   Interest: Close to QCD vacuum. Cheaper than SU(3).

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[3] Other results (overlap quarks): Topological Summation in Lattice Gauge Theory - Bietenholz W. et al. Eur. Phys. J. C72 (2012) 1938,

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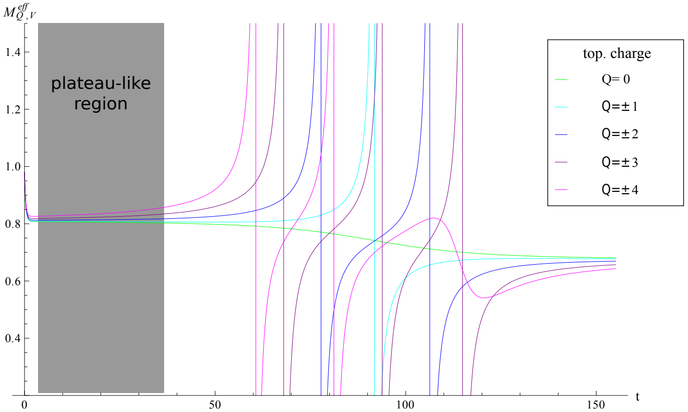
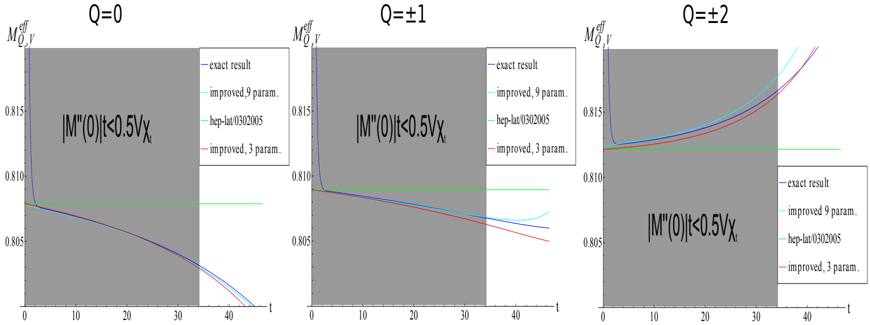
The C2 condition

$$\mathcal{H} = \frac{P^2}{2I} - U_{Sq\_Well}$$



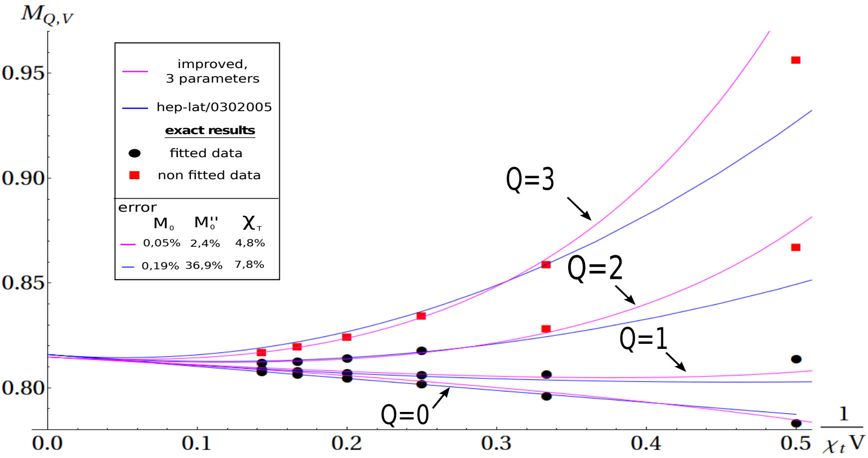
- In contrast to QFT at unfixed Top. The effective mass is not a plateau.
- Expansion is valid in the plateau like region at not too big t

Comparison of formulas



- Equations are good approximations for small t
- Improved equations reproduce the behavior of the mass with much better precision
  - The deviation from a plateau is reproduced
- Approximate range of validity of equations :
  - $M^{(2)}(\theta = 0)t < 0.5\chi_T V$ and $|Q| < \chi_T V$

Extracting result

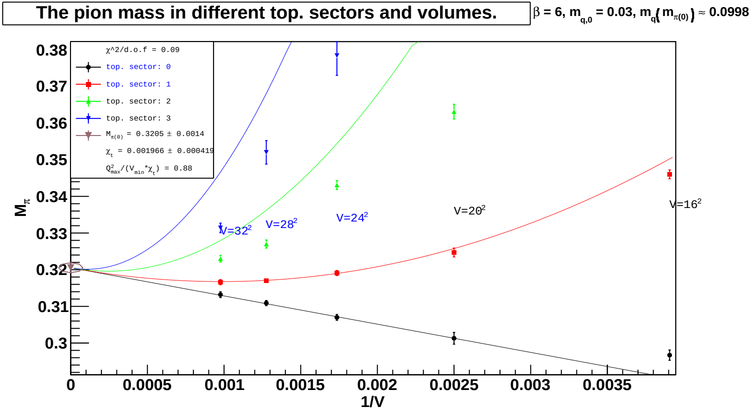


- Error: Difference between exact analytical results and results obtained by fitting
- Using the improved formula reduced significantly the errors

Outline

Schwinger Model

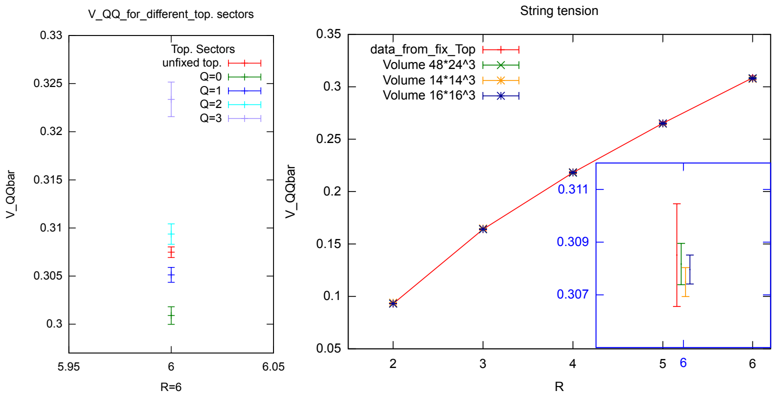
$$\mathscr{L}(\psi,\bar{\psi},A_\mu) = \bar{\psi}(x)(\gamma_\mu(\partial_\mu + igA_\mu(x)) + m)\psi(x) + \frac{1}{2}F_{\mu\nu}(x)F_{\mu\nu}(x)$$



- Good agreement with unfixed top. lattice results
- Large statistical error on the top. susceptibility. (improved by simultaneous fit for several observables)

SU(2) Model

$$\mathscr{L}_{YM} = -\frac{1}{4} F_{\mu\nu}(x) F_{\mu\nu}(x)$$



- Discrepancy between top. sector (left plot)
- Good agreement between fixed and unfixed top. simulation

Summary

1. Developed and improved method to work at fixed topology which is important for :
   1)QCD simulation for $a \lesssim 0.05fm$, 2) Overlap fermions and 3) Mixed action approaches
2. Demonstrated the effectiveness of the method to extract unfixed top. results for our 3 models
3. A large statistical error on topological susceptibility

Outlook

- Test the method in full QCD
- Other method to obtain the top. susceptibility[4] (can work on only one top sector)

[4] Topological Summation in Lattice Gauge Theory - Bietenholz, Wolfgang et al. Eur. Phys. J. C72 (2012) 1938,