Studying and removing effects of fixed topology

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Outline

1 Introduction
   - Introduction to topology
   - Motivation

2 Working at fixed Topology
   - From real QCD to fixed topological Sector
   - Method to obtain QCD results

3 Results
   - 3 different models
   - QM: Particle on a circle with well potential
   - QFT Model: Schwinger and SU(2)
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Topology and vacuum

- To avoid infinite action, one needs \( \lim_{r \to \infty} A_\mu = G \partial_\mu G^{-1} \), where \( G \) is an element of the group of symmetry.

- For each \( G \), you have a **classical vacuum** which can be classified by a topological **charge** \( Q \) ("winding number")
  - same \( Q \): one topological sector: no energy wall
  - different \( Q \): different topological sector: energy wall

- Tunneling is possible between topological sectors \( \Rightarrow \text{Quantum vacuum} \) is a superposition

\[
|\Omega, \theta\rangle = \sum_{n=-\infty}^{n=+\infty} e^{-in\theta} |n\rangle
\]

where \( \theta \) is a constant angle characterizing the theory: \( \theta_{QCD} \approx 0 \)
Topology in Euclidean QCD

Instantons and topological charge

- We can include the $\theta$-parameter in the action.

\[
S^E_{QCD}(\theta) = S^E_{QCD} - iQ\theta = S^E_{QCD} - i\theta \frac{g^2}{32\pi^2} \int d^4x F_{\mu\nu} \tilde{F}_{\mu\nu}
\]  

(2)

**Definition: Instantons**

Pseudo particles which are solutions of the Euclidean EOM

Carry a topological charge $Q=+1$

- In a classical vacuum at finite volume, the topological charge is $Q = N_{\text{inst}} - N_{\text{anti}}$

- Definition: Topological Susceptibility can be defined as:

\[
\chi_T(\theta) = \frac{\partial^2 E_0(\theta)}{\partial \theta^2} = \frac{\langle Q^2 \rangle}{V} \quad \text{if one assumes} \quad \langle Q \rangle = 0.
\]  

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**Topology on lattice**

- Generate **field configurations** with Monte-Carlo algorithms
- Each configuration has finite volume and one can assign a topological charge to it (not a unique definition on a lattice)
- To simulate QCD ⇒ ”sum” on field configurations with different topological charge

**Problem**

Topology freezes for a too small lattice spacing $a \lesssim 0.05 \text{fm}$

$\Rightarrow \text{Find a way to get from one sector a QCD observable}$

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**Other uses of fixed topology:**

- High quality fermions $\Rightarrow$ Overlap fermions
- Mixed action, ...
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Fixed Topological sectors
From fixed $\theta$ to fixed $Q$

- Topological sectors: path integral superposition of classical field configuration with the same $Q$
  \[ \Rightarrow \text{No Hamiltonian! (}Q\text{ non-local in time)} \]
- Partition function at fixed $Q$

\[
Z_Q = \int DAD \psi D \bar{\psi} \delta_{Q,Q[A]} e^{-S_{QCD}^E[A,\bar{\psi},\psi]} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \ Z(\theta)e^{iQ\theta}. \tag{4}
\]

where $Z(\theta)$ is the partition function of $S_{QCD}^E(\theta)$
- Correlator (two-point correlation function) transformation

\[
Z_Q C_Q = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \ C(\theta)Z(\theta)e^{iQ\theta}. \tag{5}
\]
Correlator at fixed topology

Using Saddle point approximation for large volume:

\[ Z_Q = \frac{e^{-TE_0(0)}}{\sqrt{2\pi\chi t V}} e^{-\frac{1}{2} \frac{Q^2}{\chi t V}} \left(1 + O\left(\frac{1}{\chi TV}\right)\right) \]  \hspace{1cm} (6)

\[ M_Q(t) := -\frac{d}{dt} \ln(C_Q(t)) \] \hspace{1cm} (7)

- From one topological sector we can extract the mass of real QCD\(^{12}\)

\[ M_Q = M(0) + \frac{M''(0)}{2\chi TV} \left(1 - \frac{Q^2}{\chi TV}\right) + O\left(\frac{1}{(\chi TV)^2}\right) \] \hspace{1cm} (8)

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Method

General Method:

1. Compute correlator at fixed topology for different volumes and different $Q$
2. Fit formula (3 parameters)
3. Extract $M(\theta = 0)$ and $\chi_T$

Conditions:

(C1) $1/\chi_T V \ll 1$ and $|Q|/\chi_T V \ll 1$: Taylor development and saddle point approximation

(C2) $|M_H^{(2)}(0)t|/\chi_T V \ll 1$: Taylor development

(C3) $m_\pi L \gg 3$: Avoid non topological finite size effect (Lattice simulation)

(C4) $(M^*_H - M_H)t \gg 1$ and $M_H(T - 2t) \gg 1$: Avoid exited state, and periodicity contribution.
Motivations of improvement:

1. Increase precision on results
2. Perhaps increase range of validity (C1 and C2)

Improved Formula

\[
C_{Q,V}(t) = \alpha(0) \exp \left( -M_H(0)t - \frac{1}{\mathcal{E}_2 V} \frac{x_2}{2} - \frac{1}{(\mathcal{E}_2 V)^2} \left( \frac{x_4 - 2(\mathcal{E}_4/\mathcal{E}_2)x_2 - 2x_2^2}{8} - \frac{x_2}{2} Q^2 \right) \right.
\]
\[
- \frac{1}{(\mathcal{E}_2 V)^3} \left( \frac{16(\mathcal{E}_4/\mathcal{E}_2)^2 x_2 + x_6 - 3(\mathcal{E}_6/\mathcal{E}_2)x_2 - 8(\mathcal{E}_4/\mathcal{E}_2)x_4 - 12x_2x_4}{48} 
+ \frac{18(\mathcal{E}_4/\mathcal{E}_2)x_2^2 + 8x_3^2}{48} - \frac{x_4 - 3(\mathcal{E}_4/\mathcal{E}_2)x_2 - 2x_2^2}{4} Q^2 \right) 
\left. + \mathcal{O}\left( \frac{1}{(\mathcal{E}_2 V)^4}, \frac{1}{(\mathcal{E}_2 V)^4} Q^2, \frac{1}{(\mathcal{E}_2 V)^4} Q^4 \right) \right) .
\]

Cost of improvement: 11 parameters for the correlator and \( M_Q(t) \) is a function of time!
Improvement

Motivations of improvement:

1. Increase precision on results
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Improved Formula

\[ C_{Q,V}(t) = \alpha(0) \exp \left( -M_H(0)t - \frac{1}{2} \frac{x_2}{\mathcal{E}_2 V} - \frac{1}{8} \frac{x_4}{(\mathcal{E}_2 V)^2} \left( \frac{2(\mathcal{E}_4/\mathcal{E}_2)x_2 - 2x^2_2}{8} - \frac{x_2}{2} Q^2 \right) \right. \]

\[ \left. - \frac{1}{(\mathcal{E}_2 V)^3} \left( \frac{16(\mathcal{E}_4/\mathcal{E}_2)^2 x_2 + x_6 - 3(\mathcal{E}_6/\mathcal{E}_2)x_2 - 8(\mathcal{E}_4/\mathcal{E}_2)x_4 - 12x_2x_4}{48} \right. \right. \]

\[ \left. \left. + \frac{18(\mathcal{E}_4/\mathcal{E}_2)x_2^2 + 8x^3_2}{48} - \frac{x_4 - 3(\mathcal{E}_4/\mathcal{E}_2)x_2 - 2x^2_2}{4} Q^2 \right) \right) \]

\[ + \theta \left( \frac{1}{(\mathcal{E}_2 V)^4}, \frac{1}{(\mathcal{E}_2 V)^4} Q^2, \frac{1}{(\mathcal{E}_2 V)^4} Q^4 \right). \]

(9)

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Exploratory studies on 3 different models (Aim is full QCD)

1. A QM Model: A particle on a circle with a square well potential:

   **Interest:** Solvable with Mathematica. Allowed to compare formula with exact results, test conditions.

   **Method:** Compute exact energy differences ($M^{\text{eff}}$) at fixed $Q$, using formula to extract $M^{\text{eff}}$ at fixed $\theta$. Compare with the exact results.

2. Schwinger Model with Wilson fermions: QED at 2 dimension:
   Interest: Share property with QCD (as confinement) and have fermions. Cheap to compute.
   Method: Compute fixed top. masses. Extracted the mass at fixed $\theta$ and infinite volume from formula. Compare with classical lattice result obtained at unfixed top.

3. SU(2) Model:
   Interest: Close to QCD vacuum. Cheaper than SU(3).
   Method: Same as Schwinger Model
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   - **Interest**: Share property with QCD (as confinement) and have fermions. Cheap to compute.
   - **Method**: Compute fixed top. masses. Extracted the mass at fixed $\theta$ and infinite volume from formula. Compare with usual lattice result obtained at unfixed top.

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   - Interest: Close to QCD vacuum. Cheaper than SU(3).
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The C2 condition

\[ \mathcal{H} = \frac{P^2}{2T} - U_{Sq\_Well} \]

In contrast to QFT at unfixed Top. The effective mass is not a plateau.

Expansion is valid in the plateau like region at not too big $t$. 

\[ M_{Q,V}^{\text{eff}} \]
Equations are good approximations for small $t$.

Improved equations reproduce the behavior of the mass with much better precision.

- The deviation from a plateau is reproduced.

Approximate range of validity of equations:

- $M^{(2)}(\theta = 0) t < 0.5 \chi_T V$ and $|Q| < \chi_T V$.
Extracting results

- Error: Difference between exact analytical results and results obtained by fitting
- Using the improved formula reduced significantly the errors
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Schwinger Model

\[ \mathcal{L}(\psi, \bar{\psi}, A_\mu) = \bar{\psi}(x)(\gamma_\mu(\partial_\mu + igA_\mu(x)) + m)\psi(x) + \frac{1}{2}F_{\mu\nu}(x)F_{\mu\nu}(x) \]

The pion mass in different top. sectors and volumes.

- Good agreement with unfixed top. lattice results
- Large statistical error on the top. susceptibility. (improved by simultaneous fit for several observables)
SU(2) Model

\[ \mathcal{L}_{YM} = -\frac{1}{4} F_{\mu \nu}(x) F^{\mu \nu}(x) \]

- Discrepancy between top. sector (left plot)
- Good agreement between fixed and unfixed top. simulation
Summary

1. Developed and improved method to work at fixed topology which is important for:
   1) QCD simulation for $a \lesssim 0.05\text{fm}$, 2) Overlap fermions and 3) Mixed action approaches
2. Demonstrated the effectiveness of the method to extract unfixed top. results for our 3 models
3. A large statistical error on topological susceptibility

Outlook

- Test the method in full QCD
- Other method to obtain the top. susceptibility\(^4\) (can work on only one top sector)