Intriguing relations between the LECs of Wilson χ-PT and spectra of the Wilson Dirac operator

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Outline

- Motivation-The Goals
- Wilson Fermions- Wilson $\chi-$PT
- Introduction of the Model
- LECs and the spectrum of D_W
- Conclusions and Outlook

Motivation-The Goals

- Facilitate simulations in the deep chiral regime by an exact, analytical understanding of the average behavior of the smallest eigenvalues
- Chiral symmetry breaking from lattice spacing
- Stability of lattice simulations

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Wilson Fermions- Wilson $\chi-$PT

Fermions on the lattice- Origin of doublers

- The momentum space propagator (free theory)

$$D(p)|_{m=0}^{-1} = \frac{-ia^{-1}\sum_\mu \gamma_\mu \sin(p_\mu a)}{a^{-2}\sum_\mu \sin(p_\mu a)^2} \xrightarrow{a \to 0} \frac{-i\sum_\mu \gamma_\mu p_\mu}{p^2}$$

- In the continuum one pole at $p = (0,0,0,0)$
- On the lattice additional poles whenever all components are either $p_\mu = 0$ or $p_\mu = \pi/a$
- Our lattice Dirac operator has 15 unphysical poles (doublers) at $p = (\pi/a, 0, 0, 0), (0, \pi/a, 0, 0), ..., (\pi/a, \pi/a, \pi/a, \pi/a)$

A No-go theorem

Nielsen and Ninomiya (1980)

It is not possible to construct a lattice fermion action that is

- Local
- Undoubled
- correct continuum limit
- chirally symmetric $\{D, \gamma_5\} = 0$

Wilson Fermions

Break chiral symmetry explicitly

Wilson (1977)

- add the lattice discretization of the Laplacian $-\frac{a}{2}\partial_\mu\partial_\mu$
- $D(p) = m\mathbf{1} + \frac{i}{a}\sum_{\mu=1}^{4}\gamma_\mu \sin p_\mu a + \mathbf{1}\frac{1}{a}\sum_{\mu=1}^{4}(1-\cos p_\mu a)$
- for components with $p_\mu = 0$ it vanishes
- for each component with $p_\mu = \pi/a$ provides an extra contribution $2/a$
- It acts like an additional "effective" mass term so the total mass of the doublers is $m + 2l/a$
- in the naive continuum limit $a \to 0$ the doublers become very heavy and decouple from the theory

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Wilson Chiral Perturbation Theory

- Wilson term breaks $\chi-$ symmetry explicitly
- Lattice spacing effects lead to new terms in $\chi - PT$

  Sharpe and Singleton (1998), Rupak and Shoresh (2002), Baer,Rupak and Shoresh (2004)
- $\epsilon-$ regime where in the thermodynamic, chiral and continuum limit $mV\Sigma$, $zV\Sigma$ and a^2VW_i kept fixed
- At order a^2 it involves three Low Energy Constants (LECs)

$$Z_{N_f}(m,z;a) = \int_{\mathcal{M}} dU \det{}^{\nu} U \; e^{-S[U]},$$

where the action is

$$S = -\frac{m}{2}\Sigma V \mathrm{tr}\,(U+U^\dagger) - \frac{z}{2}\Sigma V \mathrm{tr}\,(U-U^\dagger) + a^2 V W_6 [\mathrm{tr}\,\left(U+U^\dagger\right)]^2$$
$$+ a^2 V W_7 [\mathrm{tr}\,\left(U-U^\dagger\right)]^2 + a^2 V W_8 \mathrm{tr}\,(U^2+U^{\dagger 2}).$$

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Random Matrix Theory

- RMT applied to Physics was born in Nuclear Physics
- Description of the statistical properties of excited energy levels in complex nuclei Wigner (1955)
- Complex systems, very complicated or even unknown dynamics
- Replace the Hamiltonian by a random matrix H with the same GLOBAL symmetries
- Compute observables by averaging over the ensemble
- Identify universal quantities (independent of the probability distribution)

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γ_5- hermiticity

- $D_W = \frac{1}{2}\gamma_\mu(\nabla_\mu + \nabla^*_\mu) - \frac{1}{2}a\nabla^*_\mu\nabla_\mu$
- At $a \neq 0$ is non-Hermitian but retains γ_5 -Hermiticity $D_W^\dagger = \gamma_5 D_W \gamma_5$
- Eigenvalues of D_W because of the γ_5 -Hermiticity occur in complex conjugate pairs or are real.
- ONLY eigenvectors corresponding to real eigenvalues have non vanishing chirality $\langle k|\gamma_5|k\rangle$

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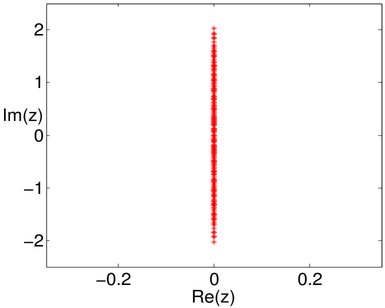
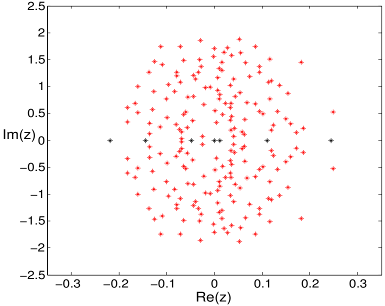
Eigenvalues of D_W with $\nu = 5$



$\hat{a}_8 = 0$



$\hat{a}_8 = 1$

- Partition function of D_W with N_f flavors :

$$Z_{N_f}^{RMT,\nu} = \int dD_W \det{}^{N_f}(D_W + m) P(D_W)$$

- $P(D_W) \to$ is a Gaussian
- $D_W = \begin{pmatrix} aA & W \\ W^\dagger & aB \end{pmatrix} + am_6 + a\lambda_7\gamma_5$ (Damgaard et al (2010),Akemann et al (2010), Kieburg et al (2011,2012)
- A : $n \times n$ Hermitian
- B : $(n+\nu) \times (n+\nu)$ Hermitian
- W : $n \times (n+\nu)$ Complex
- m_6 and λ_7 scalar random variables
- At $a = 0$: D_W has ν generic zero modes
- At finite a : definition of the index through spectral flow lines or equivalently $\nu = \sum_{\lambda_k^W \in \mathbb{R}} \mathrm{sign}(\langle k|\gamma_5|k\rangle)$ Itoh et al (1987)

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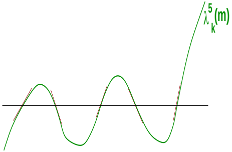
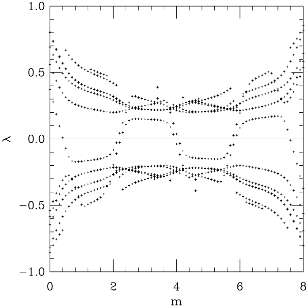


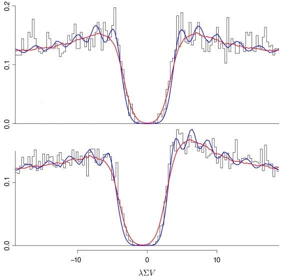
Schematic spectral flow of $D_5(m)$ (Figure courtesy of Splittorff and Verbaarschot (2010))

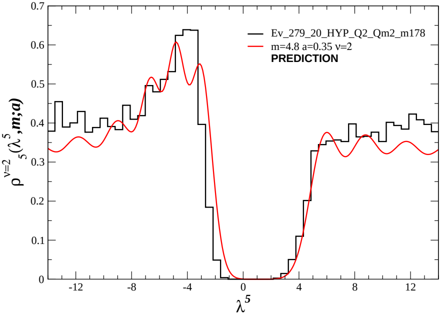


Spectral flow of $D_5(m)$ for $0 \leq m \leq 8$ for a single instanton on a 8^4 lattice. (Figure courtesy of Edwards, Heller, Narayanan (1998))

The Eigenvalue Densities

Lattice results vs RMT





$\widehat{a}_6 = \widehat{a}_7 = 0.25,\ \widehat{a}_8 = 0.7$
$\widehat{m} = 5.3$
$\nu = 0$ (top) and
$\nu = 1$ (bottom)

(Deuzeman,Wenger and Wuilloud (2011))

$\hat{m} = 4.8,\ \nu = 2$

(Damgaard,Heller and Splittorff (2011))

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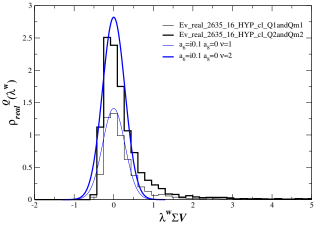
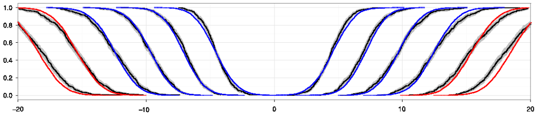


The density of real eigenvalues of D_W

Damgaard,Heller and Splittorff (2012))



Cumulative eigenvalue distributions of D_5 with all $W_{6/7/8}$ included at $\nu = 0$

(Deuzeman, Wenger and Wuilloud (2011))

The effects of W_6 and W_7 when $W_8 = 0$

- $\hat{a}_6$ and $\hat{a}_7$ introduced through the addition of the Gaussian stochastic variable $\widehat{m}_6 + \widehat{\lambda}_7\gamma_5$ to D_W
- $D = D_W + (m + \widehat{m}_6)\mathbf{1} + \widehat{\lambda}_7\gamma_5$
- When $\hat{a}_8 = 0$ D_W is anti-Hermitian,
- the eigenvalues of $D_W(\widehat{\lambda}_7, \widehat{m}_6) = D - m$ are given by

$$\widehat{z}_\pm = \widehat{m}_6 \pm i\sqrt{\lambda_W^2 - \widehat{\lambda}_7^2}$$

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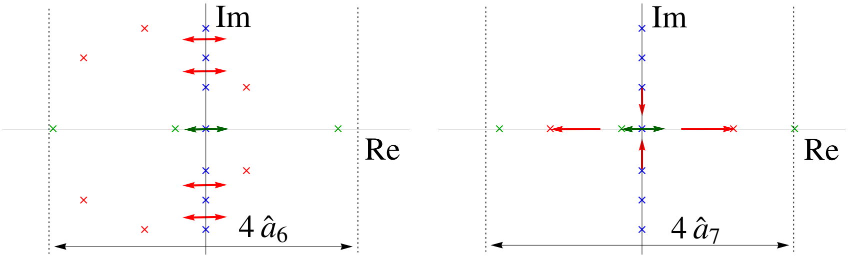
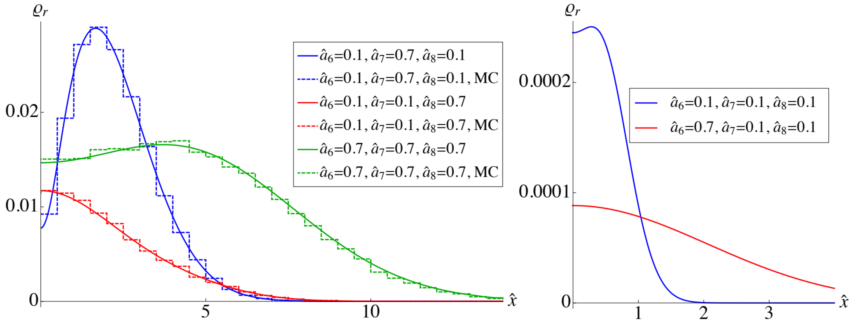
The effects of W_6 and W_7



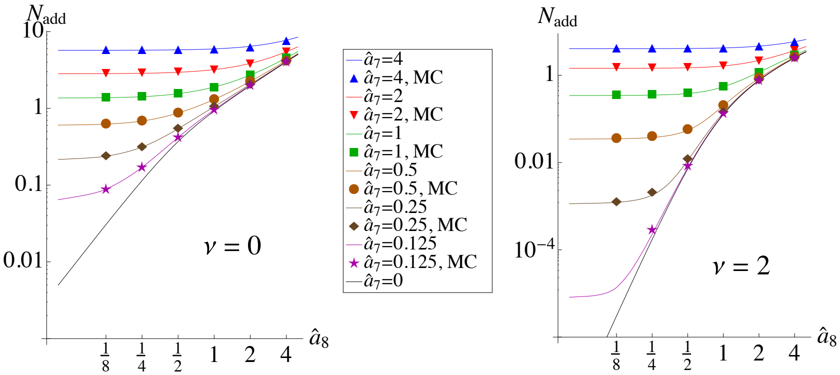
Schematic plots of the effects of W_6 (left plot) and of W_7 (right plot). W_6 broadens the spectrum parallel to the real axis according to a Gaussian with width $4\hat{a}_6$, but does not change the continuum spectrum in a significant way. When $W_7 \neq 0$ and $W_6 = 0$ the purely imaginary eigenvalues invade the real axis through the origin and only the real (green crosses) are broadened by a Gaussian with width $4\widehat{a}_7$

Distribution of additional real modes for $\nu = 1$



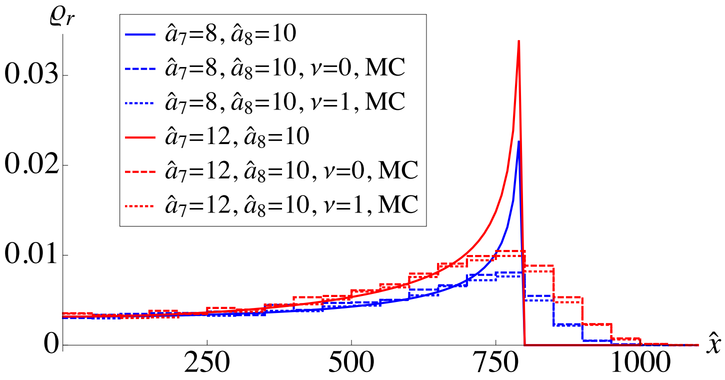
Notice that the two curves for $\widehat{a}_7 = \widehat{a}_8 = 0.1$ (right plot) are two orders smaller than the other curves (left plot). Notice the soft repulsion of the additional real modes from the origin at large $\widehat{a}_7$. The parameter $\widehat{a}_6$ smooths the distribution.

Log-Log plots of additional real modes vs $\hat{a}$ for $\nu = 0, 2$



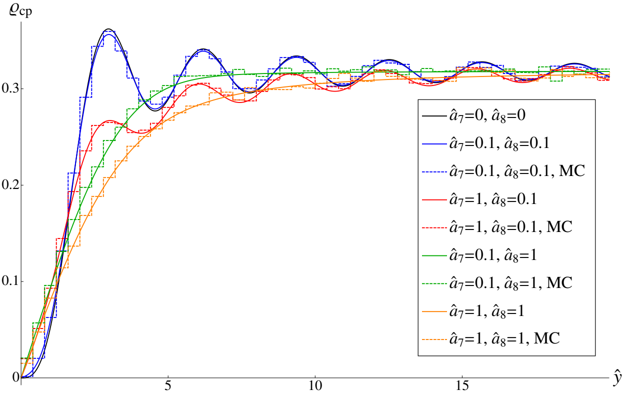
Log-log plots of N_{add} as a function of $\widehat{a}_8$ for $\nu = 0$ (left plot) and $\nu = 2$ (right plot). W_6 has no effect on N_{add}. Saturation around zero due to a non-zero value of $\widehat{a}_7$. For $\widehat{a}_7 = 0$ (lowest curves) the average number of additional real modes behaves like $\widehat{a}_8^{2\nu+2}$. Kieburg, Verbaarschot and SZ (2011)

Distribution of additional real modes for $\hat{a} >> 1$



At $\hat{a} >> 1$ ρ_r develops square root singularities at the boundaries. Finite matrix size+ finite lattice spacing $\rightarrow \rho_r$ has a tail dropping off much faster than the size of the support. The dependence on W_6 and ν is completely lost.

Projected distribution of the complex eigenvalues for $\nu = 1$



The distribution of the complex eigenvalues projected onto the imaginary axis for $\nu = 1$. Notice that $\widehat{a}_6$ does not affect this distribution. The comparison of $\widehat{a}_7 = \widehat{a}_8 = 0.1$ with the continuum result (black curve) shows that ρ_{cp} is still a good quantity to extract the chiral condensate Σ at small lattice spacing.

Distribution of the chiralities over the real eigenvalues of D_W

- Consider the chiral condensate $\Sigma(m) \equiv \left\langle \mathrm{Tr} \frac{1}{D_W + m - i\epsilon\gamma_5} \right\rangle$.
- How does it relate to the spectrum of D_W. The discontinuity of $\Sigma(m)$ across the real axis is given by

$$\rho_\chi(m) \equiv \frac{1}{2\pi i} \left\langle \mathrm{Tr} \left[\frac{1}{(D_W + m) - i\epsilon\gamma_5} - \frac{1}{(D_W + m) + i\epsilon\gamma_5} \right]_{\epsilon \to 0} \right\rangle$$

- $\rho_\chi(m) = \frac{1}{\pi} \left\langle \sum_k \frac{\epsilon \langle k|\gamma_5|k\rangle}{(\lambda_k^5(m))^2 + \epsilon^2} \Big|_{\epsilon \to 0} \right\rangle =$ $\left\langle \sum_{\lambda_k^W \in \mathbb{R}} \delta(\lambda_k^W + \lambda^W) \, \mathrm{sign}(\langle k|\gamma_5|k\rangle) \right\rangle$.

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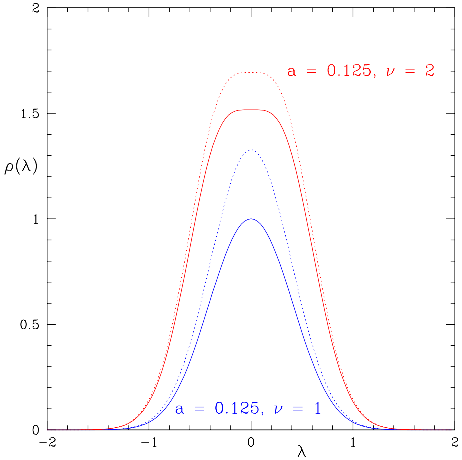
Distribution of the chiralities and inverse chiralities over the real eigenvalues of D_W

- Similarly, $\rho_{\frac{1}{\chi}}(\lambda^W) = \rho_5(\lambda^5 = 0, m; a) = \left\langle \sum_{\lambda_k^W \in \mathbb{R}} \frac{\delta(\lambda_k^W + m)}{|\langle k|\gamma_5|k\rangle|} \right\rangle$.
  Because $|\langle k|\gamma_5|k\rangle| \leq 1$ we have the inequality

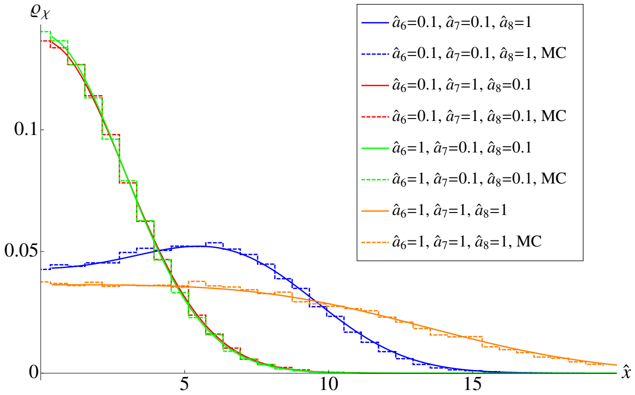
$$\rho_\chi(\lambda^W) \leq \rho_{\rm real}(\lambda^W) \leq \rho_5(\lambda^5 = 0, m = \lambda^W; a)$$

Distribution of the chiralities over the real eigenvalues of D_W



Lower and upper bounds on $\rho_{\mathrm{real}}(\lambda^W)$

Chirality distribution for $\nu = 1$

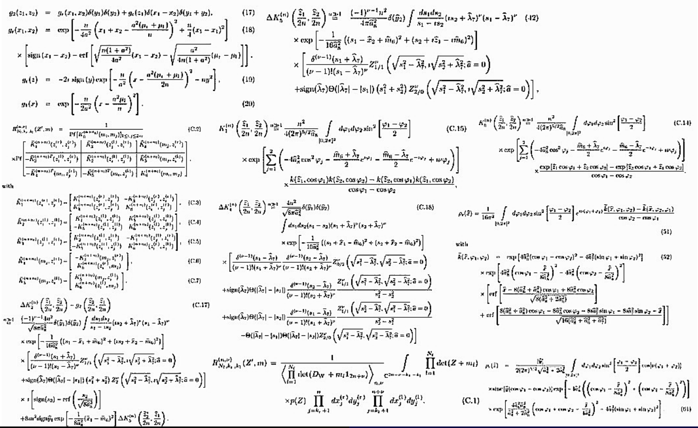


The distribution is symmetric around the origin. At small $\widehat{a}_8$ the distributions for $(\widehat{a}_6, \widehat{a}_7) = (1, 0.1), (0.1, 1)$ are almost the same Gaussian as the analytical result predicts. At large $\widehat{a}_8$ the maximum reflects the predicted square root singularity which starts to build up. We have not included the case $\widehat{a}_{6/7/8} = 0.1$ since it exceeds the other curves by a factor of 10 to 100.

Extracting the LECs of Wilson chPT

Please do not read this

(Figure courtesy of M. Kieburg)



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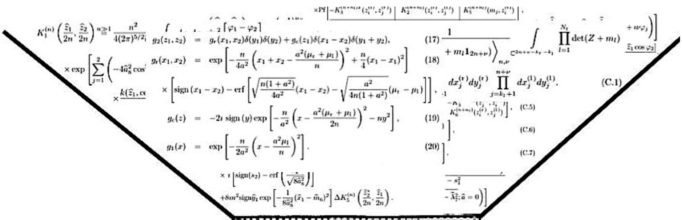

- the average number of the additional real modes for the lowest index:

$$N_{\rm add}^{\nu=0} \overset{\tilde{a}\lesssim 1}{=} 2V\tilde{a}^2(W_8 - 2W_7), \quad (75)$$

- the width of the Gaussian shaped strip of complex eigenvalues:

$$2\sigma \overset{\tilde{a}\lesssim 1}{=} 4\tilde{a}\sqrt{\frac{W_8 - 2W_6}{V\Sigma^2}}, \quad (76)$$

- the variance of the distribution of chirality over the real eigenvalues:

$$\langle (V\Sigma\tilde{x})^2\rangle_{\rho_\chi} \overset{\tilde{a}\lesssim 1}{=} 8V\tilde{a}^2(\nu W_8 - W_6 - W_7), \ \nu > 0. \quad (77)$$

(Figure courtesy of M. Kieburg)

Conclusions

- Studied the effect of the three LECs on the spectrum of D_W.
- W_6 and W_7 can be interpreted as collective fluctuations of the spectrum while W_8 induces interactions among all modes.
- Analytical and numerical results of the eigenvalue densities of D_W
- At small lattice spacing we propose the following quantities for the extraction of LECs

$$\tilde{a}^2 V \begin{bmatrix} 0 & -2 & 1 \\ -2 & 0 & 1 \\ -1 & -1 & 1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} W_6 \\ W_7 \\ W_8 \end{bmatrix} = \frac{\pi^2}{8} \begin{bmatrix} 4N_{\rm add}^{\nu=0}/\pi^2 \\ 2\sigma^2/\Delta^2 \\ \langle \tilde{x}^2 \rangle_{\rho_\chi}^{\nu=1}/\Delta^2 \\ \langle \tilde{x}^2 \rangle_{\rho_\chi}^{\nu=2}/\Delta^2 \end{bmatrix}$$

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for upcoming results ...

Thank you for your attention!

Collaborators:
Mario Kieburg, Universität Bielefeld
Jacobus Verbaarschot, Stony Brook University