Intriguing relations between the LECs of Wilson $\chi$-PT and spectra of the Wilson Dirac operator

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Outline

- Motivation-The Goals
- Wilson Fermions- Wilson $\chi$–PT
- Introduction of the Model
- LECs and the spectrum of $D_W$
- Conclusions and Outlook
Motivation - The Goals

- Facilitate simulations in the deep chiral regime by an exact, analytical understanding of the average behavior of the smallest eigenvalues
- Chiral symmetry breaking from lattice spacing
- Stability of lattice simulations
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Wilson Fermions- Wilson $\chi$–PT
The momentum space propagator (free theory)

\[
D(p)\big|_{m=0}^{-1} = -ia^{-1} \sum_{\mu} \gamma_{\mu} \sin(p_{\mu}a) - i \sum_{\mu} \gamma_{\mu} p_{\mu} \\
= \frac{1}{a^{-2} \sum_{\mu} \sin(p_{\mu}a)^2} a \to 0 \frac{1}{p^2}
\]

In the continuum one pole at \( p = (0, 0, 0, 0) \)

On the lattice additional poles whenever all components are either \( p_{\mu} = 0 \) or \( p_{\mu} = \pi/a \)

Our lattice Dirac operator has 15 unphysical poles (doublers) at \( p = (\pi/a, 0, 0, 0), (0, \pi/a, 0, 0), \ldots, (\pi/a, \pi/a, \pi/a, \pi/a) \)
Nielsen and Ninomiya (1980)

It is not possible to construct a lattice fermion action that is

- Local
- Undoubled
- correct continuum limit
- chirally symmetric $\{D, \gamma_5\} = 0$
Wilson Fermions

Break chiral symmetry explicitly

Wilson (1977)

- add the lattice discretization of the Laplacian \(-\frac{a}{2} \partial_\mu \partial_\mu\)

\[
D(p) = m1 + \frac{i}{a} \sum_{\mu=1}^{4} \gamma_\mu \sin p_\mu a + \frac{1}{a} \sum_{\mu=1}^{4} (1 - \cos p_\mu a)
\]

- for components with \(p_\mu = 0\) it vanishes
- for each component with \(p_\mu = \pi/a\) provides an extra contribution \(2/a\)
- It acts like an additional "effective" mass term so the total mass of the doublers is \(m + 2l/a\)
- in the naive continuum limit \(a \to 0\) the doublers become very heavy and decouple from the theory
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Wilson Chiral Perturbation Theory

- Wilson term breaks $\chi -$ symmetry explicitly
- Lattice spacing effects lead to new terms in $\chi - PT$
  

- $\epsilon -$ regime where in the thermodynamic, chiral and continuum limit $mV\Sigma$, $zV\Sigma$ and $a^2VW_i$ kept fixed

- At order $a^2$ it involves three Low Energy Constants (LECs)

\[
Z_{Nf}(m, z; a) = \int_\mathcal{M} dU \det U e^{-S[U]},
\]

where the action is

\[
S = -\frac{m}{2} \Sigma V \text{tr}(U + U^\dagger) - \frac{z}{2} \Sigma V \text{tr}(U - U^\dagger) + a^2 VW_6[\text{tr}(U + U^\dagger)]^2
\]
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+ a^2 VW_7[\text{tr}(U - U^\dagger)]^2 + a^2 VW_8 \text{tr}(U^2 + U^\dagger^2).
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  \[ \text{Sharpe and Singleton (1998), Rupak and Shoresh (2002), Baer,Rupak and Shoresh (2004)} \]
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Random Matrix Theory

RMT applied to Physics was born in Nuclear Physics

- Description of the statistical properties of excited energy levels in complex nuclei (Wigner (1955))
- Complex systems, very complicated or even unknown dynamics
- Replace the Hamiltonian by a random matrix $H$ with the same GLOBAL symmetries
- Compute observables by averaging over the ensemble
- Identify universal quantities (independent of the probability distribution)
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Replace the Hamiltonian by a random matrix \( H \) with the same GLOBAL symmetries

Compute observables by averaging over the ensemble

Identify universal quantities (independent of the probability distribution)
\[ D_W = \frac{1}{2} \gamma_\mu (\nabla_\mu + \nabla^*_\mu) - \frac{1}{2} a \nabla^*_\mu \nabla_\mu \]

At \( a \neq 0 \) is non-Hermitian but retains \( \gamma_5 \)-Hermiticity
\[ D_W^\dagger = \gamma_5 D_W \gamma_5 \]

Eigenvalues of \( D_W \) because of the \( \gamma_5 \)-Hermiticity occur in complex conjugate pairs or are real.

ONLY eigenvectors corresponding to real eigenvalues have non-vanishing chirality \( \langle k | \gamma_5 | k \rangle \)
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Eigenvalues of $D_W$ with $\nu = 5$

$\hat{a}_8 = 0$

$\hat{a}_8 = 1$
Partition function of $D_W$ with $N_f$ flavors:

$$Z_{N_f}^{RMT,\nu} = \int dD_W \det^{N_f}(D_W + m)P(D_W)$$

$P(D_W) \rightarrow$ is a Gaussian

$$D_W = \begin{pmatrix} aA & W \\ W^\dagger & aB \end{pmatrix} + am_6 + a\lambda_7\gamma_5$$


- $A: n \times n$ Hermitian
- $B: (n + \nu) \times (n + \nu)$ Hermitian
- $W: n \times (n + \nu)$ Complex
- $m_6$ and $\lambda_7$ scalar random variables
- At $a = 0: D_W$ has $\nu$ generic zero modes
- At finite $a$: definition of the index through spectral flow lines or equivalently $\nu = \sum_{\lambda^W_k \in \mathbb{R}} \text{sign}(\langle k|\gamma_5|k\rangle)$

Itoh et al (1987)
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Wilson Dirac operator and RMT

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Schematic spectral flow of $D_5(m)$ (Figure courtesy of Splittorff and Verbaarschot (2010))

Spectral flow of $D_5(m)$ for $0 \leq m \leq 8$ for a single instanton on a $8^4$ lattice. (Figure courtesy of Edwards, Heller, Narayanan (1998))
The Eigenvalue Densities
\[ \hat{\alpha}_6 = \hat{\alpha}_7 = 0.25, \hat{\alpha}_8 = 0.7 \]

\[ \hat{m} = 5.3 \]

\[ \nu = 0 \text{ (top) and } \nu = 1 \text{ (bottom)} \]

(Deuzeman, Wenger and Wuilloud (2011))

\[ \hat{m} = 4.8, \nu = 2 \]

(Damgaard, Heller and Splittorff (2011))
Lattice results vs RMT

The density of real eigenvalues of $D_W$

Damgaard, Heller and Splittorff (2012)

Cumulative eigenvalue distributions of $D_5$ with all $W_{6/7/8}$ included at $\nu = 0$

(Deuzeman, Wenger and Wuilloud (2011))
The effects of $W_6$ and $W_7$ when $W_8 = 0$

- $\hat{a}_6$ and $\hat{a}_7$ introduced through the addition of the Gaussian stochastic variable $\hat{m}_6 + \hat{\lambda}_7\gamma_5$ to $D_W$

- $D = D_W + (m + \hat{m}_6)1 + \hat{\lambda}_7\gamma_5$

- When $\hat{a}_8 = 0$ $D_W$ is anti-Hermitian,

- the eigenvalues of $D_W(\hat{\lambda}_7, \hat{m}_6) = D - m$ are given by

$$\hat{z}_\pm = \hat{m}_6 \pm i\sqrt{\lambda_W^2 - \hat{\lambda}_7^2}$$

where $i\lambda_W$ is an eigenvalue of $D_W$
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The effects of $W_6$ and $W_7$

Schematic plots of the effects of $W_6$ (left plot) and of $W_7$ (right plot). $W_6$ broadens the spectrum parallel to the real axis according to a Gaussian with width $4\hat{a}_6$, but does not change the continuum spectrum in a significant way. When $W_7 \neq 0$ and $W_6 = 0$ the purely imaginary eigenvalues invade the real axis through the origin and only the real (green crosses) are broadened by a Gaussian with width $4\hat{a}_7$. 

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Wilson RMT
Notice that the two curves for $\hat{a}_7 = \hat{a}_8 = 0.1$ (right plot) are two orders smaller than the other curves (left plot). Notice the soft repulsion of the additional real modes from the origin at large $\hat{a}_7$. The parameter $\hat{a}_6$ smooths the distribution.
Log-Log plots of additional real modes vs $\hat{a}$ for $\nu = 0, 2$

Log-log plots of $N_{\text{add}}$ as a function of $\hat{a}_8$ for $\nu = 0$ (left plot) and $\nu = 2$ (right plot). $W_6$ has no effect on $N_{\text{add}}$. Saturation around zero due to a non-zero value of $\hat{a}_7$. For $\hat{a}_7 = 0$ (lowest curves) the average number of additional real modes behaves like $\hat{a}_8^{2\nu+2}$. Kieburg, Verbaarschot and SZ (2011)
At \( \hat{a} \gg 1 \) \( \rho_r \) develops square root singularities at the boundaries. Finite matrix size plus finite lattice spacing \( \rightarrow \rho_r \) has a tail dropping off much faster than the size of the support. The dependence on \( W_6 \) and \( \nu \) is completely lost.
The distribution of the complex eigenvalues projected onto the imaginary axis for \( \nu = 1 \). Notice that \( \hat{a}_6 \) does not affect this distribution. The comparison of \( \hat{a}_7 = \hat{a}_8 = 0.1 \) with the continuum result (black curve) shows that \( \rho_{cp} \) is still a good quantity to extract the chiral condensate \( \Sigma \) at small lattice spacing.
Consider the chiral condensate $\Sigma(m) \equiv \langle \text{Tr} \frac{1}{D_W + m - i\epsilon\gamma_5} \rangle$.

How does it relate to the spectrum of $D_W$. The discontinuity of $\Sigma(m)$ across the real axis is given by

$$\rho_{\chi}(m) \equiv \frac{1}{2\pi i} \left\langle \text{Tr} \left[ \frac{1}{(D_W + m) - i\epsilon\gamma_5} - \frac{1}{(D_W + m) + i\epsilon\gamma_5} \right] \right\rangle_{\epsilon \to 0}.$$

$$\rho_{\chi}(m) = \frac{1}{\pi} \left\langle \sum_k \frac{\epsilon\langle k|\gamma_5|k\rangle}{(\lambda^5_k(m))^2 + \epsilon^2} \right|_{\epsilon \to 0} = \left\langle \sum_{\lambda_k^W \in \mathbb{R}} \delta(\lambda_k^W + \lambda^W) \text{sign}(\langle k|\gamma_5|k\rangle) \right\rangle.$$
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\rho_{\chi}(m) = \frac{1}{\pi} \left\langle \sum_k \frac{\epsilon_k \langle k | \gamma_5 | k \rangle \delta (\lambda_k^W + \lambda^W) \text{sign}(\langle k | \gamma_5 | k \rangle)}{\lambda_k^5(m) + \epsilon^2} \right\rangle_{\epsilon \to 0} = \left\langle \sum_{\lambda_k^W \in \mathbb{R}} \delta (\lambda_k^W + \lambda^W) \text{sign}(\langle k | \gamma_5 | k \rangle) \right\rangle.
\]
Similarly, \( \rho_{\chi}^{\perp}(\lambda^W) = \rho_5(\lambda^5 = 0, m; a) = \left\langle \sum_{\lambda_k^W \in \mathbb{R}} \delta(\lambda_k^W + m) \frac{|\langle k|\gamma_5|k\rangle|}{\langle k|\gamma_5|k\rangle} \right\rangle. \)

Because \( |\langle k|\gamma_5|k\rangle| \leq 1 \) we have the inequality

\[
\rho_{\chi}(\lambda^W) \leq \rho_{\text{real}}(\lambda^W) \leq \rho_5(\lambda^5 = 0, m = \lambda^W; a)
\]
Distribution of the chiralities over the real eigenvalues of $D_W$

Lower and upper bounds on $\rho_{\text{real}}(\lambda^W)$
The distribution is symmetric around the origin. At small $\hat{a}_8$ the distributions for $(\hat{a}_6, \hat{a}_7) = (1, 0.1), (0.1, 1)$ are almost the same Gaussian as the analytical result predicts. At large $\hat{a}_8$ the maximum reflects the predicted square root singularity which starts to build up. We have not included the case $\hat{a}_6/7/8 = 0.1$ since it exceeds the other curves by a factor of 10 to 100.
Extracting the LECs of Wilson chPT
Extracting the LECs of Wilson chPT

Please do not read this

(Figure courtesy of M. Kieburg)
Extracting the LECs of Wilson chPT

\[ K(q, n) \equiv \left( \frac{2\pi}{2n+1} \right)^{2\tau} \frac{1}{2\pi} \sum_{m=1}^{n} \arg \left( \Gamma \left( \frac{1}{2} - \frac{i m}{n+1} \right) \right) \]

The average number of the additional real modes for the lowest index:

\[ N_{\text{add}}^{(2\tau)} \equiv 2VA^2(W_8 - 2W_7) \]  (75)

The width of the Gaussian shaped strip of complex eigenvalues:

\[ 2\sigma \equiv 4\tilde{a} \sqrt{\frac{W_8 - 2W_6}{V\Sigma^2}} \]  (76)

The variance of the distribution of chirality over the real eigenvalues:

\[ \langle (V\Sigma^2)^2 \rangle_{\text{ch}} \equiv 8V\tilde{a}^2(\nu W_8 - W_6 - W_7), \quad \nu > 0. \]  (77)

(Figure courtesy of M. Kieburg)
Conclusions

- Studied the effect of the three LECs on the spectrum of $D_W$.
- $W_6$ and $W_7$ can be interpreted as collective fluctuations of the spectrum while $W_8$ induces interactions among all modes.
- Analytical and numerical results of the eigenvalue densities of $D_W$
- At small lattice spacing we propose the following quantities for the extraction of LECs

\[
\tilde{a}^2 V \begin{bmatrix}
0 & -2 & 1 \\
-2 & 0 & 1 \\
-1 & -1 & 1 \\
-1 & -1 & 2 \\
\end{bmatrix} \begin{bmatrix}
W_6 \\
W_7 \\
W_8 \\
\end{bmatrix} = \frac{\pi^2}{8} \begin{bmatrix}
4N_{\text{add}}^{\nu=0}/\pi^2 \\
2\sigma^2/\Delta^2 \\
\langle \tilde{x}^2 \rangle_{\rho_x}^{\nu=1}/\Delta^2 \\
\langle \tilde{x}^2 \rangle_{\rho_x}^{\nu=2}/\Delta^2 \\
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At small lattice spacing we propose the following quantities for the extraction of LECs

$$\tilde{a}^2 V \begin{bmatrix} 0 & -2 & 1 \\ -2 & 0 & 1 \\ -1 & -1 & 1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} W_6 \\ W_7 \\ W_8 \end{bmatrix} = \frac{\pi^2}{8} \begin{bmatrix} 4N_{\text{add}}^{\nu=0} / \pi^2 \\ 2\sigma^2 / \Delta^2 \\ \langle \tilde{x}^2 \rangle_{\rho_X}^{\nu=1} / \Delta^2 \\ \langle \tilde{x}^2 \rangle_{\rho_X}^{\nu=2} / \Delta^2 \end{bmatrix}$$
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\end{bmatrix}
\]
Stay Tuned!

for upcoming results . . .
Thank you for your attention!

Collaborators:
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