The local CS equation – structure and applications

Boaz Keren-Zur

EPFL

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With R. Rattazzi, F. Baume and L. Vitale.

Characterizing RG flows: The Callan-Symanzik equation

The basic idea:

- Global scale transformation symmetry can be broken explicitly by quantum effects.
- The non-invariance can be compensated by modifications of the parameters.

Implications:

- Scale transformation can be translated into a flow in parameter space.

Characterizing RG flows: Finiteness of T (Brown and Collins 1980)

The basic idea:

- **Local** scale transformations are encoded in $T = T^{\mu}_{\mu}$.
- Compute T in terms of bare composite operators.
- Consistency condition: T is finite.

Implications:

- Non-trivial constraints on the structure of counterterms.

Characterizing RG flows: The Local Callan-Symanzik equation (Osborn 91)

The basic idea:

- **Local** scale transformations can be compensated by local transformations of the parameters.
- Define a generalization of the Weyl symmetry.
- Consistency condition: The symmetry is abelian.

Implications:

- Irreversibility of RG flows (in perturbation theory, 4D, unitary).

The Local Callan-Symanzik symmetry: ingredients

- Local scale transformations implemented using a background metric.

$$\begin{aligned} \eta^{\mu\nu} &\rightarrow g^{\mu\nu}(x) \\ \delta_\sigma g^{\mu\nu}(x) &= 2\sigma(x) g^{\mu\nu}(x) \end{aligned}$$

- The local transformation of the parameters implemented by promoting them to background fields

$$\begin{aligned} \lambda^I &\rightarrow \lambda^I(x) \\ \delta_\sigma \lambda^I(x) &= -\sigma(x)\beta^I(\lambda(x)) \end{aligned}$$

- The symmetry generator:

$$\left(\delta_\sigma g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}(x)} + \delta_\sigma \lambda^I \frac{\delta}{\delta \lambda^I(x)} + \dots\right) = \sigma \left(2 g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}(x)} - \beta^I \frac{\delta}{\delta \lambda^I(x)} + \dots\right)$$

Background fields: a useful concept

- **Sources for renormalized operators**:
  One can define $\mathcal{W}[g, \lambda, \ldots] = -i \log \mathcal{Z}[g, \lambda, \ldots]$
  a renormalized generating functional for correlation functions of composite operators

$$\frac{1}{\sqrt{-g}} \frac{\delta \mathcal{W}}{\delta \lambda^I(x)} = [\mathcal{O}_I(x)] \qquad \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{W}}{\delta g^{\mu\nu}(x)} = [T_{\mu\nu}(x)]$$

- **Spurions**:
  One can enlarge the symmetry of the theory,
  by assigning them with transformation properties.

The background dilaton: a redundant but useful notation

Introduce a background metric $g^{\mu\nu}(x)$,
and use the redundant notation

$$g^{\mu\nu}(x) = e^{2\tau(x)}\bar{g}^{\mu\nu}(x)$$

$\tau(x)$ is a source for T:

$$\frac{\delta}{\delta\tau(x)}\mathcal{W}[g]\Big| \quad = \quad 2g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}(x)}\mathcal{W}[g]\Big| \; = \; \left[T^{\mu}_{\mu}(x)\right]$$

An effective action for the dilaton
$\Rightarrow$ bookkeeping device for n-point functions of $T \equiv T^{\mu}_{\mu}$.

The local Callan-Symanzik equation

Define a symmetry generator

$$\Delta_\sigma^{CS} \quad = \quad \int d^4x\sigma(x)\left(\frac{\delta}{\delta\tau(x)} - \beta^I\frac{\delta}{\delta\lambda^I(x)} + \ldots\right)$$

$\mathcal{W}$ is invariant up to a local anomaly

$$\Delta_\sigma^{CS}\mathcal{W}[g,\lambda,\ldots] \quad = \quad \int dx\sigma\mathcal{A}[g,\lambda,\ldots]$$

- The anomaly $\mathcal{A}$ is the most general scalar which can be written using the sources and their derivatives.
- Can be written in an operator form

$$T(x) \quad = \quad \beta^I\left[\mathcal{O}_I(x)\right] + \ldots$$

Consistency conditions

- Constraints on the coefficients in the symmetry generator

$$\left[\Delta^{CS}_{\sigma_2}, \Delta^{CS}_{\sigma_1}\right] = 0$$

- Constraints on the anomaly coefficients

$$\left[\Delta^{CS}_{\sigma_2}, \Delta^{CS}_{\sigma_1}\right] \mathcal{W} \quad = \quad \Delta^{CS}_{\sigma_2} \left(\int dx_1 \sigma_1 \mathcal{A} \right) - \Delta^{CS}_{\sigma_1} \left(\int dx_2 \sigma_2 \mathcal{A} \right) = 0$$

Consistency conditions - implications

- Example:

$$\mu \frac{d}{d\mu} \tilde{a}(\lambda) = \frac{1}{8} \chi^g_{IJ} \beta^I \beta^J$$

  If χ^g_{IJ} is positive definite – **irreversibility of RG flow**!

- What about the other constraints on the anomaly (there are ~ 10 equations)?

- **New result:**
  a new formulation of the anomaly.
  $\Rightarrow$ most of the other equations do not constrain the RG flow.
  Only one additional non-trivial consistency condition (related to F^2 anomaly).

Dilaton effective action

- The dilaton effective action is a convenient bookkeeping device for correlators of T.
- The local CS equation can be used to rewrite correlators of T:

$$\begin{aligned} \langle T(x)\rangle &= \beta^I \langle \mathcal{O}_I(x)\rangle + \mathcal{A}(x) \\ \langle T(x_n)\ldots T(x_1)\rangle &\sim \beta^{I_m}\ldots\beta^{I_1}\langle \mathcal{O}_{I_m}(x_m)\ldots\mathcal{O}_{I_1}(x_1)\rangle \\ &\quad + \textit{anomaly related contact terms} \end{aligned}$$

- **New result:**
  a systematic approach for this computation.
- Requires the reformulation of the anomaly.

Characterizing RG flows: Dispersion relations for correlators of T

The basic idea:

- Study the response of the system to **local** scale transformations.
  $=$ compute **correlators** of T ("dilaton scattering amplitudes")
- Use analyticity and unitarity to write dispersion relations
  with positivity constraints.

Implications:

- Comparing between two limits of the flow, across non-perturbative regimes
  (weak irreversibility of RG flow - a theorem, KS 2011).
- Constraining the asymptotic limits of perturbative RG flows
  (All perturbative RG flows end in conformal fixed points, LPR, 2011)

Main points of the talk

- The local CS equation can be used to characterize RG flows
  1. by studying the consistency conditions.
  2. as a tool for computing correlators of T.

  There is a lot of "know-how" involved in the process....

- We have improved the formulation of the anomaly, in a way which isolates only the interesting constraints.

- The new formulation allowed us to give analytic expressions for the dilaton effective action off-criticality.

- We proved that the "scattering amplitudes" involved in the a theorem and LPR are indeed insensitive to lower dimension operators.

Outline

Introduction

The local Callan-Symanzik equation

The local Callan-Symanzik anomaly

n point functions of T (dilaton effective action)

The set-up

- Consider a 4D fixed point.
- Put the theory in a curved background metric $g^{\mu\nu}(x)$.
- Assign a dimensionless source $\lambda^I(x)$ to each of the marginal operators.
- Define a renormalized generating functional $\mathcal{W}$.
- Consider a background

$$\begin{aligned} \bar{g}^{\mu\nu} &= \eta^{\mu\nu} \\ \nabla_\mu \lambda^I &= 0 \\ \left|\lambda^I - \lambda^{I*}\right| &\ll 1 \end{aligned}$$

The β functions vanish at $\lambda^I = \lambda^{I*}$

The set-up

- The local CS equation:

$$\int d^4x\sigma \left(\frac{\delta}{\delta\tau(x)} - \beta^I \frac{\delta}{\delta\lambda^I(x)} \right) \mathcal{W}[g,\lambda] = \int dx\sigma\mathcal{A}$$

- In general there are other operators $\mathcal{O}_\alpha(x)$ of dimension d_α.
  We could add sources m^α of dimension $4 - d_\alpha$ to all these operators and write

$$\int d^4x\sigma \left(\frac{\delta}{\delta\tau(x)} - \beta^I \frac{\delta}{\delta\lambda^I(x)} + m^\beta \left(d^\alpha_\beta + \gamma^\alpha_\beta \right) \frac{\delta}{\delta m^\alpha(x)} \right) \mathcal{W}[g,\lambda^I,m^\alpha] = \int dx\sigma\mathcal{A}$$

  In a background where $m^\alpha = 0$ this will have no effect on our computations.

- Except...

The set-up

- When writing the basis of renormalized dimension 4 scalar operators, we must take into account

$$[\mathcal{O}_I(x)]\,,\quad \nabla_\mu\,[J^\mu_A]\,,\quad \nabla^2\,[\mathcal{O}_a]$$

- We therefore have to add more background fields $A^A_\mu(x)$ and $m^a(x)$:

$$\frac{1}{\sqrt{-g}}\frac{\delta\mathcal{W}}{\delta m^a(x)} = [\mathcal{O}_a(x)] \qquad\qquad \frac{1}{\sqrt{-g}}\frac{\delta\mathcal{W}}{\delta A^A_\mu(x)} = \left[J^A_\mu(x)\right]$$

- All derivatives are promoted to be covariant derivatives:

$$\nabla_\mu \;=\; \partial_\mu + A_\mu$$

- evaluate all derivatives in the background

$$\begin{aligned} \bar{g}^{\mu\nu} &= \eta^{\mu\nu} \\ m = A = \nabla_\mu\lambda^I &= 0 \\ \left|\lambda^I - \lambda^{I*}\right| &\ll 1 \end{aligned}$$

- Now we are ready to write the most general symmetry allowed by dimensional analysis:

The local CS symmetry

$$\Delta_\sigma^{CS}(x) \quad = \quad \Delta_\sigma^W(x) - \Delta_\sigma^\beta(x)$$

where

$$\begin{aligned}
\Delta_\sigma^W &= \int d^4x \left[\sigma \frac{\delta}{\delta\tau(x)} \right] \\
\Delta_\sigma^\beta &= \int d^4x \left[\sigma \left(\beta^I \frac{\delta}{\delta\lambda^I(x)} + \rho_I^A \nabla_\mu \lambda^I \frac{\delta}{\delta A_\mu^A(x)} \right) - \nabla_\mu\sigma \left(S^A \frac{\delta}{\delta A_\mu^A(x)} \right) \right. \\
& \quad - \sigma \left(m^b \left(2\delta_b^a + \gamma_b^a\right) + \frac{1}{3}\eta^a R + d_I^a \nabla^2 \lambda^I + \frac{1}{2}\epsilon_{IJ}^a \nabla_\mu \lambda^I \nabla^\mu \lambda^J \right) \frac{\delta}{\delta m^a(x)} \\
& \quad \left. + \nabla_\mu\sigma \left(\theta_I^a \nabla^\mu \lambda^I \frac{\delta}{\delta m^a(x)} \right) - \nabla^2\sigma \left(t^a \frac{\delta}{\delta m^a(x)} \right) \right]
\end{aligned}$$

The local CS equation

$$\boxed{\Delta_\sigma^{CS} \mathcal{W} = \int dx \sigma \mathcal{A}}$$

The local CS equation - operator form

$$\int d^4x \Big[\sigma \left(\frac{\delta}{\delta \tau(x)} - \beta^I \frac{\delta}{\delta \lambda^I(x)} - \rho_I^A \nabla_\mu \lambda^I \frac{\delta}{\delta A_\mu^A(x)} \right) + \nabla_\mu \sigma \left(S^A \frac{\delta}{\delta A_\mu^A(x)} \right)$$
$$+ \sigma \left(m^b \left(2\delta_b^a + \gamma_b^a \right) + \frac{1}{3}\eta^a R + d_I^a \nabla^2 \lambda^I + \frac{1}{2}\epsilon_{IJ}^a \nabla_\mu \lambda^I \nabla^\mu \lambda^J \right) \frac{\delta}{\delta m^a(x)}$$
$$- \nabla_\mu \sigma \left(\theta_I^a \nabla^\mu \lambda^I \frac{\delta}{\delta m^a(x)} \right) + \nabla^2 \sigma \left(t^a \frac{\delta}{\delta m^a(x)} \right) \Big] \mathcal{W} = \int dx \sigma \mathcal{A}$$

The operator form of the equation:

($\sigma(x) = \delta(x)$, flat background)

$$T \quad = \quad \beta^I [\mathcal{O}_I] \quad - S^A \nabla_\mu [J_A^\mu] \quad - t^a \nabla^2 [\mathcal{O}_a]$$

The running of the renormalized operators:

$$\begin{aligned} (\mathcal{D}-4)[\mathcal{O}_I] &= \partial_J \beta^I [\mathcal{O}_J] \; - \rho_I^A \; \nabla_\mu [J_A^\mu] \; - d_I^a \nabla^2 [\mathcal{O}_a] \\ (\mathcal{D}-2)[\mathcal{O}_a] &= \gamma_a^b [\mathcal{O}_b] \end{aligned}$$

(Comment: the currents are renormalized because the symmetry is explicitly broken by the sources)

The local CS symmetry - details

$$\int d^4x \Big[\sigma \left(\frac{\delta}{\delta \tau(x)} - \beta^I \frac{\delta}{\delta \lambda^I(x)} - \rho_I^A \nabla_\mu \lambda^I \frac{\delta}{\delta A_\mu^A(x)} \right) + \nabla_\mu \sigma \left(S^A \frac{\delta}{\delta A_\mu^A(x)} \right)$$

$$+ \sigma \left(m^b \left(2\delta_b^a + \gamma_b^a \right) + \frac{1}{3} \eta^a R + d_I^a \nabla^2 \lambda^I + \frac{1}{2} \epsilon_{IJ}^a \nabla_\mu \lambda^I \nabla^\mu \lambda^J \right) \frac{\delta}{\delta m^a(x)}$$

$$- \nabla_\mu \sigma \left(\theta_I^a \nabla^\mu \lambda^I \frac{\delta}{\delta m^a(x)} \right) + \nabla^2 \sigma \left(t^a \frac{\delta}{\delta m^a(x)} \right) \Big]$$

1. Renormalization and improvement schemes.
2. Ambiguities.
3. Consistency conditions.
4. Transformation properties of functions.

Renormalization scheme

Using a different basis

$$\mathcal{W}[g,\lambda,A,m] = \mathcal{W}'[g,\lambda,A',m']$$

where a new basis of sources can be parameterized by

$$\begin{aligned} A_\mu^{A\prime} &= A_\mu^A + f_I^A \nabla_\mu \lambda^I \\ m^{a\prime} &= m^a + \frac{1}{6} f^a R + \ldots \end{aligned}$$

Corresponds to a change of basis of renormalized operators (scheme)

$$[\mathcal{O}'_I(x)] = \frac{1}{\sqrt{-g}} \frac{\delta \mathcal{W}'}{\delta \lambda^I(x)} = [\mathcal{O}_I(x)] - f_I^A \nabla_\mu [J_A^\mu(x)] + \ldots$$

Also modifies the coefficients of the local CS symmetry.

Two parameters (t^a and θ_I^a) can be set to zero.

Improvement scheme

A theory is defined up to improvement of the energy-momentum tensor

$$\begin{aligned} T_{\mu\nu} &\sim T_{\mu\nu} + \frac{1}{3}\left(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu\right)\mathcal{O}_a \\ T &\sim T + \Box\mathcal{O}_a \end{aligned}$$

In a curved background, this is determined by

$$\mathcal{W} \supset \int \sqrt{-g}d^4xR\mathcal{O}_a$$

This effect is taken into account by

$$T = \beta^I\left[\mathcal{O}_I\right] \quad - S^A\nabla_\mu\left[J^\mu_A\right] \quad \color{red}{-t^a\nabla^2\left[\mathcal{O}_a\right]}$$

t^a describes the choice of "improvement" scheme in the theory.

Ambiguities in the presence of global symmetries

Add the Ward identity to the local CS equation

$$\Delta_\alpha^{Global}\mathcal{W} = \int d^4x\left[\alpha^A(T_A\lambda)^I\frac{\delta}{\delta\lambda^I(x)} - \nabla_\mu\alpha^A\frac{\delta}{\delta A_\mu^A(x)}\right]\mathcal{W} = 0$$

$$\Rightarrow\left(\Delta_\sigma^{CS} - \Delta_\alpha^{Global}\right)\mathcal{W} = \int dx\sigma\mathcal{A}$$

choosing $\alpha^A = \sigma w^A(\lambda)$ we can rewrite the symmetry generator as

$$\begin{aligned}\Delta_\sigma^\beta + \Delta_{\sigma\omega}^{Global} &= \int d^4x\left[\sigma\left(\beta^I + (\omega^A T_A\lambda)^I\right)\frac{\delta}{\delta\lambda^I(x)}\right.\\ &\quad + \sigma\left(\rho_I^A - \partial_I\omega^A\right)\nabla_\mu\lambda^I\frac{\delta}{\delta A_\mu^A(x)} - \nabla_\mu\sigma\left(\left(S^A + \omega^A\right)\frac{\delta}{\delta A_\mu^A(x)}\right) + \ldots\end{aligned}$$

S^A can be set to zero.

Ambiguities in the presence of global symmetries

The β function is ambiguous.

$$\beta^I \to \beta^I + (\omega^A T^A \lambda)^I \qquad \rho_I^A \to \rho_I^A - \partial_I \omega^A \qquad S^A \to S^A + \omega^A$$

Invariant functions

$$\begin{aligned} B^I &= \beta^I - \left(S^A T^A \lambda\right)^I \\ P_I^A &= \rho_I^A + \partial_I S^A \end{aligned}$$

choosing the gauge $\omega^A = -S^A$

$$T = \beta^I \left[\mathcal{O}_I\right] + S^A \nabla_\mu \left[J_A^\mu\right] \qquad \to \qquad T = B^I \left[\mathcal{O}_I\right]$$

$$\text{CFT} \Leftrightarrow \; B^I = 0$$

There are CFTs with non-zero β functions (FGS, 2012).

Consistency conditions

The symmetry is abelian

$$\left[\Delta_{\sigma'}^{CS}, \Delta_{\sigma}^{CS}\right] = 0$$

This leads to three consistency equations:

- Two equations which can be used to eliminate η^a and d_I^a.
- $B^I P_I = 0$

Implications of consistency conditions

If we work in the basis where T is orthogonal to $\nabla_\mu [J^\mu_A]$

$$T = B^I [\mathcal{O}_I]$$

this orthogonality is preserved along the RG flow:

$$\begin{aligned} \mathcal{D}[\mathcal{O}_I] &\supset P_I^A \nabla_\mu [J^\mu_A] \\ \mathcal{D}[T] \sim \mathcal{D}\left[B^I \mathcal{O}_I\right] &\supset \cancel{B^I P_I^A} \nabla_\mu [J^\mu_A] \end{aligned}$$

Conclusion

- It is not necessary to know anything about correlators of J^μ_A in order to compute correlators of T.
- A similar argument can be given for the anomalies - correlators of T are not sensitive to the gauge field appearing in the anomaly.

Transformation properties of functions

$$\begin{aligned}
\Delta_\sigma^{CS}\left(Y_I \nabla^\mu \lambda^I\right) &= \sigma\left(-\mathcal{L}[Y_I]\nabla_\mu \lambda^I\right) + \nabla_\mu \sigma \left(-B^I Y_I\right) \\
\Delta_\sigma^{CS}\left(Y_I \nabla^2 \lambda^I\right) &= \sigma\left(-\mathcal{L}[Y_I]\nabla^2 \lambda^I\right) + \nabla^2 \sigma \left(-B^I Y_I\right) + \dots
\end{aligned}$$

Y_I is an arbitrary function of the sources.

- $\mathcal{L}[\dots]$ is a Lie derivative in parameter space, defined along a direction which describes the RG flow

  $$\mathcal{L}[Y_{IJ\dots}] = B^K \partial_K Y_{IJ\dots} + \gamma_I^K Y_{KJ\dots} + \gamma_J^K Y_{IK\dots} + \dots$$

  where $\gamma_I^J = \partial_I B^J + P_I^A (T_A \lambda)^J$
- The Lie derivative satisfies

  $$B^I \mathcal{L}[Y_I] = \mathcal{L}[B^I Y_I]$$

Dimension 2 covariant functions

$$\Pi^{IJ} = \nabla_\mu\lambda^I\nabla^\mu\lambda^J - B^{(I}\left(U^{-1}\right)_K^{J)}\left(\nabla^2\lambda^K + \frac{1}{6}B^K R\right)$$

$$M^a = m^a - t^a\frac{R}{6} - \frac{1}{2}\theta^a_J\left(U^{-1}\right)_K^J\left(\nabla^2\lambda^K + \frac{1}{6}B^K R\right)$$

These combinations of $(\nabla\lambda)^2,\ \nabla^2\lambda,\ m$ and R transform covariantly under the local CS symmetry:

$$\Delta_\sigma^{CS}\left(Y_{IJ}\Pi^{IJ}\right) = \sigma\left(2Y_{IJ}\Pi^{IJ} - \mathcal{L}[Y_{IJ}]\Pi^{IJ} + Y_{IJ}\gamma^{IJ}_{KL}\Pi^{KL}\right)$$

$$\Delta_\sigma^{CS}(M^a) = \sigma\left(\left(2\delta^a_b + \gamma^a_b\right)M^b + \gamma_{IJ}^{\ a}\Pi^{IJ}\right)$$

No derivatives of σ!

$$U_I^J = \delta_I^J + \partial_I B^J + \frac{1}{2}P_I^A(T_A\lambda)^J$$

$$\gamma^I_{JK} = B^{(I}(U^{-1})^{J)}_L\left(\partial_{(K}\Delta^L_{L)} + P^A_{(K}(T_A)^L_{L)}\right)$$

The local CS symmetry - summary

- By a choice of basis for the renormalized operators, adding Ward identities, and imposing consistency conditions

$$\begin{aligned}\Delta_\sigma^\beta &= \int d^4x \Bigg[\sigma \left(B^I \frac{\delta}{\delta \lambda^I(x)} + P_I^A \nabla_\mu \lambda^I \frac{\delta}{\delta A_\mu^A(x)} \right) \\ &\qquad - \sigma \left(M^b \left(2\delta_b^a + \Gamma_b^a \right) + \frac{1}{2} \epsilon^a_{IJ} \Pi^{IJ} \right) \frac{\delta}{\delta M^a(x)} \Bigg]\end{aligned}$$

- The consistency conditions make sure the correlators of T are independent of $[J_A^\mu]$.
- We defined dimension 2 covariant functions of the sources, Π and M.

Outline

The Weyl anomaly

$$\frac{1}{\sqrt{-g}}\sigma\mathcal{A} \quad = \quad \sigma\left(aE_4 - bR^2 - cW^2 - d\nabla^2 R\right)$$

- $\nabla^2 R$: can be set to zero by adding local terms to the action (not a genuine anomaly).
- E_4: "type A" - vanishes when integrated over space-time.
- W^2: "type B" - does not vanish when integrated over space-time.
- R^2: not allowed due to the WZ consistency condition.

$$\left[\Delta^W_{\sigma_2}, \Delta^W_{\sigma_1}\right]\mathcal{W} \quad = \quad \Delta^W_{\sigma_2}\left(\int dx_1\sigma_1\mathcal{A}\right) - \Delta^W_{\sigma_1}\left(\int dx_2\sigma_2\mathcal{A}\right) \propto b\int \sigma_{[1}\nabla^2\sigma_{2]}R \neq 0$$

In dim reg:

- E_4: "type A" - related to evanescent terms in the effective action.
- W^2: "type B" - related to counterterms with evanescent variations.

The Weyl anomaly

This classification cannot be used in our formalism, because $a(\lambda)E_4$ is not a total derivative.

Generalizing the classification in the presence of background sources:

- "type B": Manifestly consistent
  (variation contains no derivative of σ, so the commutator always vanishes).
- "type A": Not consistent in the presence of background sources,
  unless imposing non-trivial relations between the different anomalies.

The local CS anomaly

$$
\begin{aligned}
\frac{1}{\sqrt{-g}}\Delta_\sigma^{CS}\mathcal{W} \quad = \quad & \sigma\left(aE_4 - cW^2 + \frac{1}{9}bR^2\right) - \nabla^2\sigma\left(\frac{1}{3}dR\right) \\
& +\sigma\Big(\frac{1}{3}\chi^e_I\nabla_\mu\lambda^I\nabla^\mu R + \frac{1}{6}\chi^f_{IJ}\nabla_\mu\lambda^I\nabla^\mu\lambda^J R + \frac{1}{2}\chi^g_{IJ}G^{\mu\nu}\nabla_\mu\lambda^I\nabla_\nu\lambda^J \\
& \quad + \frac{1}{2}\chi^a_{IJ}\nabla^2\lambda^I\nabla^2\lambda^J + \frac{1}{2}\chi^b_{IJK}\nabla_\mu\lambda^I\nabla^\mu\lambda^J\nabla^2\lambda^K \\
& \quad + \frac{1}{4}\chi^c_{IJKL}\nabla_\mu\lambda^I\nabla^\mu\lambda^J\nabla_\nu\lambda^K\nabla^\nu\lambda^L \\
& +\nabla^\mu\sigma\Big(G_{\mu\nu}w_I\nabla^\nu\lambda^I + \frac{1}{3}RY_I\nabla_\mu\lambda^I + S_{IJ}\nabla_\mu\lambda^I\nabla^2\lambda^J + \frac{1}{2}T_{IJK}\nabla_\nu\lambda^I\nabla^\nu\lambda^J\nabla_\mu\lambda^K\Big) \\
& -\nabla^2\sigma\left(U_I\nabla^2\lambda^I + \frac{1}{2}V_{IJ}\nabla_\nu\lambda^I\nabla^\nu\lambda^J\right) \\
& +\sigma\Big(\frac{1}{2}p_{ab}\hat{m}^a\hat{m}^b + \hat{m}^a\left(\frac{1}{3}q_aR + r_{aI}\nabla^2\lambda^I + \frac{1}{2}s_{aIJ}\nabla_\mu\lambda^I\nabla^\mu\lambda^J\right)\Big) \\
& +\nabla_\mu\sigma\left(\hat{m}^a j_{aI}\nabla^\mu\lambda^I\right) - \nabla^2\sigma\left(\hat{m}^a k_a\right)
\end{aligned}
$$

where $\hat{m}^a = m^a - \frac{1}{6}t^aR$

The consistency conditions for the local CS anomaly

The 25 anomaly coefficients are functions of λ,
constrained by ~ 10 differential equations derived from the consistency condition

$$\left[\Delta^{CS}_{\sigma_2}, \Delta^{CS}_{\sigma_1}\right] \mathcal{W} \quad = \quad \Delta^{CS}_{\sigma_2}\left(\int dx_1 \sigma_1 \mathcal{A}\right) - \Delta^{CS}_{\sigma_1}\left(\int dx_2 \sigma_2 \mathcal{A}\right) = 0$$

e.g.,

$$\frac{1}{\sqrt{-g}}\sigma\mathcal{A} \quad = \quad \ldots \sigma\left(\hat{m}^a\left(\frac{1}{3} q_a R + r_{aI} \nabla^2 \lambda^I\right)\right) - \nabla^2\sigma\left(k_a \hat{m}^a\right)\ldots$$

the vanishing of the $\sigma_{[1}\nabla^2\sigma_{2]}\hat{m}^a$ term in the commutator leads to

$$q_a - \frac{1}{2}\left(B^I \partial_I k_a - \gamma_a^b k_b + r_{aI} B^I\right) = 0$$

There are ~ 10 such equations.

The consistency conditions for the local CS anomaly

One of these consistency condition has physical significance.

$$\frac{1}{\sqrt{-g}}\sigma\mathcal{A} \quad = \quad \ldots\sigma\left(aE_4+\frac{1}{2}\chi^g_{IJ}G^{\mu\nu}\nabla_\mu\lambda^I\nabla_\nu\lambda^J\right)+\nabla^\mu\sigma\left(G_{\mu\nu}w_I\nabla^\nu\lambda^I\right)\ldots$$

The vanishing of the $\sigma_{[1}\nabla_\mu\sigma_{2]}G^{\mu\nu}\nabla_\nu\lambda^I$:

$$\mathcal{L}[w_I] \quad = \quad -8\partial_I a+\chi^g_{IJ}B^J$$

multiplying by B^I:

$$\mu\frac{d}{d\mu}\widetilde{a}=B^I\partial_I\tilde{a}=\frac{1}{8}\chi^g_{IJ}B^IB^J$$

where $\tilde{a}=a+\frac{1}{8}B^Jw_J$.
If χ^g_{IJ} is positive definite then we have a function which changes monotonously along the RG flow.

What about all the other equations??

Do they have interesting implications?

Solving the consistency conditions

Step 1: Remove scheme dependent anomalies

- The coefficients $d, U_I, V_{IJ}, S_{(IJ)}, T_{IJK}, k_a, j_{aI}$ can be set to zero.
- Most differential equations are replaced by algebraic constraints!
  e.g.

$$q_a - \frac{1}{2}\left(\cancel{B^I \partial_I k_a - \gamma_a^b k_b} + r_{aI} B^I\right) = 0$$

Step 2: Impose algebraic constraints

- The coefficients $\beta_c, Y_I, \chi_I^e, \chi_{IJ}^f, \chi_{IJ}^a, \chi_{IJK}^b, q_a, r_{aI}$ can be eliminated.
- We find the "generalized Weyl anomaly":

$$\mathcal{A} \quad = \quad \mathcal{A}_{R^2} + \mathcal{A}_{W^2} + \mathcal{A}_{E_4} + \mathcal{A}_{F^2} + \mathcal{A}_{\nabla^2 R}$$

- Only ~ 2 consistency conditions left.

The generalized W^2 anomaly

$$\frac{1}{\sqrt{-g}}\mathcal{A}_{W^2} = -cW^2$$

- The only difference with respect to the Weyl anomaly - c is a function of λ.
- Manifestly consistent (W^2 is invariant, c transforms without derivatives) - type B anomaly.

The generalized R^2 anomaly

$$\frac{1}{\sqrt{-g}}\sigma\mathcal{A}_{R^2} \quad = \quad \sigma\left(\frac{1}{2}b_{ab}M^aM^b + \frac{1}{2}b_{aIJ}M^a\Pi^{IJ} + \frac{1}{4}b_{IJKL}\Pi^{IJ}\Pi^{KL}\right)$$

- The "meaning" of the consistency conditions:
  The most general bilinear scalar constructed from Π and M.
- Manifestly consistent - type B anomaly.
- Unimproved fixed points have an R^2 anomaly

$$\frac{1}{\sqrt{-g}}\mathcal{A}_{R^2}\Big|_{\nabla\lambda=B=m=0} \quad = \quad \frac{1}{72}b_{ab}t^at^bR^2$$

(relevant for a theorem and Buican's conjecture)

The generalized E_4 anomaly

$$\begin{aligned}
\frac{1}{\sqrt{-g}}\sigma\mathcal{A}_{E_4} &= \sigma\left(aE_4 + \chi^g_{IJ}\left(\frac{1}{2}J_{\mu\nu}\nabla^\mu\lambda^I\nabla^\nu\lambda^J - \frac{1}{4}U^I_K\Lambda^K\Lambda^J\right)\right) \\
&+\nabla^\mu\sigma\left(w_I G_{\mu\nu}\nabla^\nu\lambda^I\right) + \frac{1}{2}\partial_{[J}w_{I]}\Lambda^I\left(\Delta^{CS}_\sigma\Lambda^J\right) \\
&+\frac{1}{2}\sigma\overline{\chi}^g_{IJK}\Omega^{IJK}
\end{aligned}$$

- a, χ^g_{IJ} and w_I are related by a differential equation:

$$\mathcal{L}[w_I] = -8\partial_I a + \chi^g_{IJ}B^J$$

This is a genuine constraint on the QFT. Irreversibility!

$$\begin{aligned}
\Lambda^I &= \left(U^{-1}\right)^I_J\left(\nabla^2\lambda^J + \tfrac{1}{6}B^J R\right) & J_{\mu\nu} &= G_{\mu\nu} + \tfrac{R}{6}g_{\mu\nu} \\
\Omega^{IJK} &= \left(\Pi^{IJ} + \tfrac{1}{2}B^{(I}\Lambda^{J)}\right)\Lambda^K & \overline{\chi}^g_{IJK} &= -\partial_{(J}\chi^g_{KI)} + \tfrac{1}{2}\partial_K\chi^g_{IJ}
\end{aligned}$$

The generalized F^2 anomaly

$$
\begin{aligned}
\frac{1}{\sqrt{-g}}\sigma\mathcal{A}_{F^2} \quad = \quad & \sigma\left(\frac{1}{4}\kappa_{AB}F^A_{\mu\nu}F^{B\mu\nu}+\frac{1}{2}\zeta_{AIJ}F^A_{\mu\nu}\nabla^\mu\lambda^I\nabla^\nu\lambda^J\right) \\
& +\nabla^\mu\sigma\left(\eta_{AI}F^A_{\mu\nu}\nabla^\nu\lambda^I\right)+\frac{1}{2}\eta_{A[I}P^A_{J]}\Lambda^I\left(\Delta^{CS}_\sigma\Lambda^J\right) \\
& +\sigma\left(\frac{1}{2}P^A_I\zeta_{AJK}+\eta_{AI}\partial_{[J}P^A_{K]}\right)\Omega^{IJK}
\end{aligned}
$$

- κ_{AB}, ζ_{AIJ} and η_{AI} are related by the equations:

$$
\begin{aligned}
\mathcal{L}[\eta_{AI}] \quad &= \quad \kappa_{AB}P^B_I+\zeta_{AIJ}B^J-\chi^g_{IJ}(T_A\lambda)^J \\
0 \quad &= \quad \eta_{AI}B^I+w_I(T_A\lambda)^I
\end{aligned}
$$

- Resemblance to the E_4 anomaly,
- The Lie derivative of the second equation is the consequence of the other two consistency conditions.
- Is there interesting information in this equation?

The local CS anomaly – summary

- Most of the consistency conditions are eliminated when using covariant functions to write the anomaly.
- One of the remaining equations is related to the irreversibility of the flow, the interpretation of the other is still unclear.
- The CS anomaly in 3 dimensions (Nakayama 2013) can be simplified by using the analogues of Π and M.
- The new form of the anomaly is a good starting point for computing the dilaton effective action.

Outline

n-point functions of T (dilaton effective action)

n-point functions of T:

$$\langle T(x_1)\ldots T(x_n)\rangle = \frac{\delta}{\delta\tau(x_n)}\cdots\frac{\delta}{\delta\tau(x_1)}\mathcal{W}\Big|$$

Bookkeeping device: an effective action for the dilaton

$$\begin{aligned}\Gamma[\tau] &= \sum_{n=0}^{\infty}\frac{1}{n!}\int dx_n\ldots\int dx_1\tau(x_n)\ldots\tau(x_1)\frac{\delta}{\delta\tau(x_n)}\cdots\frac{\delta}{\delta\tau(x_1)}\mathcal{W}\Bigg| \\ &\equiv \sum_{n=0}^{\infty}\frac{1}{n!}\Delta_\tau^W\ldots\Delta_\tau^W\mathcal{W}\Big|\end{aligned}$$

where

$$\Delta_\tau^W = \int d^4x\left[\tau\frac{\delta}{\delta\tau(x)}\right] = \Delta_\tau^{CS}+\Delta_\tau^\beta \qquad \Delta_\tau^\beta = \int d^4x\left[\tau\beta^I\frac{\delta}{\delta\lambda^I(x)}+\ldots\right]$$

1 and 2-point functions of T (dilaton effective action)

$$\Delta_\tau^W \mathcal{W} = \Delta_\tau^\beta \mathcal{W} + \int d^4x \tau \mathcal{A}$$

$$\begin{aligned}
\Delta_\tau^W \Delta_\tau^W \mathcal{W} &= \Delta_\tau^W \Delta_\tau^\beta \mathcal{W} + \Delta_\tau^W \int d^4x \tau \mathcal{A} \\
&= \Delta_\tau^\beta \Delta_\tau^W \mathcal{W} + \left[\Delta_\tau^W, \Delta_\tau^\beta\right] \mathcal{W} + \Delta_\tau^W \int d^4x \tau \mathcal{A} \\
&= \underbrace{\Delta_\tau^\beta \Delta_\tau^\beta \mathcal{W} + \left[\Delta_\tau^W, \Delta_\tau^\beta\right] \mathcal{W}}_{\mathcal{D}_2 \mathcal{W}} + \underbrace{\Delta_\tau^\beta \int d^4x \tau \mathcal{A} + \Delta_\tau^W \int d^4x \tau \mathcal{A}}_{\mathcal{C}_2}
\end{aligned}$$

$\mathcal{D}_2\mathcal{W}$: contribution from the composite operators of the theory.
$\mathcal{C}_2$: anomaly related ultra-local terms.

$$\begin{aligned}
\mathcal{D}_2 \mathcal{W}\Big| &= \int d^4x \sqrt{-g}\, \tau(x) \int d^4y \sqrt{-g}\, \tau(y) B^I(x) B^J(y) \left\langle \mathcal{O}_I(x) \mathcal{O}_J(y) \right\rangle \\
&\quad + \int dx \sqrt{-g}\, \tau^2(x) B^I \partial_I B^J \left\langle \mathcal{O}_J(x) \right\rangle . \\
\mathcal{C}_2 &= \int d^4x \sqrt{-g} \nabla^2 \tau \nabla^2 \tau (2d + B^I U_I)
\end{aligned}$$

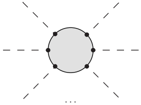
n-point functions of T (dilaton effective action)

$$\underbrace{\Delta_\tau^W \ldots \Delta_\tau^W}_{n} \mathcal{W} \quad = \quad \mathcal{D}_n \mathcal{W} + \mathcal{C}_n$$

where we used the recursive definitions

$$\begin{aligned} \mathcal{D}_n &= \mathcal{D}_{n-1} \Delta_\tau^\beta + \left[\Delta_\tau^W , \mathcal{D}_{n-1} \right] \\ \mathcal{C}_n &= \Delta_\tau^W \mathcal{C}_{n-1} + \mathcal{D}_{n-1} \int dx \tau \mathcal{A} \end{aligned}$$

$\langle T(x_1) \ldots T(x_n) \rangle =$



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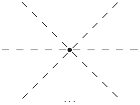


Anomaly related contact terms

Generalized W^2 anomaly

Vanishes in a flat background.

Generalized F^2 anomaly

Vanishes in the $S^A = 0$ gauge, and flat background, due to the consistency condition

$$\begin{aligned} \Delta_\tau^{CS} A_\mu^A \Big| &= -\tau P_I^A \nabla_\mu \lambda^I \Big| = 0 \\ \Delta_\tau^{CS} \Delta_\tau^{CS} A_\mu^A \Big| &= \tau \nabla_\mu \tau B^I P_I^A \Big| = 0 \end{aligned}$$

$$\Delta_\sigma^\beta \;\; = \int d^4x \Big[\sigma \left(B^I \frac{\delta}{\delta \lambda^I(x)} + P_I^A \nabla_\mu \lambda^I \frac{\delta}{\delta A_\mu^A(x)} + \ldots \right)$$

Anomaly related contact terms

Generalized R^2 anomaly

$$\Gamma[\tau] \quad \supset \quad \sum_{k=1}^{\infty} \frac{1}{k!} \widetilde{b}_k \int dx \tau^k \left(\nabla^2 \tau - (\nabla \tau)^2\right)^2$$

Generalized $\nabla^2 R$ anomaly

$$\Gamma[\tau] \quad \supset \quad -\widetilde{d} \int dx \left(\nabla^2 \tau - (\nabla \tau)^2\right)^2$$

Both types of interactions vanish when using the on-shell condition

$$\nabla^2 \phi \propto \nabla^2 \tau - (\nabla \tau)^2 = 0$$

$(e^{-\tau} = 1 + \phi)$

Anomaly related contact terms

Generalized E_4 anomaly

$$\Gamma[\tau] \supset \sum_{k=0}^{\infty} \frac{1}{k!} \left(B^I \partial_I\right)^k \widetilde{a} \int dx \tau^k \left(-4\nabla^2\tau\nabla_\mu\tau\nabla^\mu\tau + 2\left(\nabla_\mu\tau\nabla^\mu\tau\right)^2\right)$$
$$-\frac{3}{8}\sum_{k=0}^{\infty} \frac{1}{k!} \left(B^I \partial_I\right)^{k+1} \left(B^I w_I\right) \int dx \tau^k \left(\nabla_\mu\tau\nabla^\mu\tau\right)^2$$

$$= \widetilde{a} \int dx \left(-4\nabla^2\tau\nabla_\mu\tau\nabla^\mu\tau + 2\left(\nabla_\mu\tau\nabla^\mu\tau\right)^2\right) + O(B^2)$$

The corrections to the fixed point WZ action begin at order $(B^I)^2$.
(need to use the consistency condition $B^I\partial_I\widetilde{a} = \frac{1}{8}\chi^g_{IJ}B^I B^J$).

The couplings of the dilaton to composite operators

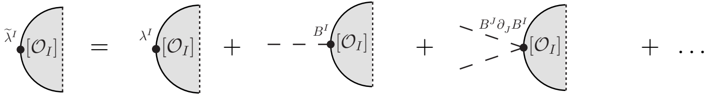
$$\Gamma[\tau] \supset \sum_{n=0}^{\infty} \frac{1}{n!}\mathcal{D}_n \mathcal{W}$$
$$= \exp\left\{ \int d^4x \sum_{k=1}^{\infty} \tau^{k_j} \left(\frac{v^I_{k_j}}{k_j!} \frac{\delta}{\delta\lambda^I(x)} + \ldots \right) \right\} \mathcal{W}$$

v^I_k: the coefficients of a coupling of k dilatons to $[\mathcal{O}_I]$

$$v^I_K = (B^J \partial_J)^{k-1} B^I$$

Agrees with the standard procedure of absorbing the dilaton into the renormalization scale

$$\lambda^I \mathcal{O}_I \rightarrow \widetilde{\lambda}^I(e^\tau \mu)\mathcal{O}_I = \lambda^I \mathcal{O}_I + \tau B^I \mathcal{O}_I + \frac{\tau^2}{2} B^J \partial_J B^I \mathcal{O}_I \ldots$$



The couplings of the dilaton to composite operators

Coupling to dim 3 vectors

These couplings are eliminated when working in the "gauge" $S^A = 0$, using B^I instead of β^I.

Coupling to dim 2 operators

Derivative couplings. More complicated...

$$
\begin{aligned}
\Gamma[\phi] \quad \supset \quad \exp\Big\{ \int d^4x \Big(& - \phi\, B^I \frac{\delta}{\delta \lambda^I(x)} + \frac{\phi^2}{2} \left(B^J \left(\delta^I_J + \partial_J B^I \right) \frac{\delta}{\delta \lambda^I(x)} + \frac{1}{2} B^J \theta^a_J \nabla^2 \frac{\delta}{\delta m^a(x)} \right) \\
& + (1-\phi) \nabla^2 \phi\, t^a \frac{\delta}{\delta m^a(x)} \\
& - \phi \nabla^2 \phi \left(2\eta^a + B^I \left(\frac{1}{2} \theta^a_I - \partial_I t^a \right) \right) \frac{\delta}{\delta m^a(x)} + \dots \Big) \Big\} \mathcal{W} \Big|
\end{aligned}
$$

$(e^{-\tau} = 1 + \phi)$

Notice the importance of the on-shell condition $\nabla^2 \phi = 0$!
Eliminates t^a which could be of order 1.

The dilaton effective action – summary

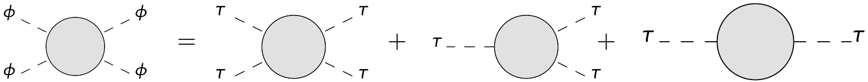
- We have a systematic approach for writing the dilaton effective action as a sum of ultra local term + correlators of composite operators.

- The on-shell condition cleans the dilaton effective action from contributions dependent on renormalization and improvement schemes.

Example - Dilaton scattering and irreversibility of RG flow

Consider the 4 point function of ϕ with on-shell kinematics

$$\phi(x) = e^{-\tau(x)} - 1 \qquad , \qquad A(s) = \frac{\delta}{\delta\phi}\frac{\delta}{\delta\phi}\frac{\delta}{\delta\phi}\frac{\delta}{\delta\phi}\mathcal{W}$$



- Close enough to the fixed point B^I is a good expansion parameter.
- The leading contribution

$$A(s) \quad \propto \quad s^2 \left(\left(\widetilde{a} + O(B^2) \right) + \left(\frac{1}{2} B^I B^J \mathcal{G}_{IJ} + O(B^3) \right) \ln s/\mu^2 \right)$$

where $\mathcal{G}_{IJ}$ is a matrix in parameter space related to the 2 point functions $\langle \mathcal{O}_I \mathcal{O}_J \rangle$ and $\theta^a_I \theta^b_J \langle \mathcal{O}_a \mathcal{O}_b \rangle$.

Example: Dilaton scattering and irreversibility of RG flow

$$A(s) \quad \propto \quad s^2 \left(\left(\widetilde{a} + O(B^2) \right) + \left(\frac{1}{2} B^I B^J \mathcal{G}_{IJ} + O(B^3) \right) \ln s/\mu^2 \right)$$

- This expression is non-trivial:
  1. The corrections to $\widetilde{a}$ begin at order B^2.
  2. All the non-local contributions begin at order B^3.
- In a unitary theory $\mathcal{G}_{IJ}$ is positive definite.
- The amplitude is independent of μ:

$$0 \quad = \quad \mu \frac{d}{d\mu} A(s) = \mu \frac{d}{d\mu} \widetilde{a} - B^I B^J \mathcal{G}_{IJ} + O(B^3)$$

Conclusion:

In a unitary theory, the change in $\widetilde{a}(\lambda)$ is monotonous
$\Rightarrow$ Irreversibility of RG flow.

Main points of the talk

- The local CS equation can be used to characterize RG flows
  1. by studying the consistency conditions.
  2. as a tool for computing correlators of T.

  There is a lot of "know-how" involved in the process....

- We have improved the notations of the formalism, and the formulation of the anomaly, in a way which isolates only the interesting constraints.

- The new formulation allowed us to give analytic expressions for the dilaton effective action off-criticality, in terms of the coefficients in the equation.

- We proved that the "scattering amplitudes" involved in the a theorem and LPR are indeed insensitive to lower dimension operators.

Open questions

- SUSY
- Consistency conditions in the presence of chiral anomalies.
- What can we learn from the constraints on the F^2 anomaly?
- Can we say something unrelate to irreversibility?
  e.g., can we constrain accidental symmetries?

Thank you