QCD calculations of radiative B decays

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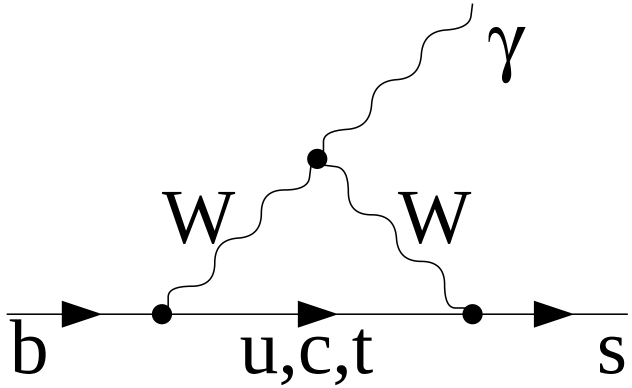
1. **Introduction**
2. **Calculation of the Wilson coefficients ($\mathcal{O}(\alpha_s^2)$ complete).**
3. **The issue of normalization**
4. **Non-applicability of the existing MSOPE results at moderate E_γ**
5. **Missing ingredients at $\mathcal{O}(\alpha_s^2)$**
6. **Summary**

Motivation for precision studies of the $b \to s\gamma$ transition.

A sample SM diagram:



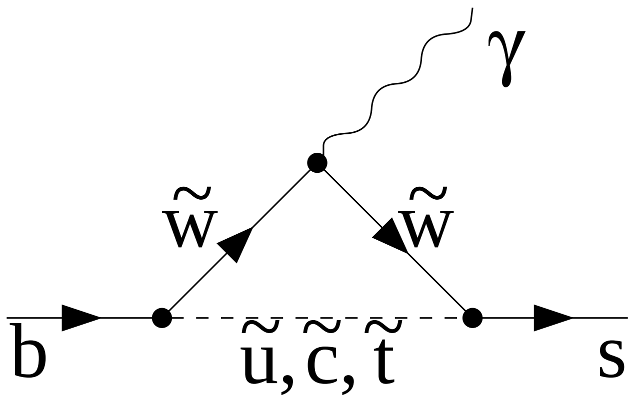
The u, c, t quarks are not degenerate at all:
$(m_t > M_W, \quad m_u, m_c \ll M_W)$

$\Rightarrow$ **No GIM suppression by mass ratios**

$$\mathrm{BR} \simeq 3.2 \times 10^{-4} \simeq 0.14 \, \frac{\alpha_{\mathrm{em}}}{\pi}.$$

$\Rightarrow$ **Large statistics, because** $\sim 10^9 \;\; b\bar{b}$ **pairs have already been produced at the** B**-factories**

A sample MSSM diagram:



Roughly: $\Delta^{\mathrm{SUSY}}\mathrm{BR} \sim \left(\frac{100GeV}{m_{\mathrm{squark}}}\right)^2 BR^{\mathrm{SM}}$

Likely: $m_{\mathrm{squark}} \sim$ **(a few hundred GeV)**

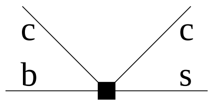
$\Rightarrow$ **A few % effects in the BR are likely.**

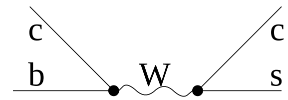
$\Rightarrow$ **Precise SM calculations are necessary.**

At present, the uncertainty in $BR[\bar{B} \to X_s\gamma]$ amounts to around $\pm 7\%$, both on the experimental and the theoretical sides (for $E_\gamma > 1.6$ GeV).

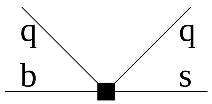
Resummation of $\left(\alpha_s \ln M_W^2/m_b^2\right)^n$ **is most conveniently performed in the framework of an effective theory that arises from the SM after decoupling of the heavy electroweak bosons and the top quark. The Lagrangian of such a theory reads:**

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\rm QCD\times QED}(u,d,s,c,b) + \frac{4G_F}{\sqrt{2}} V_{ts}^* V_{tb} \sum_{i=1}^{8} C_i(\mu)\, Q_i + \left(\begin{array}{c}\textbf{higher-dimensional,}\\ \textbf{on-shell vanishing,}\\ \textbf{evanescent}\end{array}\right).$$

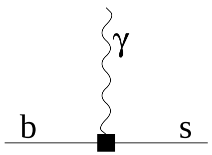
$Q_{1,2}$ = 

$= (\bar{s}\Gamma_i c)(\bar{c}\Gamma'_i b)$, **from** 

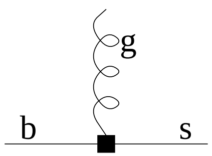
, $|C_i(m_b)| \sim 1$

$Q_{3,4,5,6}$ = 

$= (\bar{s}\Gamma_i b)\Sigma_q(\bar{q}\Gamma'_i q)$, $|C_i(m_b)| < 0.07$

Q_7 = 

$= \frac{e m_b}{16\pi^2}\, \bar{s}_L \sigma^{\mu\nu} b_R F_{\mu\nu}$, $C_7(m_b) \simeq -0.3$

Q_8 = 

$= \frac{g m_b}{16\pi^2}\, \bar{s}_L \sigma^{\mu\nu} T^a b_R G^a_{\mu\nu}$, $C_8(m_b) \simeq -0.15$

Three steps of the calculation:

Matching: **Evaluating** $C_i(\mu_0)$ **at** $\mu_0 \sim M_W$ **by requiring equality of the SM and the effective theory Green functions.**

Mixing: **Deriving the effective theory Renormalization Group Equations** $\left(C_j^{\rm bare} = C_i Z_{ij}\right)$ **and evolving** $C_i(\mu)$ **from** μ_0 **to** $\mu_b \sim m_b$.

Matrix elements: **Evaluating the on-shell amplitudes at** $\mu_b \sim m_b$.

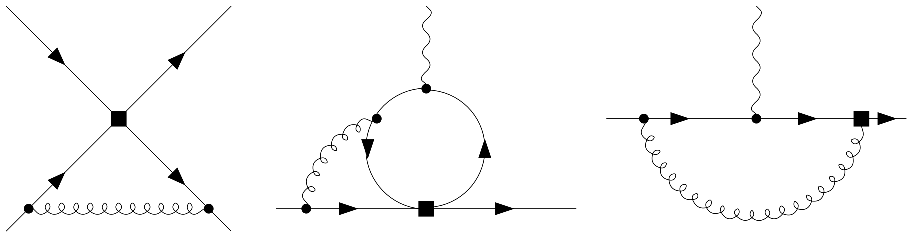
Resummation of large logarithms $\left(\alpha_s \ln \frac{M_W^2}{m_b^2}\right)^n$ in the $b \to s\gamma$ amplitude.

RGE for the Wilson coefficients: $$\mu \frac{d}{d\mu} C_j(\mu) = C_i(\mu)\gamma_{ij}(\mu)$$

The anomalous dimension matrix γ_{ij} is found from the effective theory renormalization constants, e.g.:

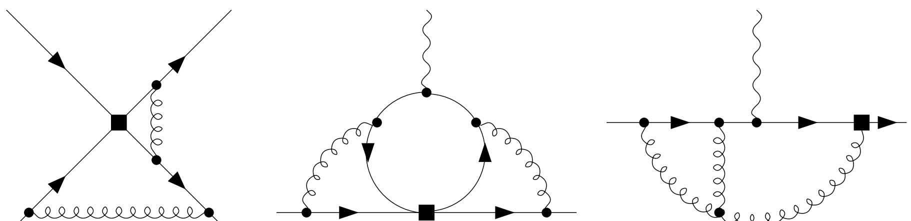
Z_{22} Z_{27} Z_{87}

LO



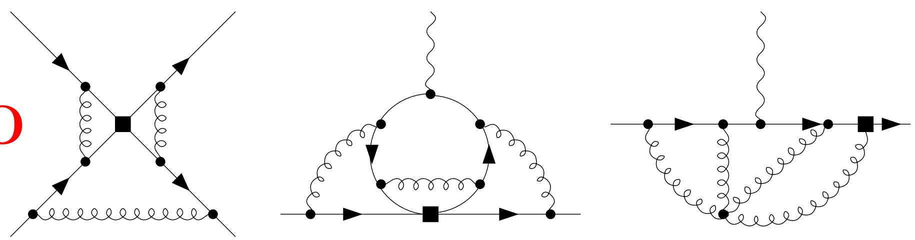
[Gaillard, Lee, 1974] [Altarelli, Maiani, 1974] — [Grinstein *et al.*, 1990] — [Shifman *et al.*, 1978] [Grigjanis *et al.*, 1988]

NLO



[Altarelli *et al.*, 1981] [Buras, Weisz, 1990] — [Chetyrkin, MM, Münz, 1997] — [MM, Münz, 1995]

NNLO

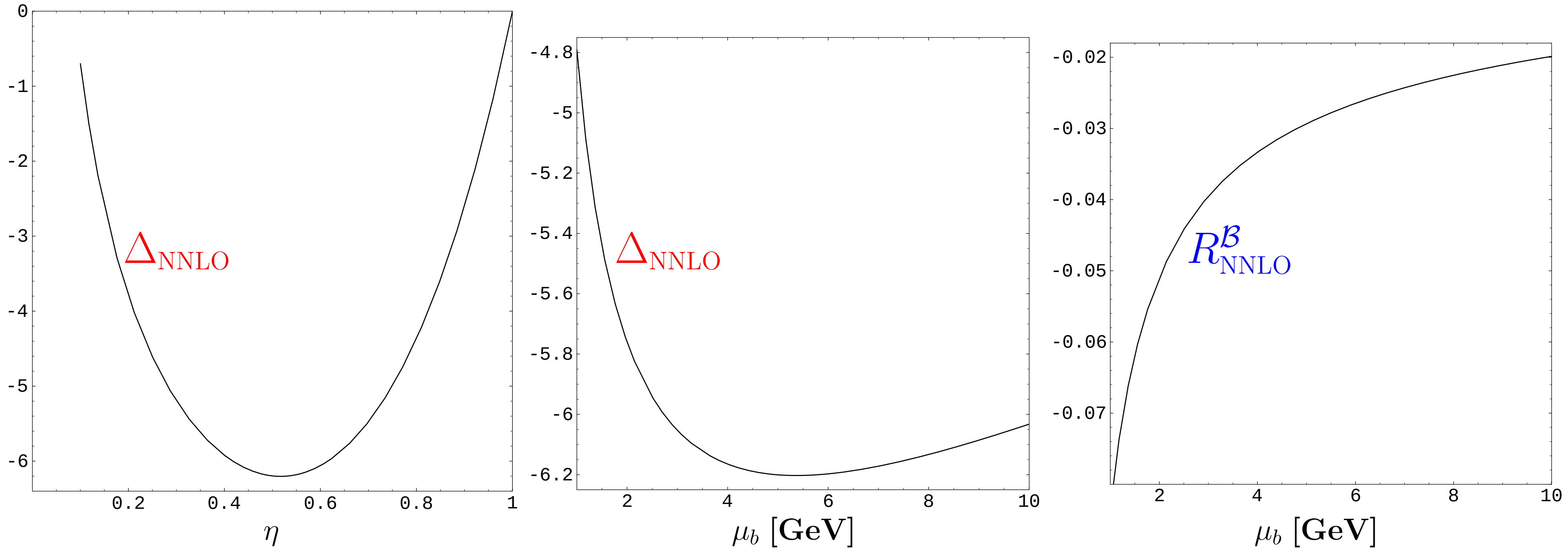


[Gorbahn, Haisch, 2004] — [Czakon, Haisch, MM, 2006] — [Gorbahn, Haisch, MM, 2005]

$\sim 2 \times 10^4$ **diagrams,**
-4% **effect in the BR**

All the Wilson coefficients $C_1(\mu_b), \ldots, C_8(\mu_b)$ are now known at the NNLO in the SM.

Numerical effect of the 4-loop mixing at the NNLO



$$R^{\mathcal{B}}_{\rm NNLO} \equiv \frac{\mathcal{B}_{\rm NNLO} - \mathcal{B}^{\rm 4L\to 0}_{\rm NNLO}}{\mathcal{B}_{\rm LO}} = \left(\frac{\alpha_s(\mu_b)}{\pi}\right)^2 \Delta_{\rm NNLO}$$

$$\Delta_{\rm NNLO} = \frac{C_7^{(2)\rm eff}(\mu_b) - \left[C_7^{(2)\rm eff}(\mu_b)\right]^{\rm 4L\to 0}}{8\, C_7^{(0)\rm eff}(\mu_b)} = \frac{h_1^{(2)}\eta^{a_1+2} + h_2^{(2)}\eta^{a_2+2} + \sum_{i=3}^{8} h_i^{(2)}\eta^{a_i}}{\eta^{a_2} C_7^{(0)}(\mu_0) + \frac{8}{3}\left(\eta^{a_1} - \eta^{a_2}\right) C_8^{(0)}(\mu_0) + \sum_{i=1}^{8} h_i^{(0)}\eta^{a_i}},$$

$$\eta = \alpha_s(\mu_0)/\alpha_s(\mu_b)$$

The weak radiative B-decay branching ratio:

$$\mathcal{B}(\bar{B} \to X_s \gamma)_{E_\gamma > E_0} = \mathcal{B}(\bar{B} \to X_c e \bar{\nu})_{\exp} \left| \frac{V_{ts}^* V_{tb}}{V_{cb}} \right|^2 \frac{6\alpha_{\rm em}}{\pi\, C} \left[\underbrace{P(E_0)}_{\text{pert.}} + \underbrace{N(E_0)}_{\text{non-pert}} \right]$$

$$N(E_0) = \mathcal{O}\left(\frac{\Lambda^2}{m_c^2}, \alpha_s \frac{\Lambda}{m_b}\right) \quad \sim 3\%!, \sim 5\%?$$

$$\frac{\Gamma[b \to X_s \gamma]_{E_\gamma > E_0}}{|V_{cb}/V_{ub}|^2\, \Gamma[b \to X_u e \bar{\nu}]} = \left| \frac{V_{ts}^* V_{tb}}{V_{cb}} \right|^2 \frac{6\alpha_{\rm em}}{\pi}\, P(E_0),$$

The semileptonic phase-space factor:

$$C = \left| \frac{V_{ub}}{V_{cb}} \right|^2 \frac{\Gamma[\bar{B} \to X_c e \bar{\nu}]}{\Gamma[\bar{B} \to X_u e \bar{\nu}]}$$

$$C = \begin{cases} 0.582 \pm 0.016, & \textbf{C. W. Bauer } \textit{et el.}\textbf{, hep-ph/0408002,} \quad \textbf{1S scheme,} \\ 0.546^{+0.023}_{-0.033}, & \textbf{P. Gambino and P. Giordano, arXiv:0805.0271,} \quad \textbf{kinetic scheme.} \end{cases}$$

$$\overline{m}_c(\overline{m}_c) = \begin{cases} 1.224 \pm 0.057, & \textbf{1S scheme,} \\ 1.267 \pm 0.056, & \textbf{kinetic scheme.} \end{cases}$$

$\frac{\partial}{\partial m_c} P(E_0) < 0 \quad \Rightarrow$ **The differences tend to cancel in the radiative branching ratio.**

The final result of the SM calculation:

$$\mathcal{B}(\bar{B} \to X_s\gamma)^{\rm NNLO}_{E_\gamma > 1.6\ {\rm GeV}} = \begin{cases} (3.15 \pm 0.23) \times 10^{-4}, & \text{hep-ph/0609232, using the 1S scheme,} \\ (3.26 \pm 0.24) \times 10^{-4}, & \text{following the kin scheme analysis of arXiv:0805.0271, but } \overline{m}_c(\overline{m}_c)^{\rm 3loop} \text{ rather than } \overline{m}_c(\overline{m}_c)^{\rm 1loop} \text{ in } P(E_0). \end{cases}$$

Contributions to the total uncertainty:

5% non-perturbative $\mathcal{O}\left(\alpha_s \frac{\Lambda}{m_b}\right)$ $\Rightarrow$ Dedicated analysis necessary

See S.J. Lee, M. Neubert, G. Paz, hep-ph/0609224 $\longrightarrow -1.5\%$.

3% parametric $\left(\alpha_s(M_Z),\ \mathcal{B}^{\rm exp}_{\rm semileptonic},\ m_c\ \&\ C,\ \ldots\right)$

2.0% (for $\alpha_s(M_Z)$), 1.6% (for $\mathcal{B}^{\rm exp}_{\rm semileptonic}$), 1.1% (1S) / 2.5% (kin) (for m_c & C)

3% m_c-interpolation ambiguity $\Rightarrow$ Complete three-loop on-shell matrix element calculation even for $m_c = 0$ should help a lot. Work in progress by R. Boughezal, M. Czakon, T. Schutzmeier.

3% higher order $\mathcal{O}(\alpha_s^3)$ $\Rightarrow$ This uncertainty will stay with us.

Currently known contributions to $\mathcal{B}(\bar{B} \to X_s\gamma)$ that have not been included in the estimate $\underset{(\pm 7.3\%)}{(3.15 \pm 0.23) \times 10^{-4}}$ in hep-ph/0609232:

- **New/old large-β_0 bremsstrahlung effects**
  [Ligeti, Luke, Manohar, Wise, 1999]
  [Ferroglia, Haisch, 2007, to be published] $\Rightarrow$ $+2.0\%$ **in the BR**

- **Four-loop mixing into the $b \to sg$ operator Q_8**
  [Czakon, Haisch, MM, hep-ph/0612329] $\Rightarrow$ -0.3% **in the BR**

- **Effects of m_c and m_b in loops on gluon lines**
  [Asatrian, Ewerth, Gabrielyan, Greub, hep-ph/0611123]
  [Boughezal, Czakon, Schutzmeier, arXiv:0707.3090] $\Rightarrow$ $+1.6\%$ **in the BR**
  [Pak, Czarnecki, arXiv:0803.0960]
  [Ewerth, arXiv:0805.3911]

- **Non-perturbative $\mathcal{O}\left(\alpha_s \frac{\Lambda}{m_b}\right)$ effects in the term $\sim C_7 C_8$**
  [Lee, Neubert, Paz, hep-ph/0609224] $\Rightarrow$ -1.5% **in the BR**

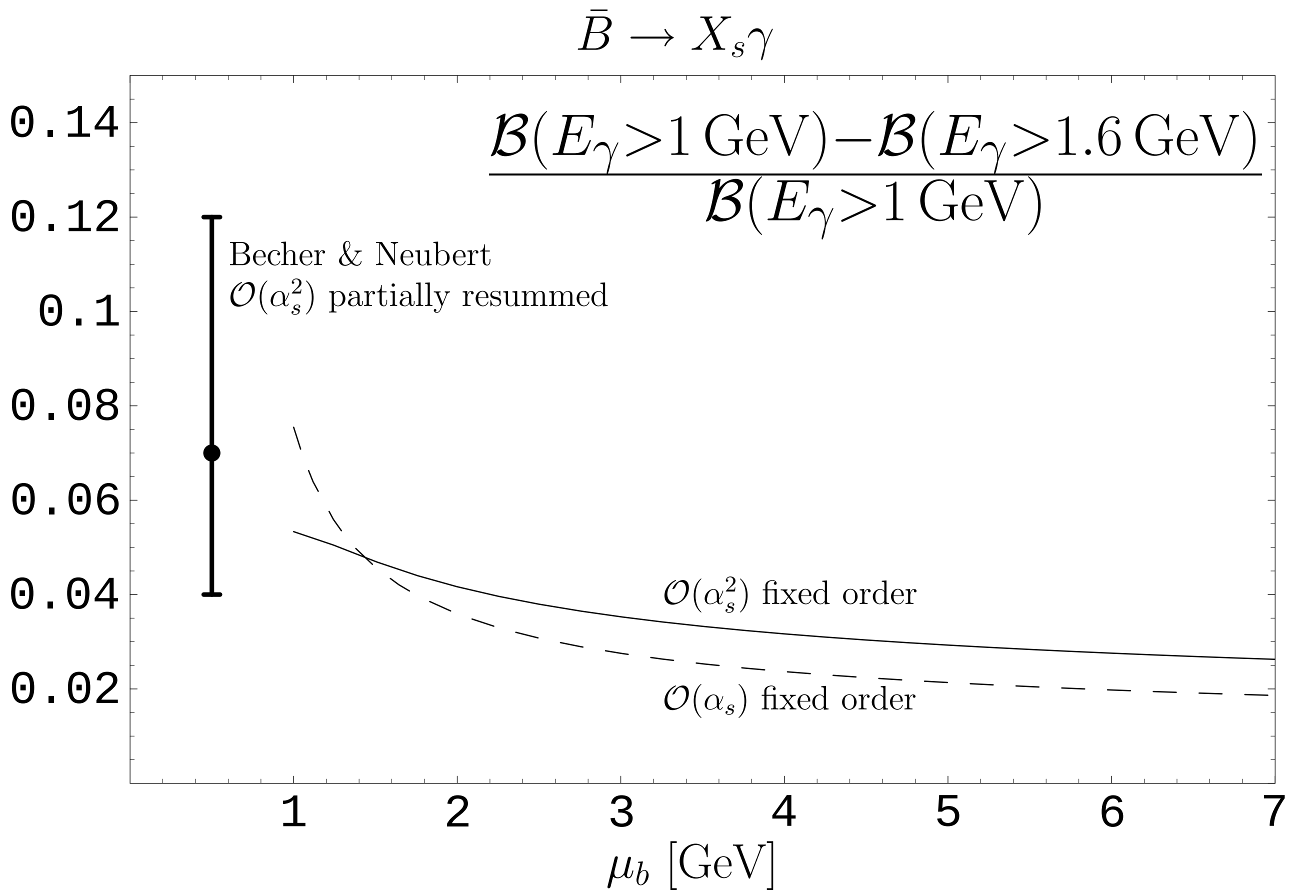
Total: $+1.8\%$ **in the BR**

Comments on the Multi-Scale OPE (MSOPE) calculation by T. Becher and M. Neubert, PRL 98 (2007) 022003 [hep-ph/0610067].

	$\mathcal{B}(E_\gamma > 1GeV)$	$\mathcal{B}(E_\gamma > 1.6GeV)$
hep-ph/0609232 ("fixed order")	3.27×10^{-4}	3.15×10^{-4}
hep-ph/0610067 ("MSOPE")	3.27×10^{-4} **(adopted from above)**	3.05×10^{-4}

before adding the -1.5% of $\mathcal{O}(\alpha_s \Lambda/m_b)$.

There is almost a factor-of-two difference in:



For simplicity, let us set $C_i(\mu_b) \to 0$ for $i \neq 7$. Then, in the "fixed order":

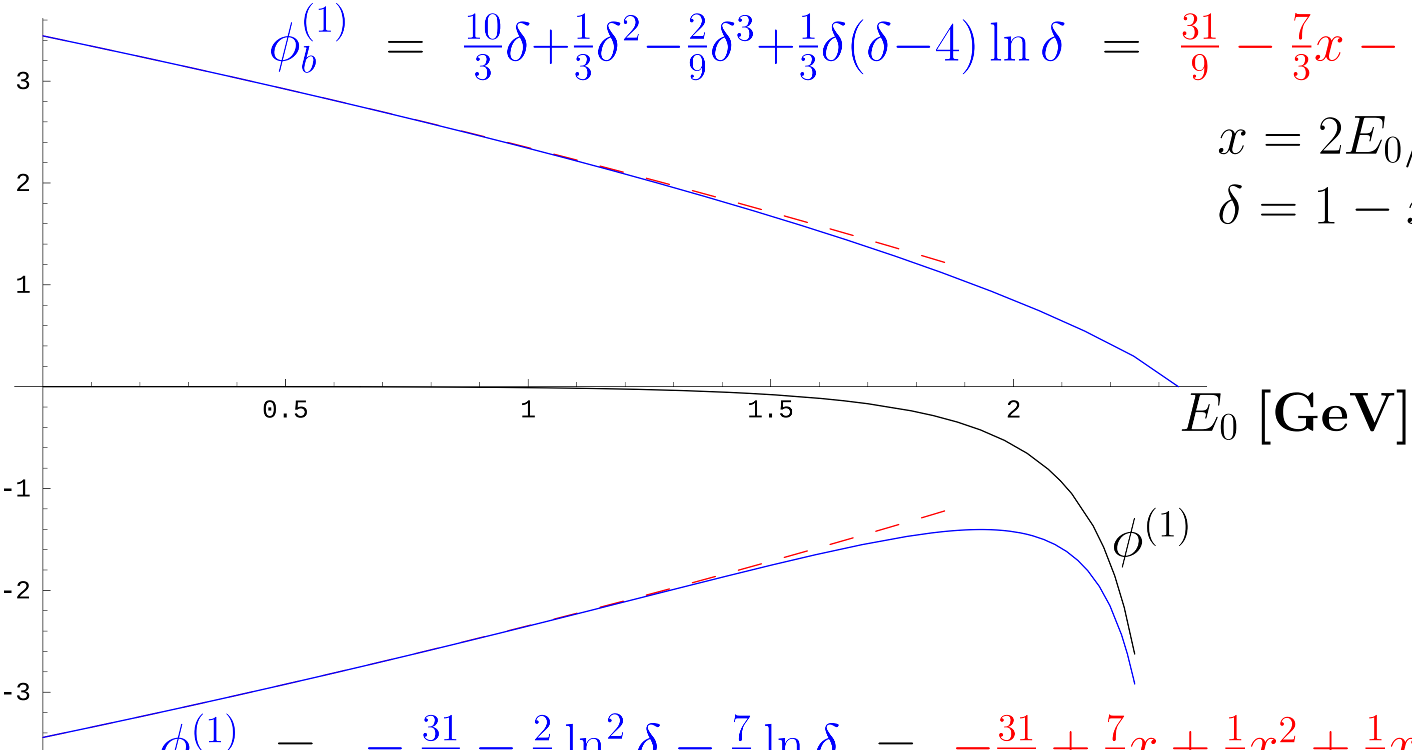
$$\mathcal{B}(E_\gamma > E_0)/\mathcal{B}_{\rm total} = 1 + \frac{\alpha_s(\mu_b)}{\pi}\phi^{(1)}(E_0) + \left(\frac{\alpha_s(\mu_b)}{\pi}\right)^2 \phi^{(2)}(E_0) + \ldots$$

$$\phi^{(1)}(E_0) = \phi_a^{(1)}(E_0) + \phi_b^{(1)}(E_0)$$

$$\phi_b^{(1)} = \tfrac{10}{3}\delta + \tfrac{1}{3}\delta^2 - \tfrac{2}{9}\delta^3 + \tfrac{1}{3}\delta(\delta-4)\ln\delta = \tfrac{31}{9} - \tfrac{7}{3}x - \tfrac{1}{2}x^2 - \tfrac{1}{9}x^3 - \tfrac{5}{36}x^4 + \mathcal{O}(x^5)$$

$$x = 2E_0/m_b$$

$$\delta = 1 - x$$



$$\phi_a^{(1)} = -\tfrac{31}{9} - \tfrac{2}{3}\ln^2\delta - \tfrac{7}{3}\ln\delta = -\tfrac{31}{9} + \tfrac{7}{3}x + \tfrac{1}{2}x^2 + \tfrac{1}{9}x^3 - \tfrac{1}{36}x^4 + \mathcal{O}(x^5)$$

Terms up to $\mathcal{O}(x^3)$ must cancel out in $\phi_a^{(1)} + \phi_b^{(1)}$. **In the current MSOPE results, the higher-order corrections to $\phi_a^{(1)}$ are resummed, but $\phi_b^{(1)}$ is retained in the "fixed order".**

$\Rightarrow$ These results are unreliable for 1 GeV $< E_0 <$ 1.6 GeV.

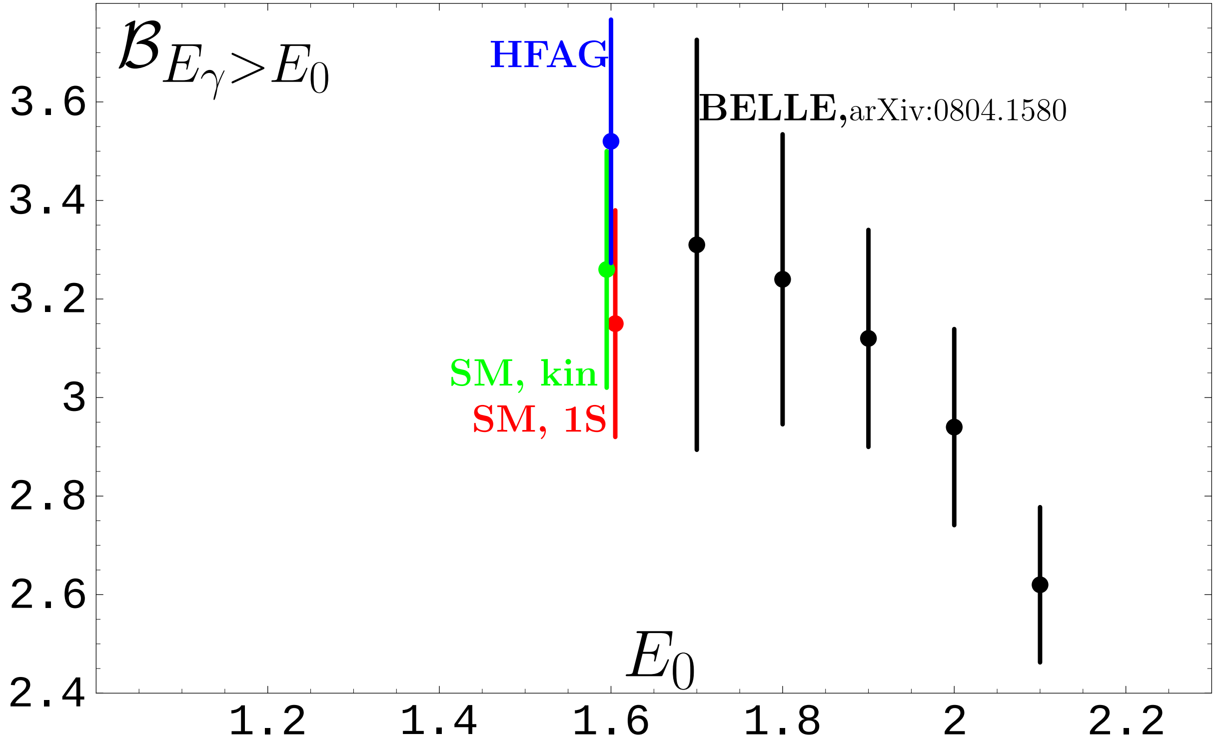
The SM result:

$$\mathcal{B}(\bar{B} \to X_s\gamma)^{\rm NNLO}_{E_\gamma>1.6\ {\rm GeV}} = \begin{cases} (3.15 \pm 0.23) \times 10^{-4}, & \text{hep-ph/0609232, using the 1S scheme,} \\ (3.26 \pm 0.24) \times 10^{-4}, & \text{following the kin scheme analysis of arXiv:0805.0271, but } \overline{m}_c(\overline{m}_c)^{\rm 3loop} \text{ rather than } \overline{m}_c(\overline{m}_c)^{\rm 1loop} \text{ in } P(E_0). \end{cases}$$

agrees within $\sim 1\sigma$ with the current experimental average

(Belle, Babar, Cleo $\longrightarrow$ HFAG)

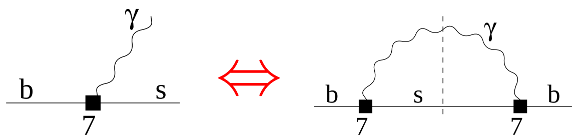
$$\mathcal{B}(\bar{B} \to X_s\gamma)_{E_\gamma>1.6\ {\rm GeV}} = (3.52 \pm \underbrace{0.23}_{\text{stat \& syst}} \pm \underbrace{0.09}_{\text{theory}}) \times 10^{-4}.$$



Missing ingredients in the perturbative NNLO matrix elements

$$\Gamma(b \to X_s^{\text{parton}}\gamma)_{E_\gamma > E_0} = \frac{G_F^2 m_b^5 \alpha_{\text{em}}}{32\pi^4} |V_{ts}^* V_{tb}|^2 \sum_{i,j=1}^{8} C_i(\mu_b) C_j(\mu_b) G_{ij}(E_0, \mu_b)$$

LO: $G_{ij} = \delta_{i7}\delta_{j7}$



$|C_{1,2}(\mu_b)| \sim 1$, $|C_{3,4,5,6}(\mu_b)| < 0.07$,
$C_7(\mu_b) \sim -0.3$, $C_8(\mu_b) \sim -0.15$.

NLO: **The most important** G_{ij} $(i, j = 1, 2, 7, 8)$ **are known since 1996.**

[Greub, Hurth, Wyler, 1996]
[Ali, Greub, 1991-1995]

The remaining G_{ij} **are known since 2002.**

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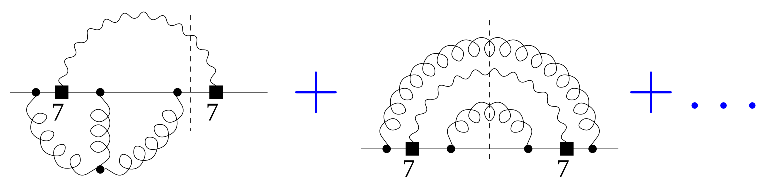
NNLO: **Only** $i, j = 1, 2, 7, 8$ **have been considered so far.**

Only G_{77} **is fully known:**



[Blokland *et al.*, 2005]
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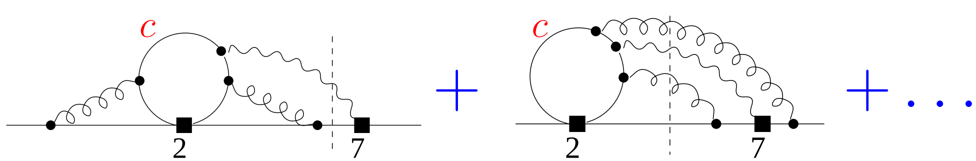
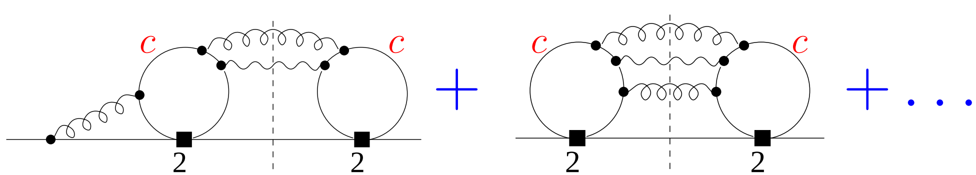
[Asatrian *et al.*, 2006-2007]

G_{27}:



G_{22}:



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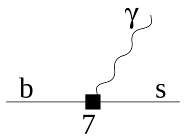
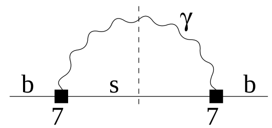


$\Leftrightarrow$



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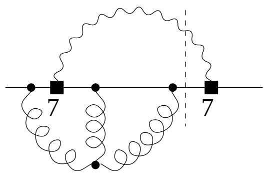
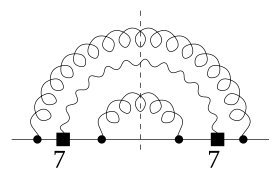


$+$



$+ \ldots$

{ [Blokland *et al.*, 2005] [Melnikov, Mitov, 2005] [Asatrian *et al.*, 2006-2007] }

Large-m_c **asymptotics of** G_{ij} $(m_c \gg m_b/2)$:

1	2	7	8	
+	+	+	+	**1**
	+	+	+	**2**
		+	−	**7**
			−	**8**

[MM, Steinhauser, 2006]

Large-β_0 **approximation for** G_{ij} **(arbitrary** m_c**)**:

1	2	7	8	
+	+	+	−	**1**
	+	+	−	**2**
		+	+	**7**
			+	**8**

[Bieri, Greub, Steinhauser, 2003]
[Ligeti, Luke, Manohar, Wise, 1999]
[Ferroglia, Haisch, 2007]

The β_0 **corr. to** G_{78}, G_{88} **are small.**

G_{18} **and** G_{28} **are small at the NLO.**

Interpolation in m_c

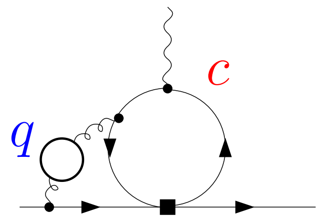
$$\mathcal{B}(\bar{B} \to X_s \gamma)_{E_\gamma > E_0} = \underset{\text{normalization}}{X} \left[\underset{\text{perturbative}}{P(E_0)} + \underset{\text{non-perturbative}}{N(E_0)} \right]$$

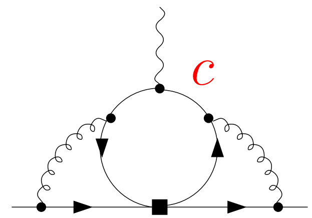
Expansion of $P(E_0)$:

$$P = \underbrace{P^{(0)} + \frac{\alpha_s(\mu_b)}{4\pi}\left(P_1^{(1)} + P_2^{(1)}(r)\right) + \left(\frac{\alpha_s(\mu_b)}{4\pi}\right)^2 \left(P_1^{(2)}}_{\text{known}} + P_2^{(2)}(r) + \underbrace{P_3^{(2)}(r)}_{\text{known}}\right)$$

$$P_1^{(1)}, P_3^{(2)} \sim C_i^{(0)} C_j^{(1)}, \qquad P_2^{(1)}, P_2^{(2)} \sim C_i^{(0)} C_j^{(0)}, \qquad P_1^{(2)} \sim \left(C_i^{(0)} C_j^{(2)},\ C_i^{(1)} C_j^{(1)}\right)$$

Moreover: $P_2^{(2)} = A\, n_f + B = -\frac{3}{2}(11 - 2/3 n_f) A + \frac{33}{2} A + B = P_2^{(2)\beta_0} + P_2^{(2)\text{rem}}$



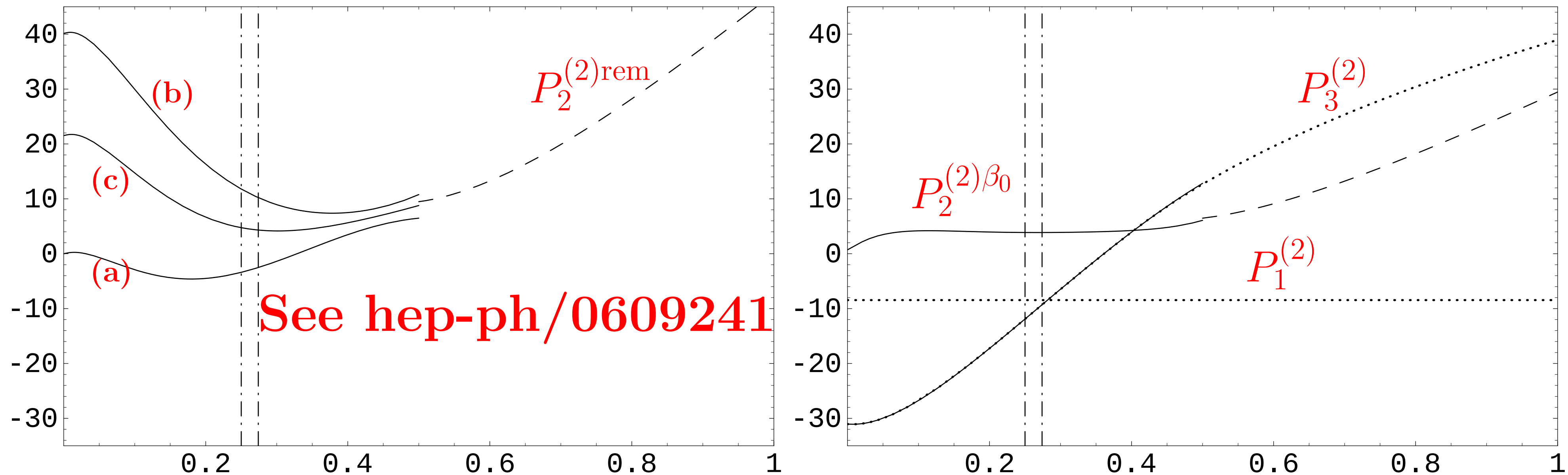


$$r = \frac{m_c(m_c)}{m_b^{1S}}$$

$P_2^{(2)\beta_0}$ **known for all** r

The complete $P_2^{(2)}$ **has been calculated only for** $r \gg \frac{1}{2}$.

The NNLO corrections $P_k^{(2)}$ as functions of $r = m_c(m_c)/m_b^{1S}$



Dotted: exact, Solid: small-r expansions, Dashed: leading large-r asymptotics.

Interpolation:

$$P_2^{(2)\mathrm{rem}}(r) = x_1 + x_2\, P_2^{(1)}(r) + x_3\, r\, \frac{d}{dr} P_2^{(1)}(r) + x_4\, P_2^{(2)\beta_0}(r) + x_5 |A_{\mathrm{NLO}}(r)|^2$$

The coefficients x_k are determined from the asymptotic behaviour at large r

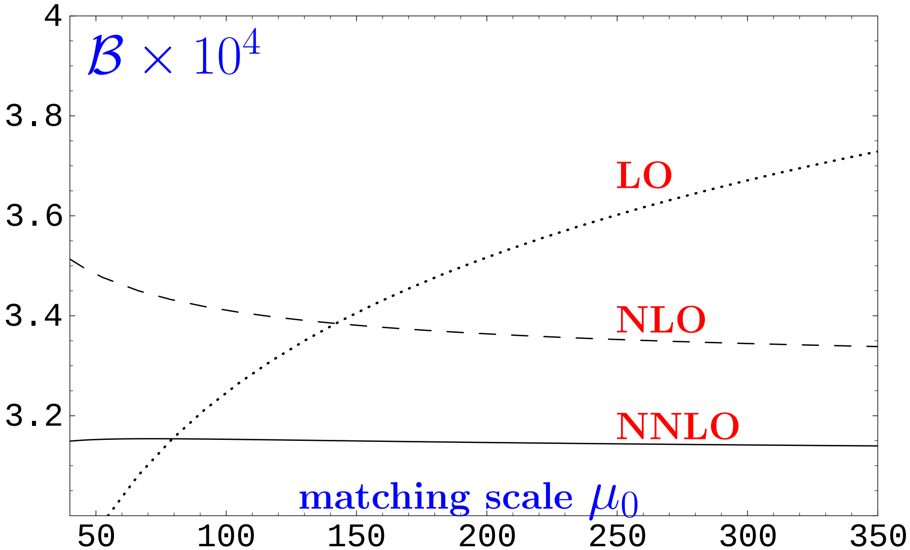
and from the requirement that either (a) $P_2^{(2)\mathrm{rem}}(0) = 0$,

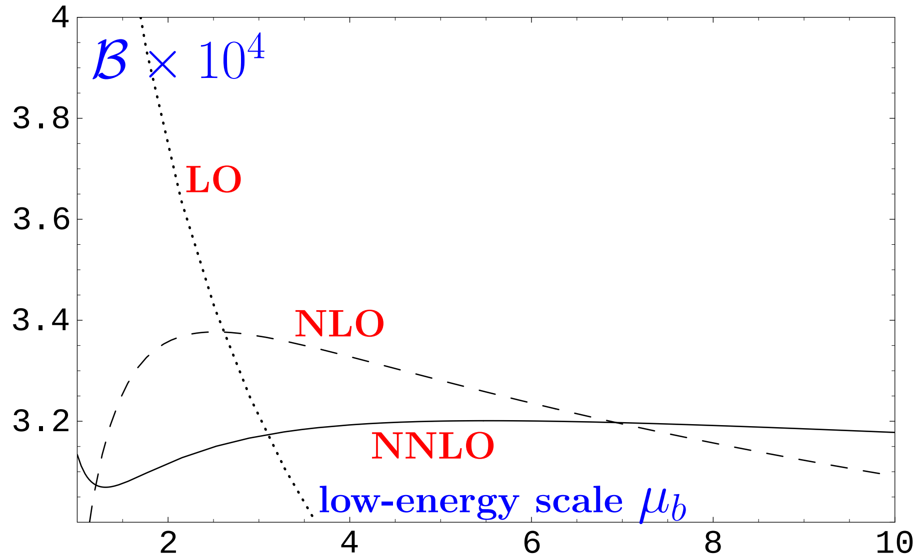
or (b) $P_1^{(2)} + P_2^{(2)\mathrm{rem}}(0) + P_3^{(2)}(0) = 0$,

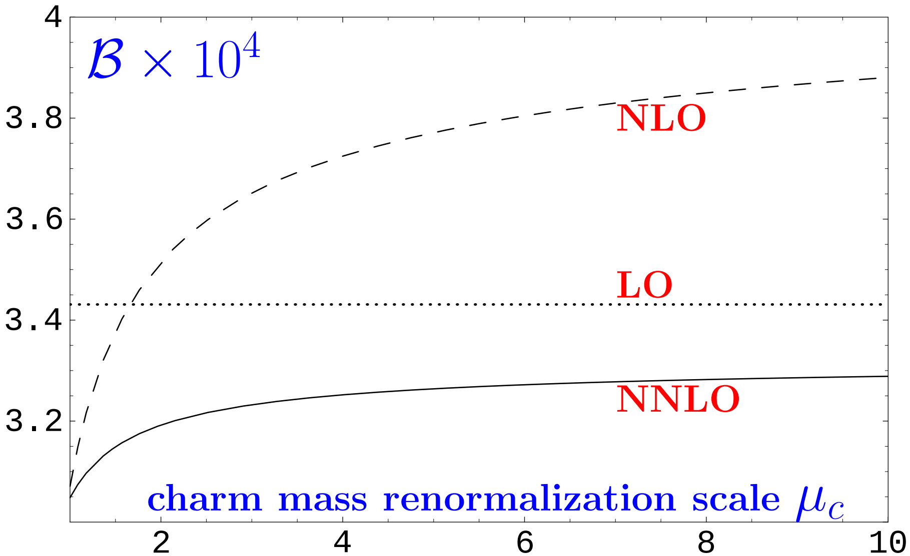
or (c) $P_2^{(2)\mathrm{rem}}(0) = \left[P_2^{(2)\mathrm{rem}}(0)\right]_{77}$.

The average of (a) and (b) is chosen to determine the central value of the NNLO branching ratio.
The difference between these two cases is used to estimate the interpolation ambiguity.

Renormalization scale dependence of $\mathcal{B}(\bar{B} \to X_s \gamma)_{E_\gamma > 1.6\ \mathrm{GeV}}$







"Central" values:

$\mu_0 = 160$ **GeV**

$\mu_b = 2.5$ **GeV**

$\mu_c = 1.5$ **GeV**

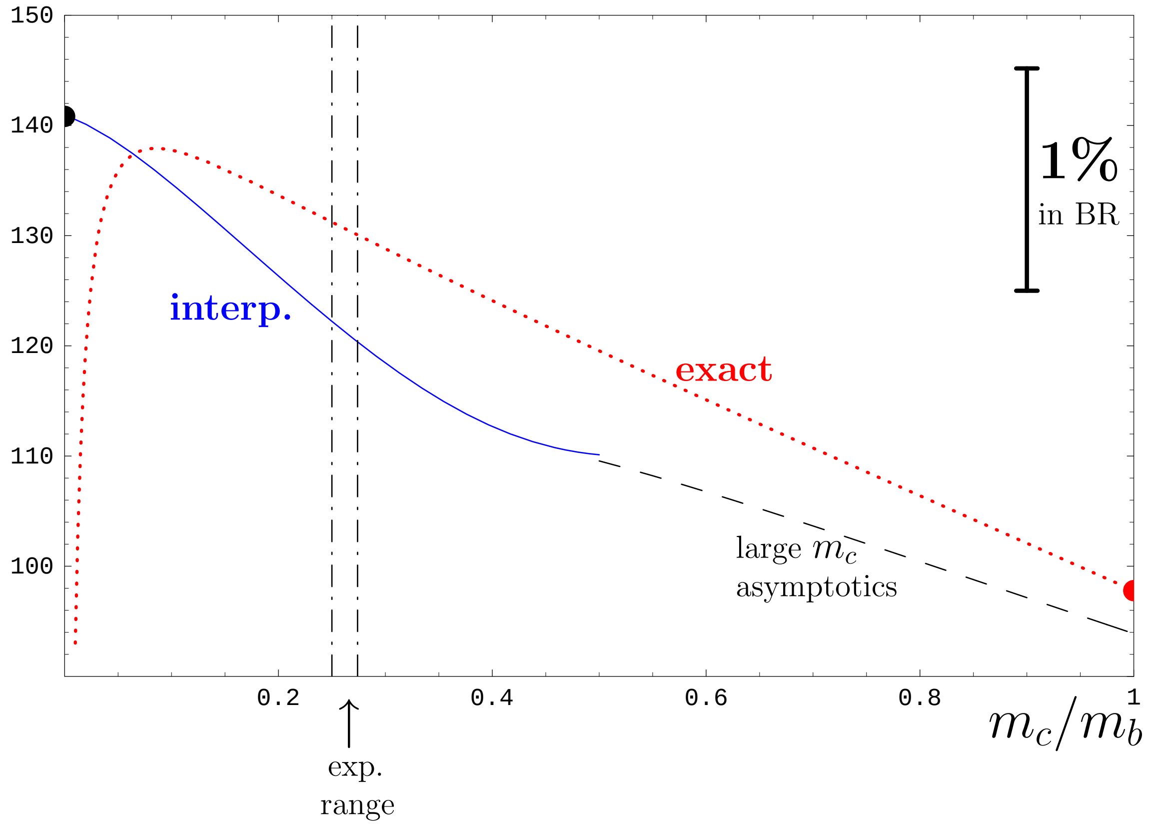
Summary

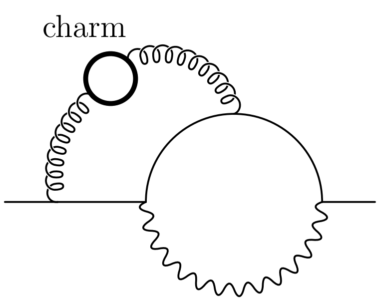
- **The NNLO calculation of the Wilson coefficients at $\mu = \mu_b$ is completed. The 4-loop terms affect the branching ration by $\sim -4\%$.**
- **An intriguing discrepancy occurs between the 1S- and kinetic scheme determinations of the normalization factor C.**
- **For $E_0 = 1.6$ GeV or lower, the MSOPE-resummed logarithmic perturbative corrections undergo a dramatic cancellation with the non-logarithmic terms. Consequently, both types of terms must be treated with the same precision. Until this is done, the fixed-order results should be considered more reliable.**
- **The $m_c = 0$ results for the NNLO contributions from 4-quark operators are awaited. They should allow to improve the the interpolation in m_c, and place it on more solid grounds.**

BACKUP SLIDES

The m_c-dependence of $P_2^{(2)\mathrm{rem}} = C_i^{(0)}(\mu_b) C_j^{(0)}(\mu_b) K_{ij}^{(2)\mathrm{rem}}(\mu_b, E_0)$.

Example: $K_{77}^{(2)\mathrm{rem}}(2.5\ \mathrm{GeV}, 1.6\ \mathrm{GeV})$ **as a function of** m_c/m_b:





Value at $m_c = 0$: **Blokland *et al.*, hep-ph/0506055** **($c\bar{c}$ production included).**

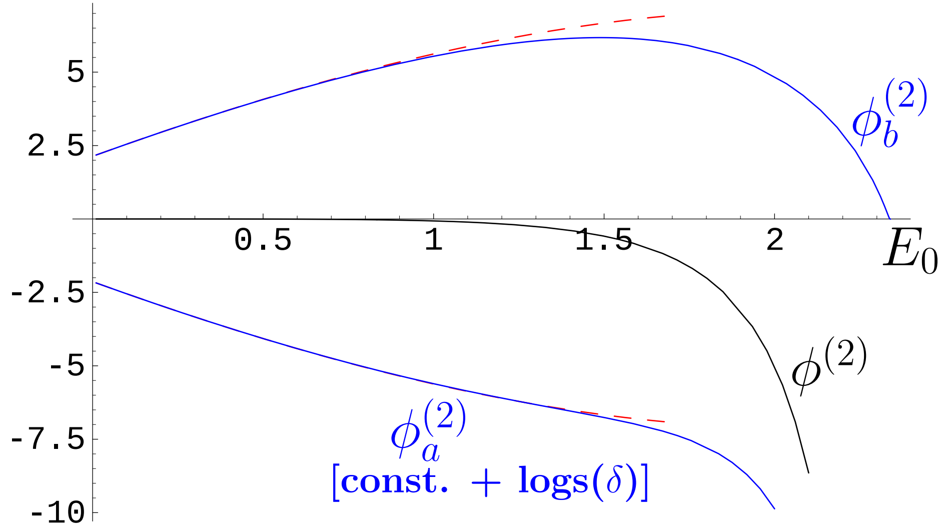
Large-m_c asymptotics: **Steinhauser, MM, hep-ph/0609241.**

Interpolation: " " " **($c\bar{c}$ production included).**

Exact $b \to X_s\gamma$: **Asatrian *et al*, hep-ph/0611123** **($c\bar{c}$ production excluded).**

Exact $b \to X_u e\bar{\nu}$: **Pak, Czarnecki, arXiv:0803.0960** **($c\bar{c}$ production included).**

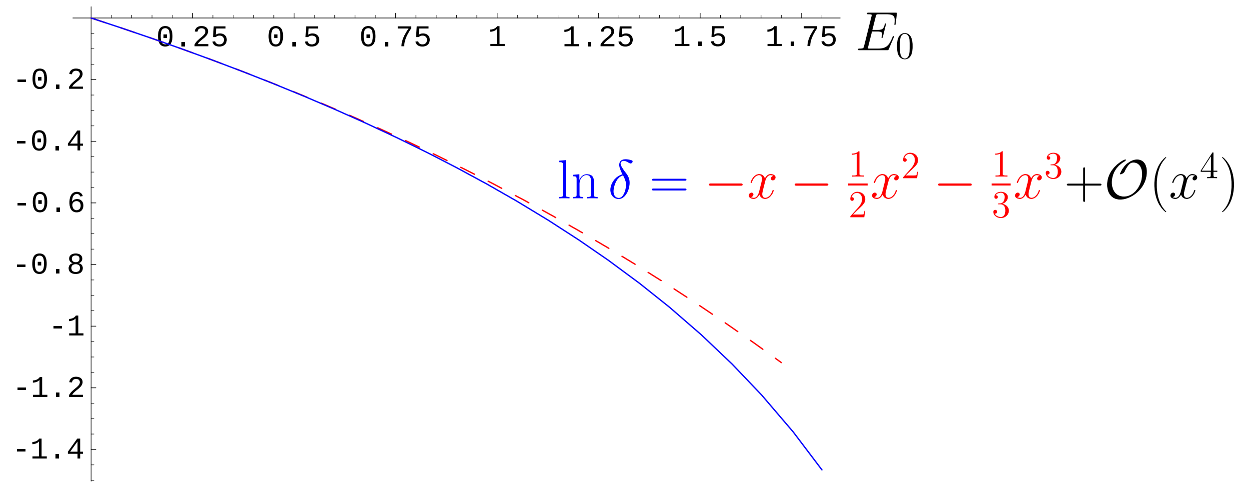
The same pattern arises at $\mathcal{O}(\alpha_s^2)$:



$x = 2E_0/m_b$

$\delta = 1 - x$

It must be the case also at higher orders because:



However, only "const + logs(δ)" have been included at orders $\mathcal{O}(\alpha_s^3)$ and higher in hep-ph/0610067.