Supersymmetric Warped AdS in Extended Topologically Massive Supergravity

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Model: $N = (1,1)$ off-shell supergravity in three dimensions

• Three dimensions is a useful laboratory.

• In off-shell supergravities local supersymmetry transformations do not depend on field equations and hence one can construct higher derivative invariants that are exactly supersymmetric by themselves. The auxiliary fields, characterized by having algebraic field equations in a 2-derivative off-shell supergravity, typically become propagating fields once higher derivative invariants are added.

The bosonic part of the Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{\rm bos} &= R - 2|Z|^2 + 2m(Z + \text{h.c.}) + 2A_\mu A^\mu + \frac{1}{\mu}\varepsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho \\ &\quad -\frac{1}{4\mu}\,\varepsilon^{\mu\nu\rho}\left(R_{\mu\nu}{}^{ab}\omega_{\rho ab} + \tfrac{2}{3}\omega_\mu^{ab}\omega_{\nu b}{}^c\omega_{\rho ca}\right),\end{aligned}$$

- The parameters m and μ are real.

- Auxiliary fields consist of a complex scalar Z and a real vector field A_μ.

We want to find supersymmetric solutions of this model. This is aided greatly by studying the off-shell Killing spinor equation. We shall determine the most general form of the metric and the auxiliary fields under the assumption that at least one off-shell Killing spinor exists. We shall then impose the field equations and find supersymmetric solutions.

The supersymmetry transformation rules are

$$\begin{aligned}
\delta e_\mu^a &= \tfrac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu + \mathrm{h.c.} \\
\delta\psi_\mu &= \left(\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\right)\epsilon - \tfrac{i}{2}A_\nu\gamma^\nu\gamma_\mu\epsilon + \tfrac{1}{2}Z\gamma_\mu\epsilon^\star\ , \\
\delta A_\mu &= \tfrac{i}{8}\bar{\epsilon}\gamma^{\nu\rho}\gamma_\mu\left(2D_{[\nu}\psi_{\rho]} - iA_\lambda\gamma^\lambda\gamma_\nu\psi_\rho + Z\gamma_\nu\psi_\rho^\star\right) + \mathrm{h.c.}\ , \\
\delta Z &= \tfrac{1}{4}\overline{\epsilon^\star}\gamma^{\mu\nu}\left(2D_{[\mu}\psi_{\nu]} - iA_\rho\gamma^\rho\gamma_\mu\psi_\nu + Z\gamma_\mu\psi_\nu^\star\right)\ ,
\end{aligned}$$

ψ_μ is the gravitino which is a complex Dirac spinor.

The field equations are

$$
\begin{aligned}
R_{\mu\nu} + 2A_\mu A_\nu + 2m^2 g_{\mu\nu} + \frac{1}{\mu}\,\varepsilon_\mu{}^{\rho\sigma}\nabla_\rho(R_{\sigma\nu} - \tfrac{1}{4}g_{\sigma\nu}R) &= 0\,, \\
\frac{1}{2\mu}\varepsilon^{\mu\nu\rho}\,F_{\nu\rho} + A^\mu &= 0\,, \\
Z - m &= 0\,,
\end{aligned}
$$

These equations reduce to those for $N = (1,0)$ topologically massive supergravity upon setting $A_\mu = 0$, $Z^\star = Z$ and $\psi^\star_\mu = \psi_\mu$.

Killing Spinor Analysis

The defining equation for ϵ to be a Killing spinor is

$$\delta\psi_\mu = \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\right)\epsilon - \frac{i}{2}A_\nu\gamma^\nu\gamma_\mu\epsilon + \frac{1}{2}Z\gamma_\mu\epsilon^\star = 0$$

Let us focus on the following spinor bilinear:

$$\bar{\epsilon}\gamma^\mu\epsilon = \overline{\epsilon^\star}\gamma^\mu\epsilon^\star \equiv K^\mu \ ,$$

where K^μ is a *real* vector.
From Fierz identities one may show that

$$K_\mu K^\mu = -f^2 \ ,$$

where f is a *real* function.

Using the Killing spinor equation one finds

$$\nabla_{(\nu} K_{\mu)} = 0 \ ,$$

thus K^μ is a Killing vector which is null or timelike.

Null Killing Vector ($f = 0$)

It is convenient to choose coordinates such that the metric takes the form

$$ds^2 = dx^2 + 2P(u,x)dudv + Q(u,x)du^2 \ ,$$

with $K^\mu \partial_\mu = \partial_v$.

In these coordinates, the Killing spinor equations imply that the auxiliary fields are given by

$$\begin{aligned} A_\mu &= -\frac{1}{2}\,\partial_\mu\theta\ , \\ Z &= -\frac{1}{2}\,e^{-i\theta}\,\partial_x \ln\left(Pe^{i\theta}\right)\ , \end{aligned}$$

in terms of an unconstrained real function $\theta(u,x)$ and the metric function $P(u,x)$.

Having found the most general form of the solution from the requirement that it admits a Killing spinor, we now examine the field equations.

Solutions with a Null Killing Vector

Killing spinor analysis $\rightarrow A_\mu = -\frac{1}{2}\partial_\mu\theta$

Field equation $\rightarrow A^\mu = -\frac{1}{2\mu}\varepsilon^{\mu\nu\rho}F_{\nu\rho}$

$\Rightarrow A_\mu = 0.$
The field equations and the Killing spinor analysis reduce to that of $N = (1, 0)$ supersymmetric topologically massive supergravity whose most general supersymmetric solutions were determined in:

- *The General Supersymmetric Solution of Topologically Massive Supergravity*, G.W. Gibbons, C.N. Pope, E. Sezgin, Class.Quant.Grav.25:205005,2008 ; arXiv:0807.2613.
- *Massive 3D Supergravity*, R. Andringa, E. A. Bergshoeff, M. de Roo, O. Hohm, E. Sezgin, P. K. Townsend Class.Quant.Grav.27:025010,2010 ; arXiv:0907.4658.

- All solutions are of pp-wave type.

Timelike Killing Vector ($f \neq 0$)

We choose a coordinate system, in which

$$K^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} ,$$

and the metric can be parametrized as

$$ds^2 = -e^{2\varphi} \left(dt + B_i dx^i\right)^2 + e^{2\sigma}(dx^2 + dy^2) ,$$

where φ, B_x, B_y and σ are arbitrary time independent functions. We have $f \equiv e^\varphi$.

Using Killing spinor equations and Fierz identities one finds

$$
\begin{aligned}
A_0 &= -\frac{1}{2}\, e^{\varphi-2\sigma} \epsilon^{ij}\, \partial_i B_j \ , \\
A_1 - iA_2 &= ie^{-\sigma}\, \partial_z \left(\varphi - \sigma + ic\right) \ , \\
Z &= -ie^{-\sigma-ic}\, \partial_z \left(\varphi + \sigma - ic\right) \ .
\end{aligned}
$$

Where c is a time independent real function and $z = x + iy$.

Again the analysis above remains valid for any further extension of the model by matter and/or higher derivative couplings since we have only made use of the off-shell Killing spinor equation. For any solution to these equations it remains to verify the model dependent field equations.

Solutions with a Timelike Killing Vector

We shall seek solutions for which

$$A_a = \text{constant} \ , \qquad A_2 = 0 \ , \qquad c = 0 \ .$$

Note that the ansatz we have made for the vector field shows that it is constant in the flat basis.

- From the field equations we have $Z = m$.

From the vector field equations one gets:

$$\left(A_0 - \frac{\mu}{2}\right) A^2 = 0 \ .$$

Therefore, we have different classes of solutions which we may distinguish as follows:

$A_0 \neq \frac{\mu}{2}$ $\longrightarrow$	$A_a = 0$		round AdS_3
$A_0 \neq \frac{\mu}{2}$ $\longrightarrow$	$A^2 = 0$	$\lvert\mu\ell\rvert \neq 2$	AdS_3 pp-wave
$A_0 = \frac{\mu}{2}$ $\longrightarrow$	$A^2 > 0$	$\iff \lvert\mu\ell\rvert < 2$	spacelike squashed AdS_3
$A_0 = \frac{\mu}{2}$ $\longrightarrow$	$A^2 = 0$	$\iff \lvert\mu\ell\rvert = 2$	null warped AdS_3
$A_0 = \frac{\mu}{2}$ $\longrightarrow$	$A^2 < 0$	$\iff \lvert\mu\ell\rvert > 2$	timelike stretched AdS_3

The second class contains warped AdS_3 solutions.

The $A_0 = \frac{\mu}{2}$ case

In this case, we find from vector field equations

$$A_0 = \frac{\mu}{2} \ , \quad A_1 = -m \ .$$

Using these in expressions we obtained from the Killing spinor analysis we get:

$$e^{\sigma} \ = \ 1 \ , \qquad e^{\varphi} = e^{-2my} \ .$$

Without loss of generality we choose the gauge $B_y = 0$ and find

$$B_x = \frac{\mu}{2m} \, e^{2my} \ ,$$

Thus, we have the metric

$$ds^2 \;=\; -e^{-4my}\left(dt + \frac{\mu}{2m}\, e^{2my}\, dx\right)^2 + dx^2 + dy^2\ .$$

From this point discussion crucially depends on the value of the parameter

$$\nu^2 \;\equiv\; \frac{1}{4}\,(\mu\ell)^2 \;=\; 1 - \frac{A^2}{m^2}\ ,$$

which as we shall see will take the role of the warping parameter of the solutions.

Spacelike squashed AdS_3

For $|\mu\ell| < 2$, which implies $A^2 > 0$, we rewrite the above metric as

$$ds^2 \;=\; \frac{A^2}{m^2}\left(dx - \frac{\mu m}{2A^2} e^{-2my} dt\right)^2 - \frac{m^2}{A^2} e^{-4my} dt^2 + dy^2 \ .$$

which can be put into the following form:

$$ds^2 = \frac{\ell^2}{4}\left[\frac{-dt'^2 + dz^2}{z^2} + \nu^2\left(dx' + \frac{dt'}{z}\right)^2\right] \ .$$

This is spacelike squashed AdS_3 with squashing parameter ν^2. The terminology of "squashed" is due to the relation $\nu^2 < 1$.

This metric can be expressed in global coordinates as:

$$ds^2 = \frac{\ell^2}{4}\left[-\cosh^2\sigma\, d\tau^2 + d\sigma^2 + \nu^2 \left(du + \sinh\sigma\, d\tau\right)^2\right] \ .$$

with $u, \sigma \in \mathbb{R}$ and $\tau \sim \tau + 4\pi$.
In these coordinates, the vector field A_μ reads

$$A = \frac{1}{2}\nu\sqrt{1-\nu^2}\left(\sinh\sigma\, d\tau + du\right) \ .$$

If we compactify along the u direction by identifying $u \sim u + 2\pi\beta$, we obtain the so-called self-dual solution with

$$ds^2 = \frac{\ell^2}{4}\left[-\cosh^2\sigma\, d\tau^2 + d\sigma^2 + \nu^2 \left(\beta\, d\phi + \sinh\sigma\, d\tau\right)^2\right]$$

with $\tau, \sigma \in \mathbb{R}$ and $\phi \sim \phi + 2\pi$.

• *Self-dual solutions of 2+1 Einstein gravity with a negative cosmological constant*, O. Coussaert, M. Henneaux, hep-th/9407181.

Extremal Black Hole

Having found supersymmetric warped AdS_3 solutions, it is natural to examine their modding out by suitable discrete subgroups of the isometry group to obtain black hole solutions. This procedure has been systematically analyzed in:

- *Warped AdS_3 Black Holes*, D. Anninos, W. Li, M. Padi, W. Song, A. Strominger, JHEP 0903:130,2009 ; arXiv:0807.3040.

where a black hole solution is obtained from the discrete quotient of the spacelike-stretched AdS_3 solution where self-dual solution is a special case. In our case no spacelike-stretched AdS_3 solution exists. Nonetheless, we have spacelike-squashed AdS_3 solutions, that include the self-dual solution and that can be modded out by a suitable discrete subgroup, yielding an extremal black hole solution without closed timelike curves.

The self-dual solution given above can be represented in the Schwarzschild coordinates (t, r, θ) as follows:

$$
\begin{aligned}
ds^2 &= \frac{3\ell^2}{4-\nu^2}\Big[\Big(dt + \frac{\sqrt{3}(\nu r - r_h)}{\sqrt{4-\nu^2}}\, d\theta\Big)^2 + \frac{4-\nu^2}{12\,(r-r_h)^2}\, dr^2 \\
&\quad - \frac{3(r-r_h)^2}{4-\nu^2}\, d\theta^2\Big] ,
\end{aligned}
$$

and the vector field is given by

$$
A = \sqrt{\frac{3\,(1-\nu^2)}{4-\nu^2}}\, dt + 3\,(\nu r + r_h)\frac{\sqrt{1-\nu^2}}{4-\nu^2}\, d\theta \, .
$$

This can be interpreted as an extremal black hole although it has no causal or geometric singularities. However, it has a Killing horizon at $r = r_h$ where g^{rr} and the determinant of (t, θ) part of the metric vanishes. Here $r > 0$ and $\theta \in \mathbb{R}$. Since $\nu^2 < 1$ for the self-dual solution, θ cannot be identified periodically, which would imply the existence of closed timelike curves for large r. Since $u \simeq u + 2\pi\beta$, thus $t \simeq t - \nu\sqrt{\frac{(4-\nu^2)}{3}}\pi\beta$. Its entropy obeys the Cardy formula like usual black holes.

Future Directions

• Supersymmetric warped AdS_3 solutions had appeared so far only in on-shell extended $N = (2,0)$ supergravity coupled to a vector multiplet. Similar solutions have been investigated in Anninos et.al. for topologically massive gravity in the absence of the vector field in which case they are not supersymmetric. In that case, critical behaviour has been observed at $\mu\ell = 3$ where the solution is round AdS_3 and the warping transitions from stretching to squashing. Interestingly, in presence of the vector field we find that this transition occurs at $\mu\ell = 2$, i.e. at vanishing norm, where the solution is null warped AdS_3, which represents the transition point between spacelike squashed and timelike stretched solutions.

• It would be interesting to study the holographic dual our extremal black hole solution.

• It is shown in

Critical solutions in topologically gauged N=8 CFTs in three dimensions, B.E.W. Nilsson, arXiv:1304.2270.

that only for certain values of $\mu\ell$, which are obtained by varying the number of scalar fields with a VEV, solutions of topological gauged $D = 3$, $\mathcal{N} = 8$ CFTs with SO(N) gauge group are also solutions of topologically massive gravity. It is interesting to note that among these, $\mu\ell = 2$ is the only even one and until now no solution with this value were known.

• Supersymmetry of these warped AdS_3 solutions may also prove useful to further analyze the stability of them.

• An interesting open problem is the possible embedding of our solutions into string theory.