On transverse momentum dependent factorization in perturbative QCD

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Generalities and jargon

- cross sections in the Bjorken limit of QCD $\quad s \to \infty\ ,\ \ Q^2 \to \infty$

  are expressed as a $1/Q^2$ "twist" expansion $\quad Q^2/s = x\ \ \text{fixed}$

$$d\sigma = \sum_p f_p \otimes d\hat{\sigma} + O(1/Q^2) \qquad f_p = \text{PDF or TMD}$$

collinear factorization: parton content of proton described by k_T-integrated distributions

sufficient approximation for most high-p_T processes

TMD factorization: involves transverse-momentum-dependent (TMD) distributions

TMDs are needed in particular cases

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- cross sections in the Regge limit of QCD $\quad s \to \infty\ ,\quad x \to 0$

  are expressed as a $1/s$ "eikonal" expansion $\quad xs = Q^2\ \ \text{fixed}$

$$d\sigma = \sum_p f_p \otimes d\hat{\sigma} + O(1/s) \qquad f_p = \text{UGD}$$

k_T factorization: parton content described by the un-integrated gluon distribution (UGD)

Collinear factorization

in standard pQCD calculations, the incoming parton transverse momenta are set to zero in the matrix element and are integrated over in the parton densities

$$d\sigma_{AB\to X} = \sum_{ij} \int dx_1 dx_2 \underbrace{f_{i/A}(x_1,\mu^2) f_{j/B}(x_2,\mu'^2)}_{k_T \text{ integrated quantities}} \, d\hat{\sigma}_{ij\to X} + \mathcal{O}\left(\Lambda^2_{QCD}/M^2\right)$$

$d\hat{\sigma}_{ij\to X}$: the incoming partons are taken collinear to the projectile hadrons

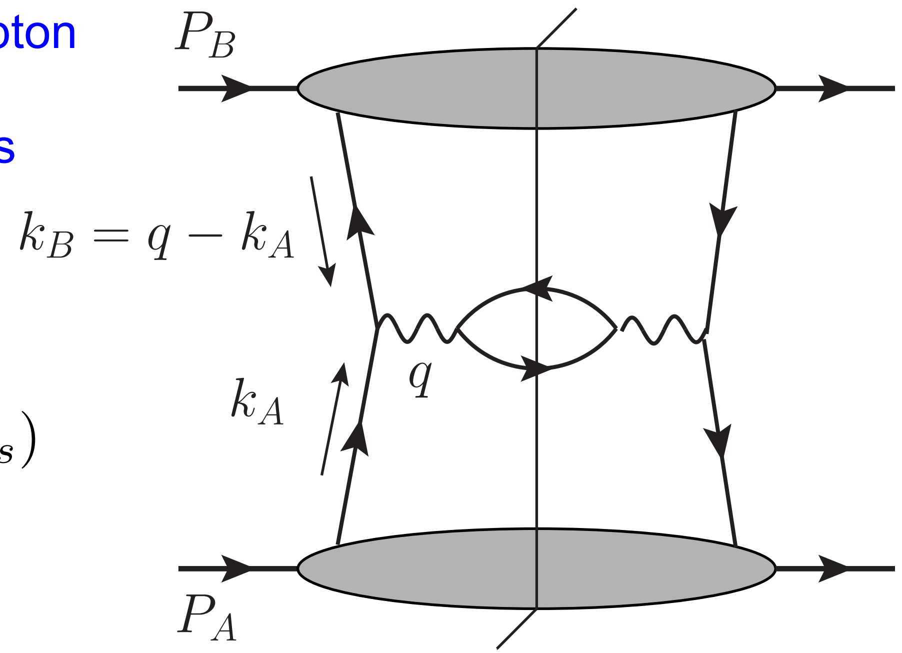
M^2: some hard scale

in general for a hard process, this approximation is accurate

in some cases however, this is not good enough, and TMD factorization is needed

Drell-Yan process

the transverse momentum of the lepton pair is the sum of the transverse momenta of the incoming partons



in collinear factorization

$$d\sigma^{AB\to l^+l^-X} \propto \delta(q_T) + \mathcal{O}(\alpha_s)$$

in this case, TMDs can be useful objects :

$$d\sigma^{AB\to l^+l^-X} = \sum_{i,j}\int dx_1 dx_2 d^2k_{1T} d^2k_{2T}\ f_{i/A}(x_1,\mathbf{k}_{1T}) f_{j/B}(x_2,\mathbf{k}_{2T})\ d\hat{\sigma}^{ij\to l^+l^-X}$$

Drell-Yan production is directly sensitive to partonic transverse momenta

$$d\hat{\sigma} \propto \delta(\mathbf{k}_{1T} + \mathbf{k}_{2T} - q_\perp)$$

A general guiding rule

- Consider an hadronic collision $A + B \to C + D + \cdots + X$

  - if one hadron is involved, TMD and k_T factorization OK

  - if two hadrons are involved, TMD factorization OK but process dependent (and soft factor needed), k_T factorization OK

  - if more than two hadrons are involved, no TMD or k_T factorization, except in some special cases and limits

  note: dilute-dense colliding hadrons (p+A or p+p at forward rapidities) only count for 1 hadron

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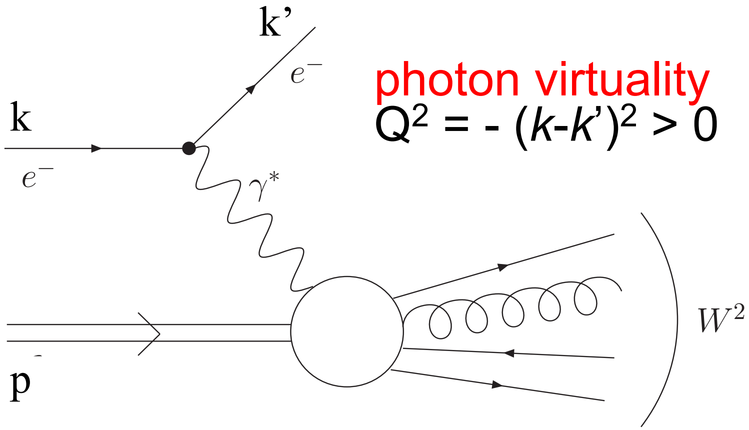
- in many cases, TMD/k_T factorization is broken at some order of the perturbation theory, here I am only discussing the validity at leading-order
- warning: in cases where TMD/k_T factorization is not valid, it is still applied in phenomenological studies, due to the lack of alternatives

One-hadron case, e.g.:

$$\gamma^* + p(A) \to X$$

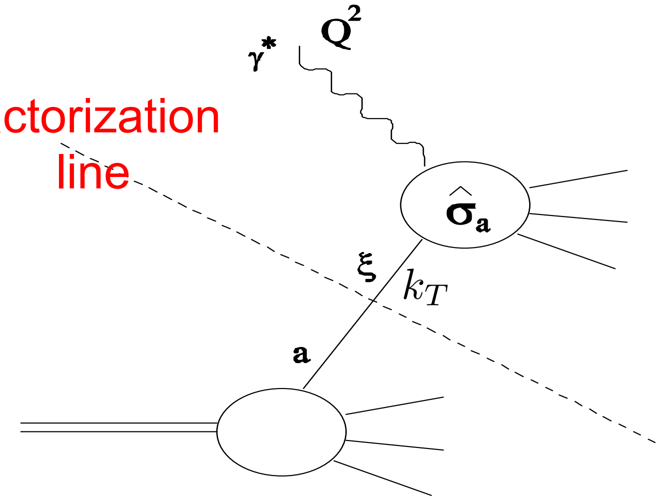
$$p + A \to \gamma^* + X$$

Deep inelastic scattering



photon virtuality
$Q^2 = -(k-k')^2 > 0$

factorization
line

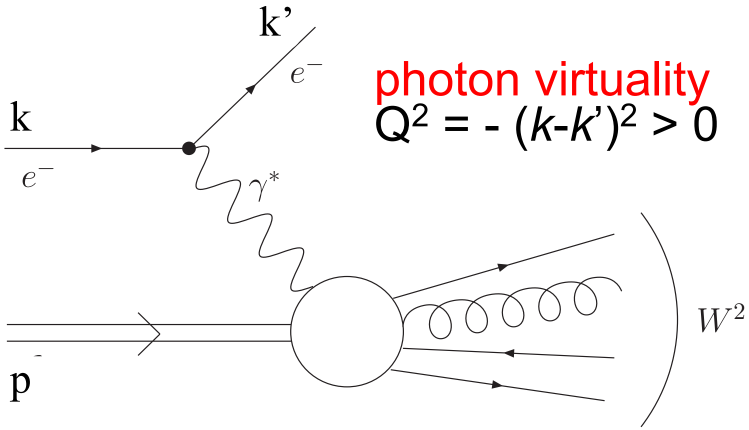


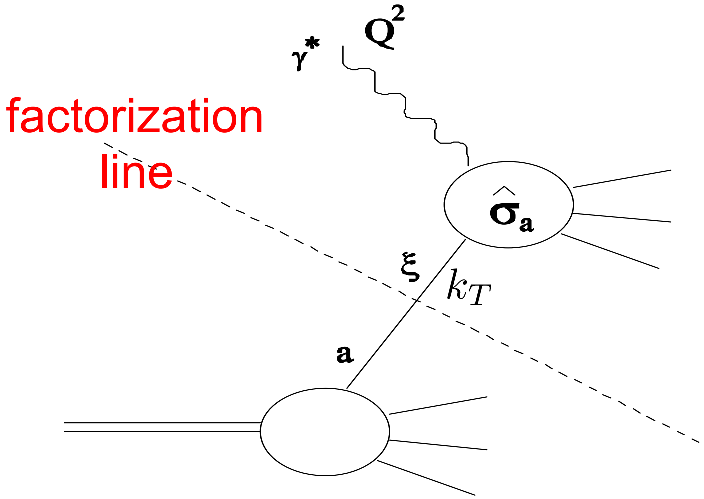
let's consider the two limits in which the cross section can be obtained:

Bjorken limit:

- set k_T=0 in ME and integrate over it in pdfs up to the factorization scale μ^2
- pdfs have ξ dependence and evolve with μ^2 (DGLAP)
- a usual scale choice is Q^2

Deep inelastic scattering





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- set k_T=0 in ME and integrate over it in pdfs up to the factorization scale μ^2
- pdfs have ξ dependence and evolve with μ^2 (DGLAP)
- a usual scale choice is Q^2

Regge limit:

- set ξ=0 in ME and integrate over it in pdfs down to the factorization "scale" x
- pdfs have k_T dependence and evolve with x (BFKL/BK)
- a usual choice is $x=x_{Bj}$

TMD/k_T factorization in DIS

- in the Bjorken limit, the natural factorization is not k_T dependent

  but it can be extended to feature ξ and k_T dependent pdfs
  (e.g. TMDs), while DGLAP evolution turns into CCS evolution

- in the Regge limit, the factorization is intrinsically k_T dependent

  and it is equivalent, after Fourier transformation to coordinate space,
  to the so-called dipole factorization

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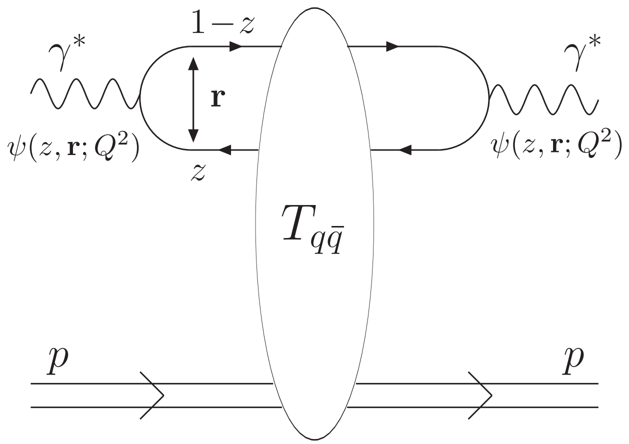
Mueller (1990), Nikolaev and Zakharov (1991)

$$\sigma_{T,L}^{\gamma^* p\to X} = 2\int d^2r\, dz\, |\psi_{T,L}(z,\mathbf{r};Q^2)|^2 \underbrace{\int d^2b\, T_{q\bar{q}}(\mathbf{r},\mathbf{b},x_B)}$$

overlap of $\gamma^* \to q\bar{q}$
splitting functions

dipole-hadron cross-section



link to the unintegrated gluon distribution

$$F(q_\perp, x_B) = \int \frac{d^2r}{(2\pi)^2}\; e^{-iq_\perp\cdot\mathbf{r}}[1 - T_{q\bar{q}}(\mathbf{r},x_B)]$$

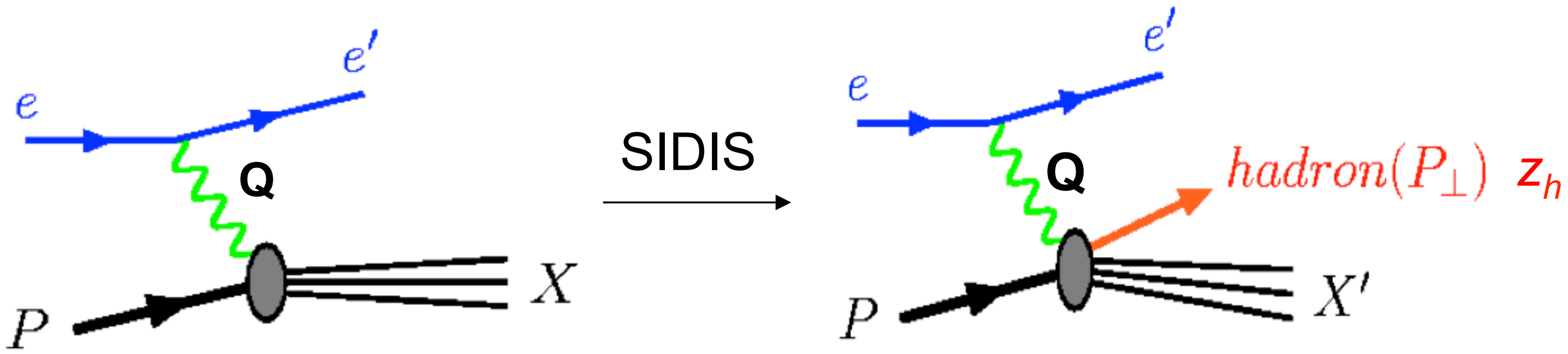
Two-hadron case, e.g.:

$$\gamma^* + p(A) \to h + X$$

$$p + p \to \gamma^* + X$$

$$p + A \to h + X$$

The dipole factorization in SIDIS

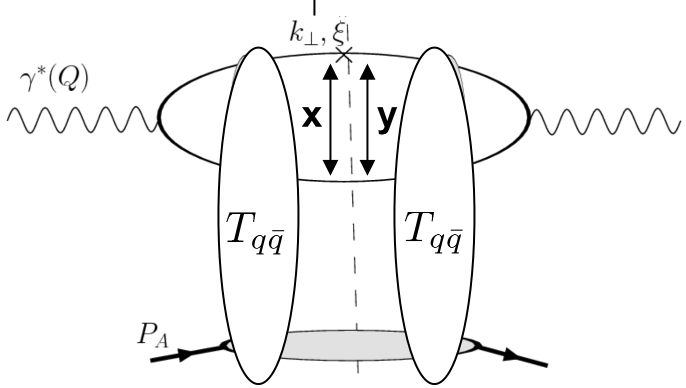


- the cross section at small x

$$\Phi(\xi, \mathbf{x}, \mathbf{y}; Q^2) = \psi(\xi, \mathbf{x}; Q^2)\psi^*(\xi, \mathbf{y}; Q^2)$$

dipoles in amplitude / conj. amplitude

fragmentation into hadron



$$\frac{d\sigma^{\gamma^* p \to hX}}{dz_h d^2P_\perp} = \frac{d\sigma_{T,L}^{\gamma^* p \to qX}}{d\xi d^2k_\perp}\left(k_\perp = \frac{\xi}{z_h}P_\perp\right) \otimes D_{h/q}(z_h/\xi)$$

$$\frac{d\sigma_{T,L}^{\gamma^* p \to qX}}{d\xi d^2k_\perp} = \int \frac{d^2x}{2\pi}\frac{d^2y}{2\pi}\, e^{-ik_\perp\cdot(\mathbf{x}-\mathbf{y})}\,\Phi_{T,L}(\xi, \mathbf{x}, \mathbf{y}; Q^2)\int d^2b\,[T_{q\bar{q}}(\mathbf{x}, x_B) + T_{q\bar{q}}(\mathbf{y}, x_B) - T_{q\bar{q}}(\mathbf{x}-\mathbf{y}, x_B)]$$

McLerran and Venugopalan, Mueller, Kovchegov and McLerran (1999)

Cross section in momentum space

- the lepto-production cross section

k_T factorization

$$\frac{d\sigma(ep\to e'hX)}{d\mathcal{P}} = \frac{\alpha_{em}^2 N_c}{2\pi^3 x_B Q^2}\sum_f e_f^2 \int_{z_h}\frac{dz}{z}\frac{D(z)}{z^2}\int d^2b d^2q_\perp F(q_\perp, x_B)\,\mathcal{H}\left(\xi = \frac{z_h}{z}, k_\perp = \frac{P_\perp}{z}\right)$$

phase space $d\mathcal{P} = dx_B dQ^2 dz_h dP_\perp^2$

UGD

F.T. of photon wave function

$\epsilon_f^2 = \xi(1-\xi)Q^2$

massless quarks

CM, Xiao and Yuan (2009)

$$\mathcal{H}(\xi, k_\perp) = \left(1 - y + \frac{y^2}{2}\right)(\xi^2 + (1-\xi)^2)\left|\frac{k_\perp}{k_\perp^2 + \epsilon_f^2} - \frac{k_\perp - q_\perp}{(k_\perp - q_\perp)^2 + \epsilon_f^2}\right|^2 \quad \text{photon T}$$

$$+(1-y)4\xi^2(1-\xi)^2Q^2\left(\frac{1}{k_\perp^2 + \epsilon_f^2} - \frac{1}{(k_\perp - q_\perp)^2 + \epsilon_f^2}\right)^2 \quad \text{photon L}$$

TMD factorization of SIDIS

- the cross section can be factorized in 4 pieces

Collins and Soper (1981), Collins, Soper and Sterman (1985), Ji, Ma and Yuan (2005)

$$\frac{d\sigma(ep \to e'hX)}{d\mathcal{P}} = \frac{4\pi\alpha_{em}^2}{Q^2}\left(1 - y + \frac{y^2}{2}\right)\int d^2k_\perp d^2p_{1\perp} d^2\lambda_\perp$$

$$q(x_B, k_\perp; x_B\zeta) D(z_h, p_{1\perp}; \hat{\zeta}/z_h)$$

$$S(\lambda_\perp;\rho) H(Q^2, x_B, z_h; \rho)\delta^{(2)}(z_h k_\perp + p_{1\perp} + \lambda_\perp - p_\perp)$$

TMD ff

TMD quark distribution

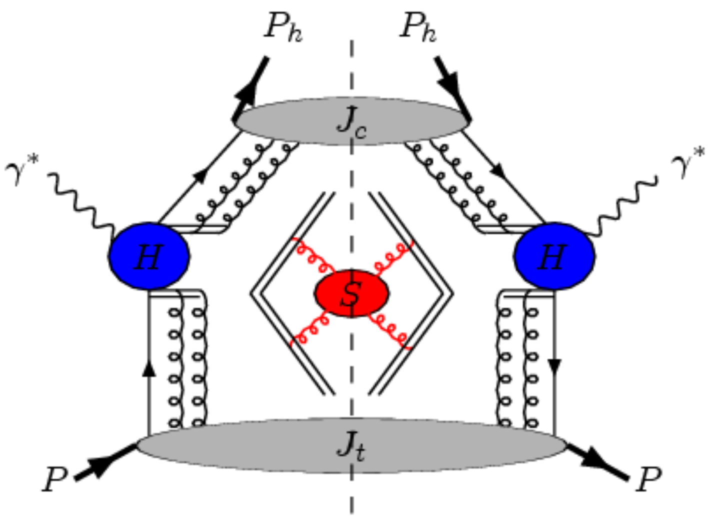
soft factor

hard part

valid to leading power in $1/Q^2$ and to all orders in α_s

(the gluon TMD piece is power-suppressed)

not the naïve factorization I wrote at the beginning (without soft factors)



The TMD quark distribution

- operator definition

$$q(x, k_\perp) = \frac{1}{2}\int \frac{d^2\xi_\perp d\xi^-}{(2\pi)^2} e^{-ixP^+\xi^- - ik_\perp\cdot\xi_\perp} \langle P|\bar{\Psi}(\xi)\mathcal{L}_\xi \gamma^+ \mathcal{L}_0 \Psi(0)|P\rangle$$

quark fields also have transverse separation

Wilson lines needed for gauge invariance

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Wilson lines needed
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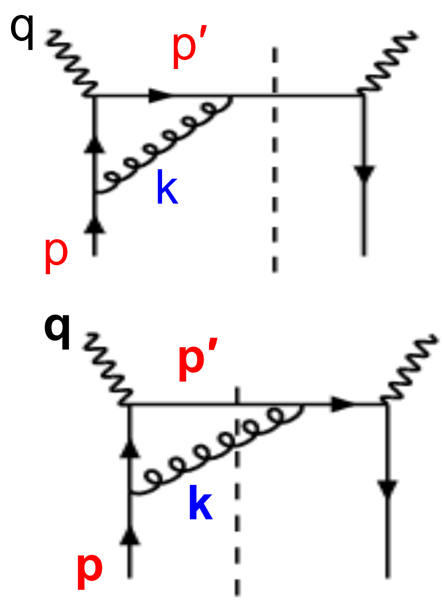
- how factorization works

possible regions for the gluon momentum

k collinear to p (parton distribution)
k collinear to p' (parton fragmentation)
k soft (soft factor)
k hard (α_s correction)



- gauge links make TMDs process dependent

e.g. $f_{1T}^{\perp}(x,\mathbf{k}_\perp)_{DY} = -f_{1T}^{\perp}(x,\mathbf{k}_\perp)_{SIDIS}$

Three-hadron case, e.g.:

$$\gamma^* + p(A) \rightarrow h_1 + h_2 + X$$

$$p + p \rightarrow h + X$$

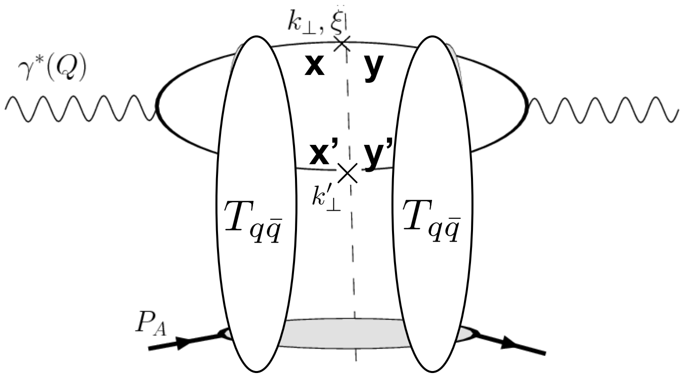
$$p + A \rightarrow h_1 + h_2 + X$$

Di-hadron production in γ*+p(A)

- the cross section in the dipole picture

$$\frac{d\sigma_{T,L}^{\gamma^* p\to q\bar{q}X}}{d^2k_\perp d^2k'_\perp}=\int\frac{d^2x}{2\pi}\frac{d^2y}{2\pi}\frac{d^2x'}{2\pi}\frac{d^2y'}{2\pi}\,e^{-ik_\perp\cdot(\mathbf{x}-\mathbf{y})}e^{-ik'_\perp\cdot(\mathbf{x}'-\mathbf{y}')}\int d\xi\;\Phi_{T,L}(\xi,\mathbf{x}-\mathbf{x}',\mathbf{y}-\mathbf{y}';Q^2)$$
$$\times\left[T_{q\bar{q}}(\mathbf{x}-\mathbf{x}',x_B)+T_{q\bar{q}}(\mathbf{y}-\mathbf{y}',x_B)-T_{q\bar{q}\bar{q}q}(\mathbf{x},\mathbf{x}',\mathbf{y}',\mathbf{y},x_B)\right]$$

because of the 4-point function $T_{q\bar{q}\bar{q}q}$, there is no k_T factorization (the cross section is not a linear function of the UGD)



- SIDIS was a special case

in SIDIS, the $k'_\perp$ integration sets **x'=y'**,
and then $T_{q\bar{q}\bar{q}q}(\mathbf{x},\mathbf{x}',\mathbf{x}',\mathbf{y},x_B)=T_{q\bar{q}}(\mathbf{x}-\mathbf{y},x_B)$

this cancellation of the interactions involving the spectator antiquark in SIDIS is what led to k_T factorization

- in the Bjorken limit, there is no TMD factorization either

Collins and Qiu (2007), Xiao and Yuan (2010)

Di-hadrons in the correlation limit

- k_T factorization is recovered in the limit $|k_\perp + k'_\perp| \ll |k_\perp|\ , |k'_\perp|$

$$d\sigma^{\gamma^* p \to q\bar{q}X} = \int d^2k_\perp d^2k'_\perp\ F_{WW}(|k_\perp + k'_\perp|, x)\ d\hat{\sigma}^{\gamma^* g \to q\bar{q}}$$

BUT : not the naïve factorization, as it involves a new operator definition of the UGD, in terms of a quadrupole (as opposed to a dipole previously)

$$F_{WW}(k_\perp, x) \propto \int \frac{d^2v}{(2\pi)^2} \frac{d^2v'}{(2\pi)^2}\ e^{-ik_\perp \cdot (v-v')} \left\langle \mathrm{Tr}\left[\partial_i U(v)\right] U^\dagger(v') \left[\partial_i U(v')\right] U^\dagger(v) \right\rangle_x$$

Dominguez, Xiao and Yuan (2010)

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Dominguez, Xiao and Yuan (2010)

- the case of di-hadron production in dilute-dense collisions

similar conclusions, and a linear combination of both UGDs is needed

	DIS and DY	SIDIS	hadron in pA	photon-jet in pA	Dijet in DIS	Dijet in pA
$G^{(1)}$ (WW)	$\times$	$\times$	$\times$	$\times$	$\surd$	$\surd$
$G^{(2)}$ (dipole)	$\surd$	$\surd$	$\surd$	$\surd$	$\times$	$\surd$

Dominguez, CM, Xiao and Yuan (2011)

Conclusions

- there are two types of transverse momentum dependent factorization in QCD (TMD and k_T factorization), established in different limits
- I gave and illustrated a general guiding principle to determine the validity (at leading order) of such factorizations
- warning: there may be exceptions to the rule …
- even when factorization is valid, there are subtleties (soft factors, different TMD/UGD definitions, …) that should be treated properly