



Beyond correlations



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CERN February 2008

This work has been supported by a Marie Curie Early Stage Research Training Fellowship of the European Community's Sixth Framework Programme under contract number MRTN-CT-2006-035606, by LPNHE Paris and the Commissariat à l'Energie Atomique and CNRS/Institut National de Physique Nucléaire et de Physique des Particules, France and by the HEPtools EU Marie Curie Research Training Network under the contract number MRTN-CT-2006-035505.

Introduction

- Joint Cumulants:

$$\kappa(X_1, \dots, X_n) = \sum_{\pi} \prod_{B \in \pi} (|B| - 1)!(-1)^{|B|-1} E \left(\prod_{i \in B} X_i \right)$$

where π runs through the list of partitions $\{1, \dots, n\}$ and B runs through the blocks of a given partition, e.g. $E(X_1, X_2) \Rightarrow B = 1$. $E(X_1)E(X_2) \Rightarrow B = 2$.

- Examples:
 $B = 1$

$$\text{Mean: } \kappa(X_1) = \overbrace{E(X_1)}^{B=1}$$

$$\text{Covariance: } \kappa(X_1, X_2) = \overbrace{E(X_1, X_2)}^{B=2} - \overbrace{E(X_1)E(X_2)}^{B=1}$$

- Higher order statistical momenta:

3rd order cumulant tensor: $\kappa(X_1, X_2, X_3)$

4th order cumulant tensor: $\kappa(X_1, X_2, X_3, X_4)$

- Correlations $\hat{=} 2$ nd order dependencies ($n \equiv 2$)
- Diagonalising covariance matrix $\hat{=} \text{whitening}$
- Removing means $\hat{=} \text{centering}$
- Normalising variances to unity $\hat{=} \text{sphereing}$

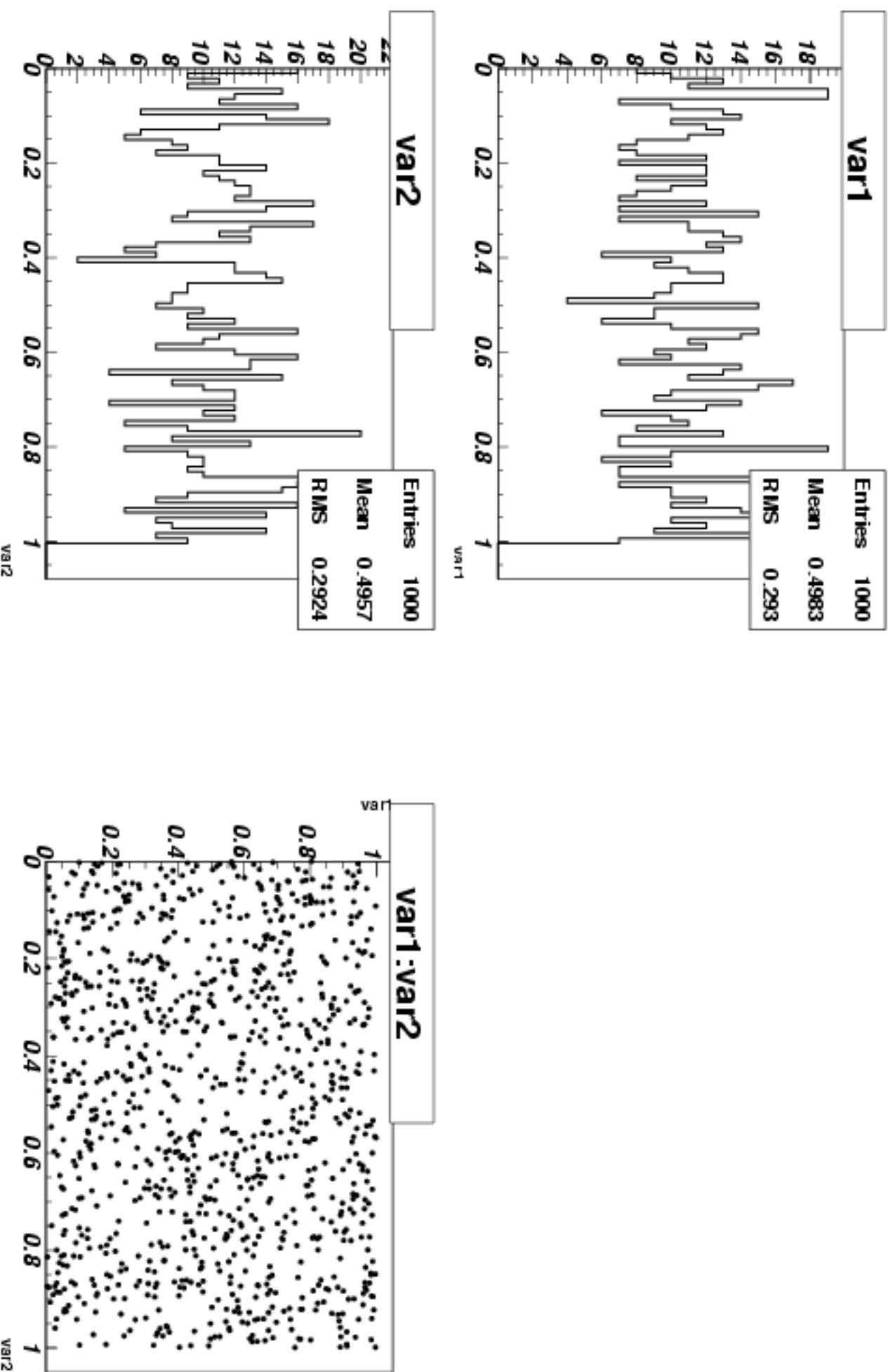
Introduction

- Covariance (2D) matrix $\hat{=} 2^{\text{nd}}$ order cumulant tensor
Diagonal elements = Variances
- 3rd order cumulant tensor $\hat{=} 3\text{D}$ matrix
Diagonal elements = Skewness
- 4th order cumulant tensor $\hat{=} 4\text{D}$ matrix
Diagonal elements = Kurtosis

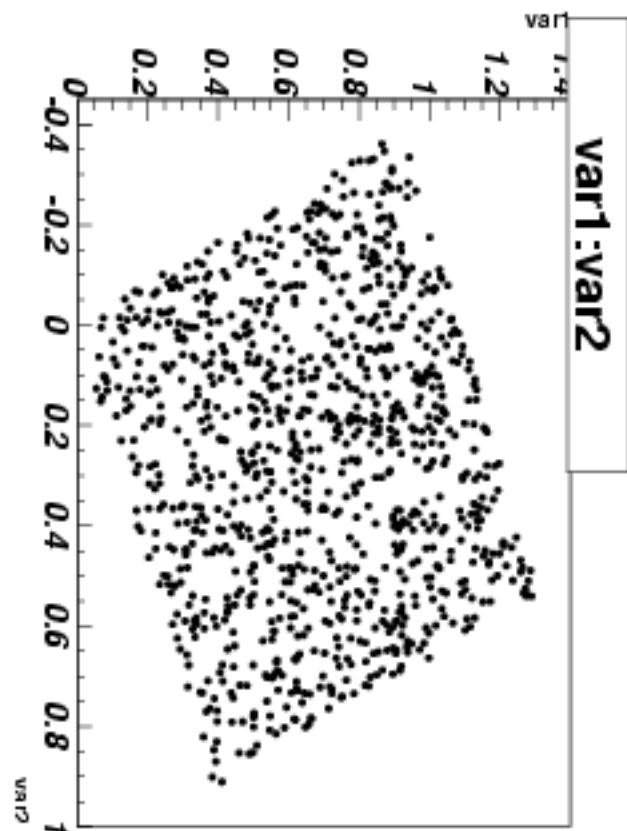
ICA

- Algorithm exists to diagonalise 3rd and 4th order cumulant tensor simultaneously
 - Diagonalisation is orthogonal to 2nd order cumulant tensor
 \Rightarrow Diagonalise 2nd, 3rd & 4th order cumulant tensor
- Gaussian variables can only have correlations (\equiv 2nd order dependencies)
 - Diagonalising covariance matrix (PCA) \equiv maximizing variances
 - Diagonalising 3rd & 4th order cumulant tensor \equiv maximizing non-gaussianity
 - (after PCA) gaussian variables have to be excluded (before ICA)

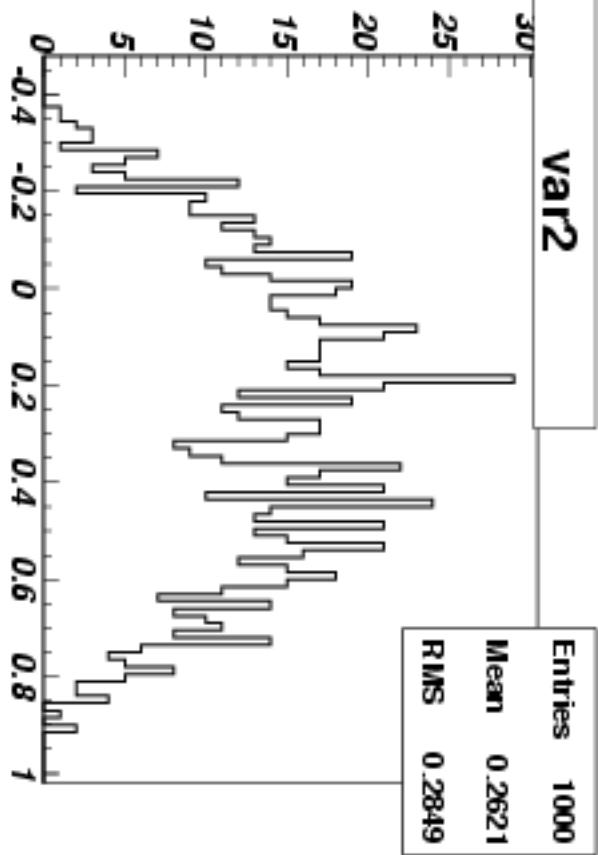
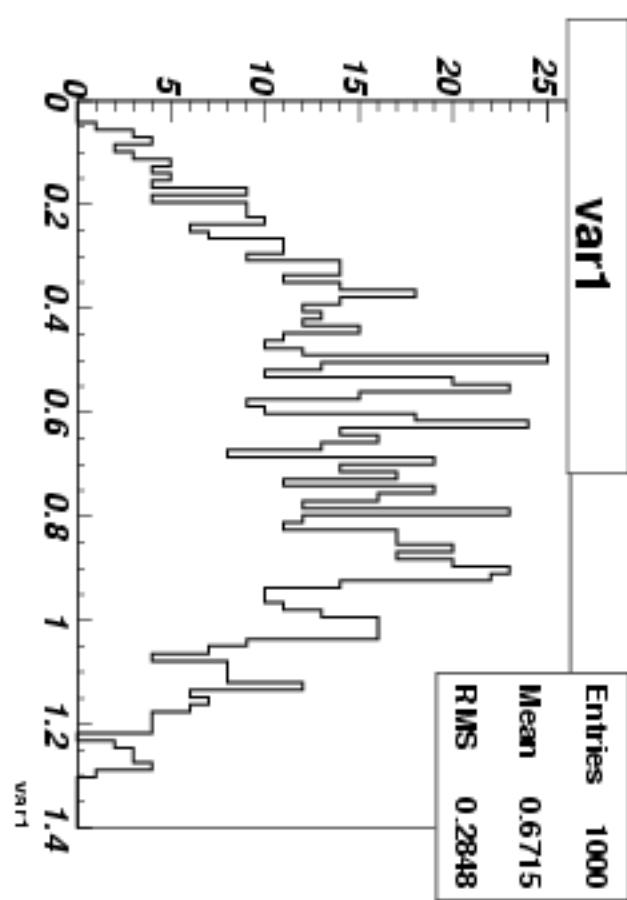
Flat independant random variables



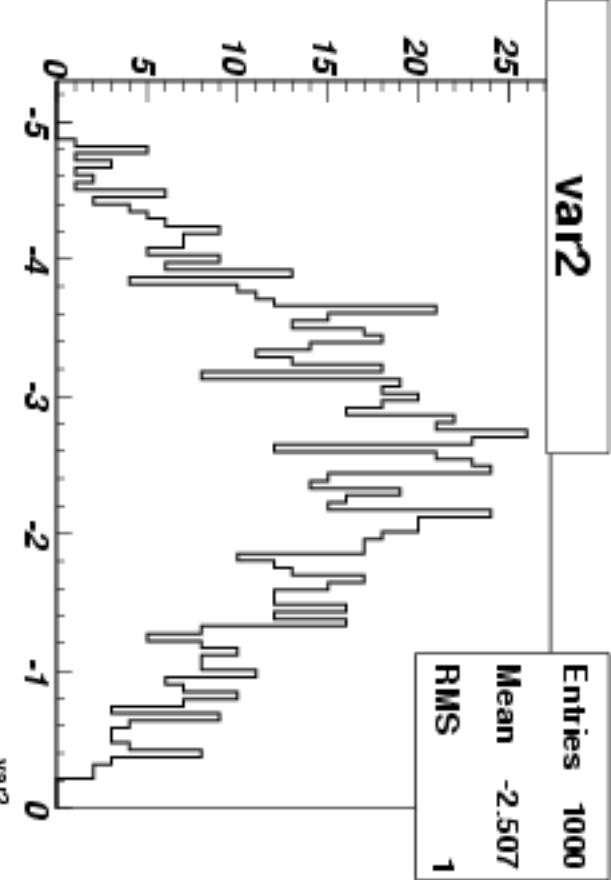
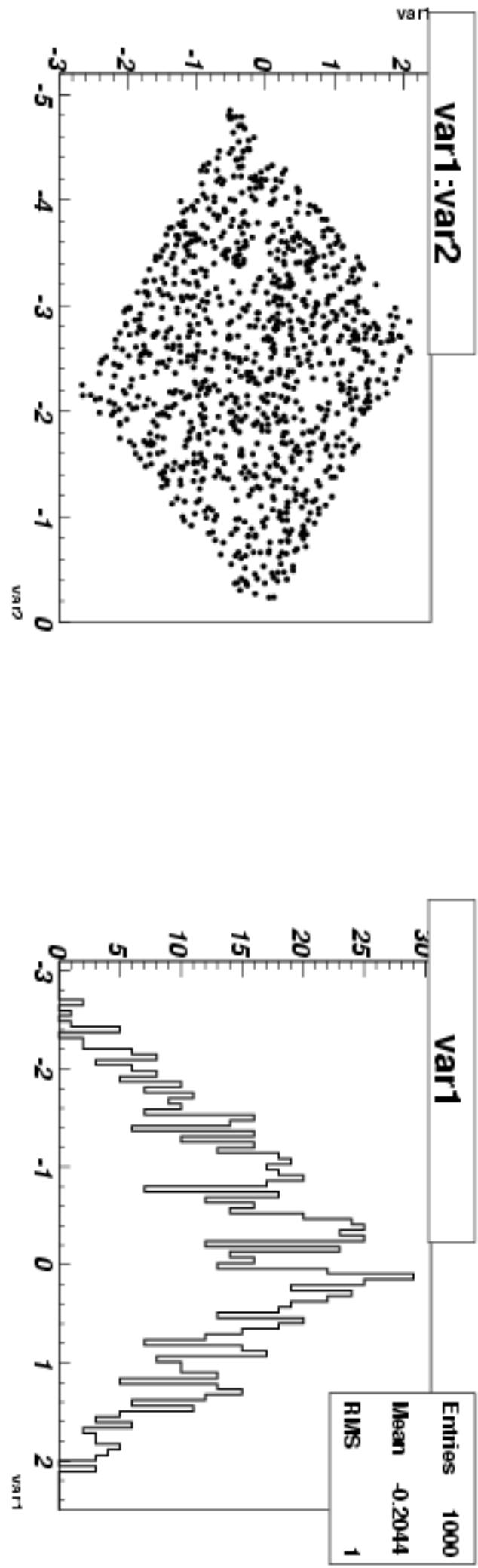
Mixing random variables



- Mixing independent variables with mixing matrix: $\begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}$, e.g. $\phi = 40^\circ$ rotation

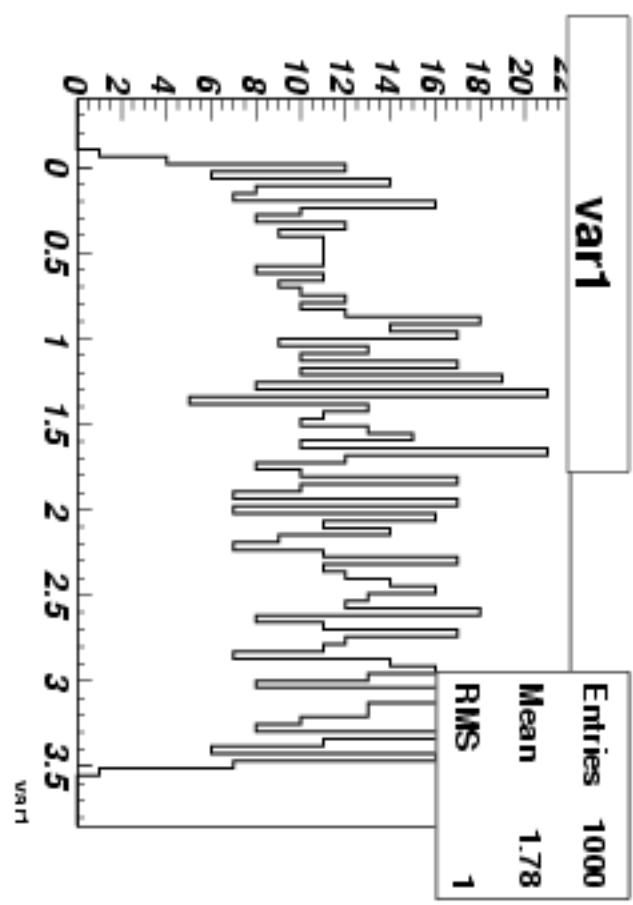
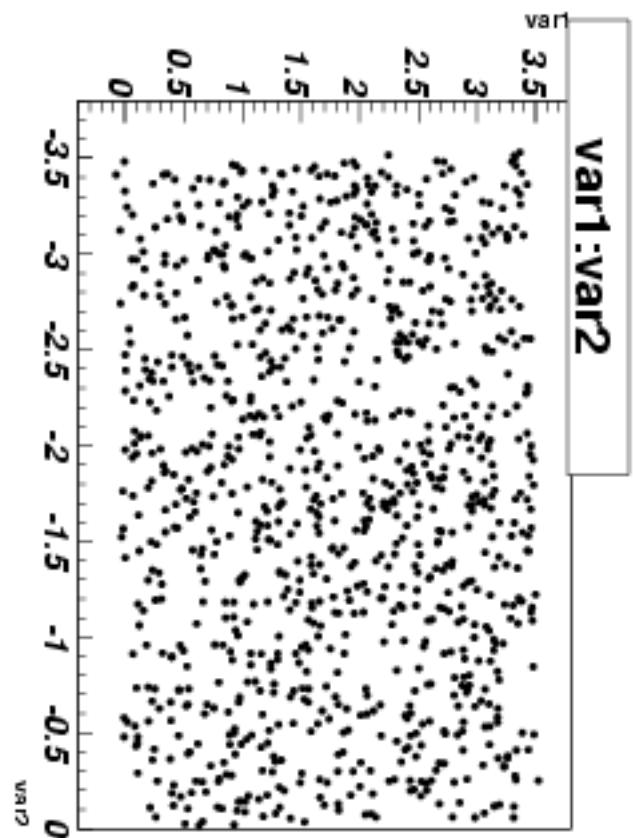


Diagonalising covariance matrix

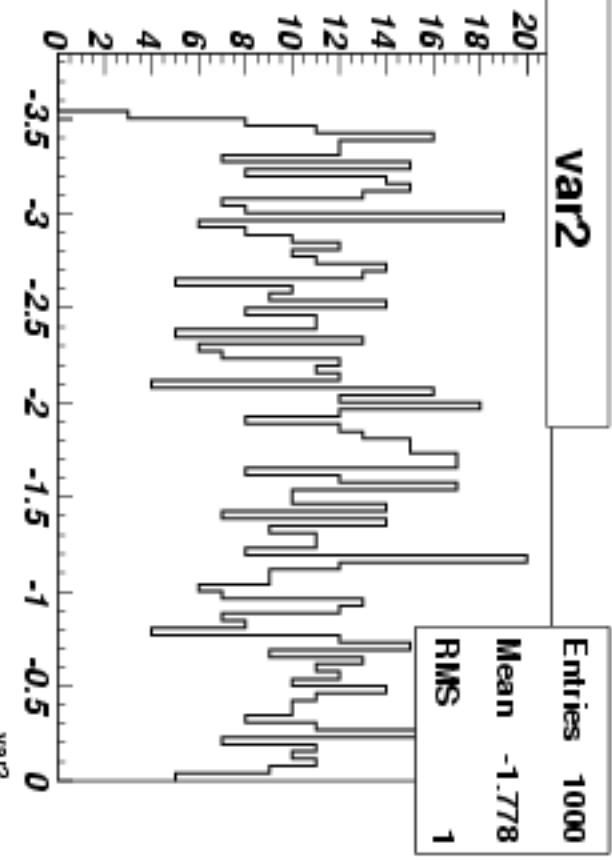


- Covariance matrix = 2nd order cumulant tensor

Diagonalising 3rd & 4th order Cumulant tensor



- Diagonalising 3rd & 4th order cumulant tensor simultaneously
- Have now Eigen-Matrices instead of Eigen-Vectors
- Diagonalisation is orthogonal to 2nd order cumulant tensor (covariance matrix)
- Removing higher order dependencies up to order four





Conclusions

- Only for gaussian variables it is guaranteed that decorrelation removes dependencies of any order
- 3rd & 4th order cumulant tensor diagonalisation
 - removes higher order dependencies (up to order 4)
 - Diagonalisation orthogonal to diagonalisation of covariance matrix
- Concerning gaussian error propagation:
 - Fine for gaussian variables (just Variances after decorrelation)
 - Asymmetric errors for skewed variables
 - Taking Kurtosis into account
 - Respecting gaussian limit (normalised Kurtosis)
 - Linear error propagation in limit of uniform distributions
 - Have to think about analytic form of (≤ 4 th order) Independent error propagation

