# Geometric Scaling and DGLAP evolution

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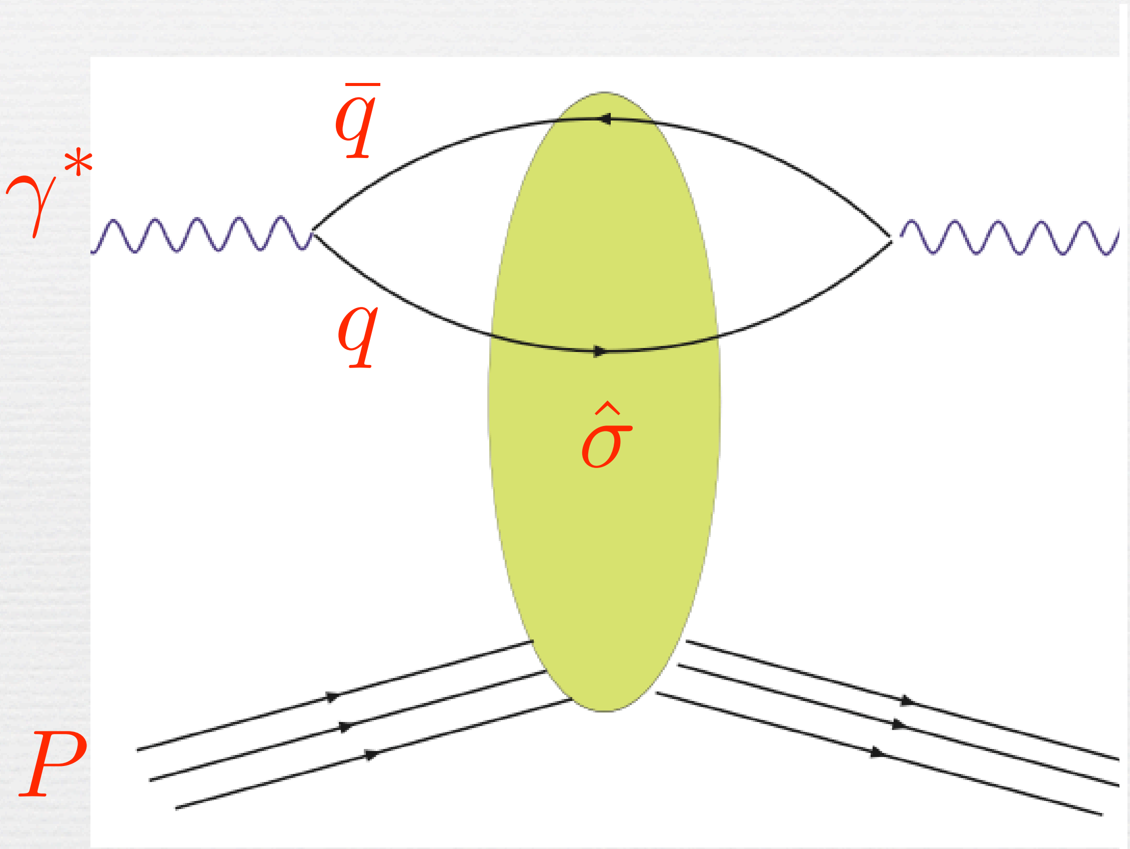
# Outline

- Dipole picture.
- Golec-Biernat and Wusthoff (GBW) model.
- Geometrical scaling.
- Evolution of the DGLAP from the saturation line.

# Virtual photon-proton scattering : dipole picture




Approximation justified for very high energy-low x.

# Virtual photon-proton total cross section

$$\sigma_{T,L}(x,Q^2) \;=\; \int d^2\mathbf{r} \int_0^1 dz \; |\Psi_{T,L}(r,z,Q^2)|^2 \; \hat{\sigma}(r,x)\,,$$

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GBW model

Saturation radius(scale)

$$R_0(x) \sim \frac{1}{Q_s(x)}$$

Dipole size $r$

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dipole cross section depends on a single variable

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2) Property of the dipole formula for the total cross section:
rescale the integration variable $r$ (neglecting the quark masses)

$$\sigma_{\gamma * p}(x, Q^2) \;=\; \sigma_{\gamma * p}(\tau)$$

$$\tau = \frac{Q^2}{Q_s^2(x)}$$

*Note that both conditions are necessary*

# Qualitative analysis from GBW model:

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$$\sigma_{\gamma * p} \sim \sigma_0 / \tau$$

Large $\tau$

*Modulo logarithmic terms*

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Geometric scaling should be valid at small x and not too large Q only!

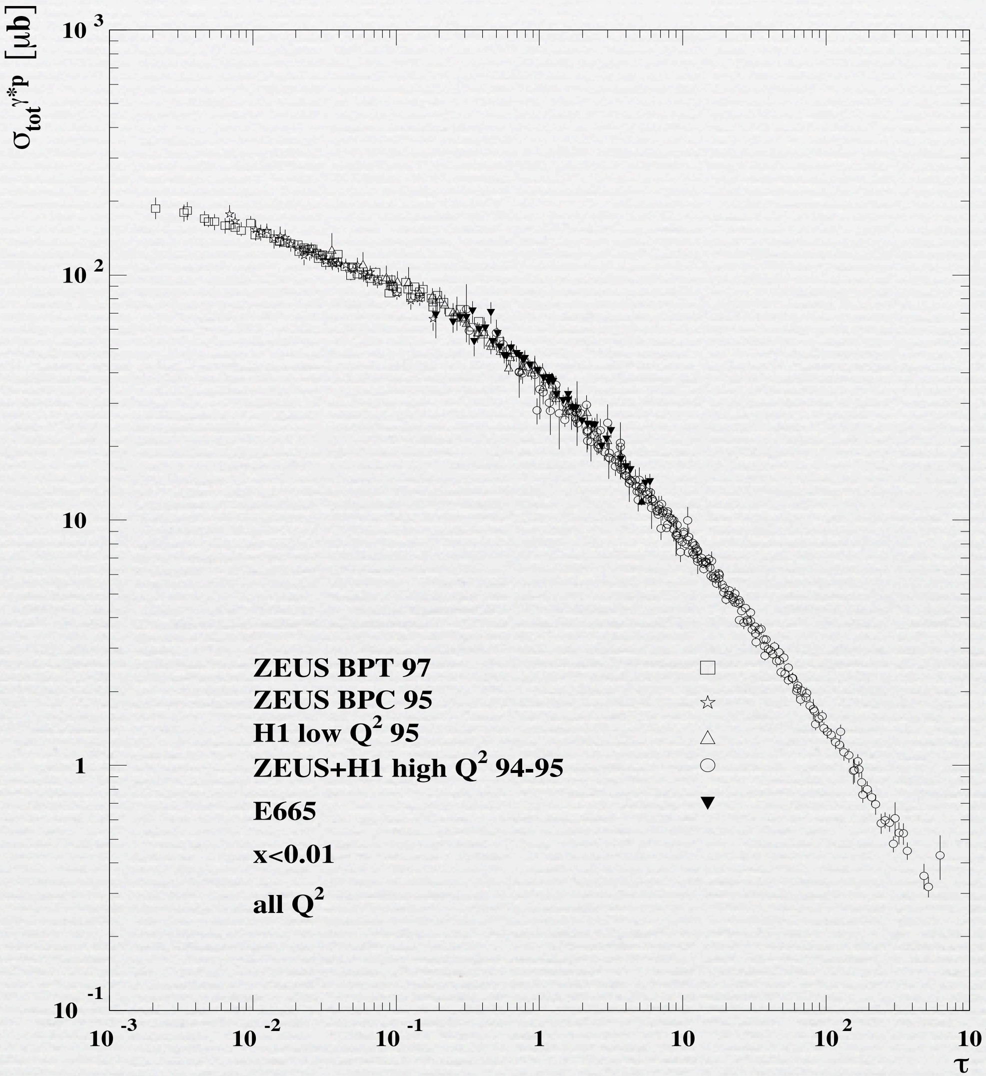
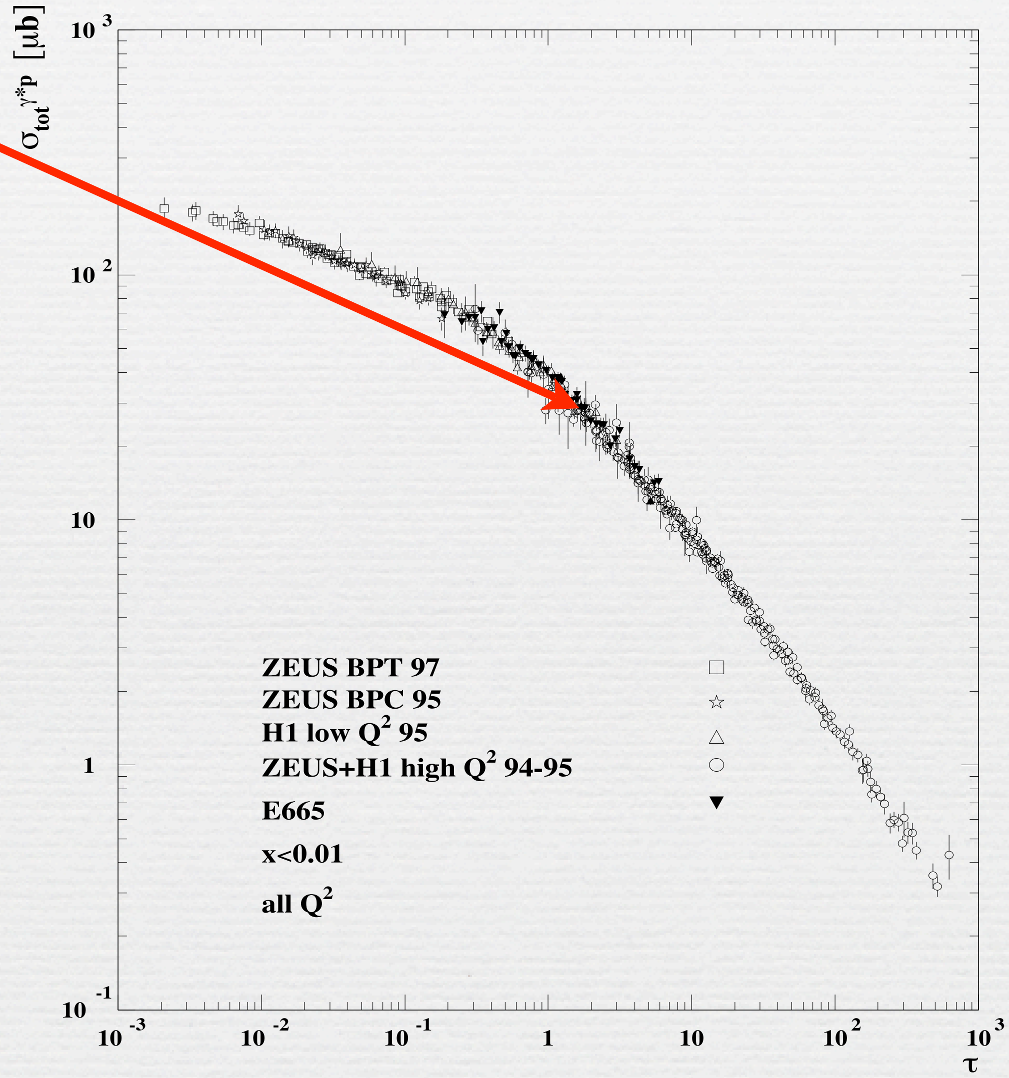



Figure 1: Experimental data on $\sigma_{\gamma^* p}$ from the region $x < 0.01$ plotted versus the scaling variable $\tau = Q^2 R_0^2(x)$.

In each bin of scaling variable are data points with different x and Q values

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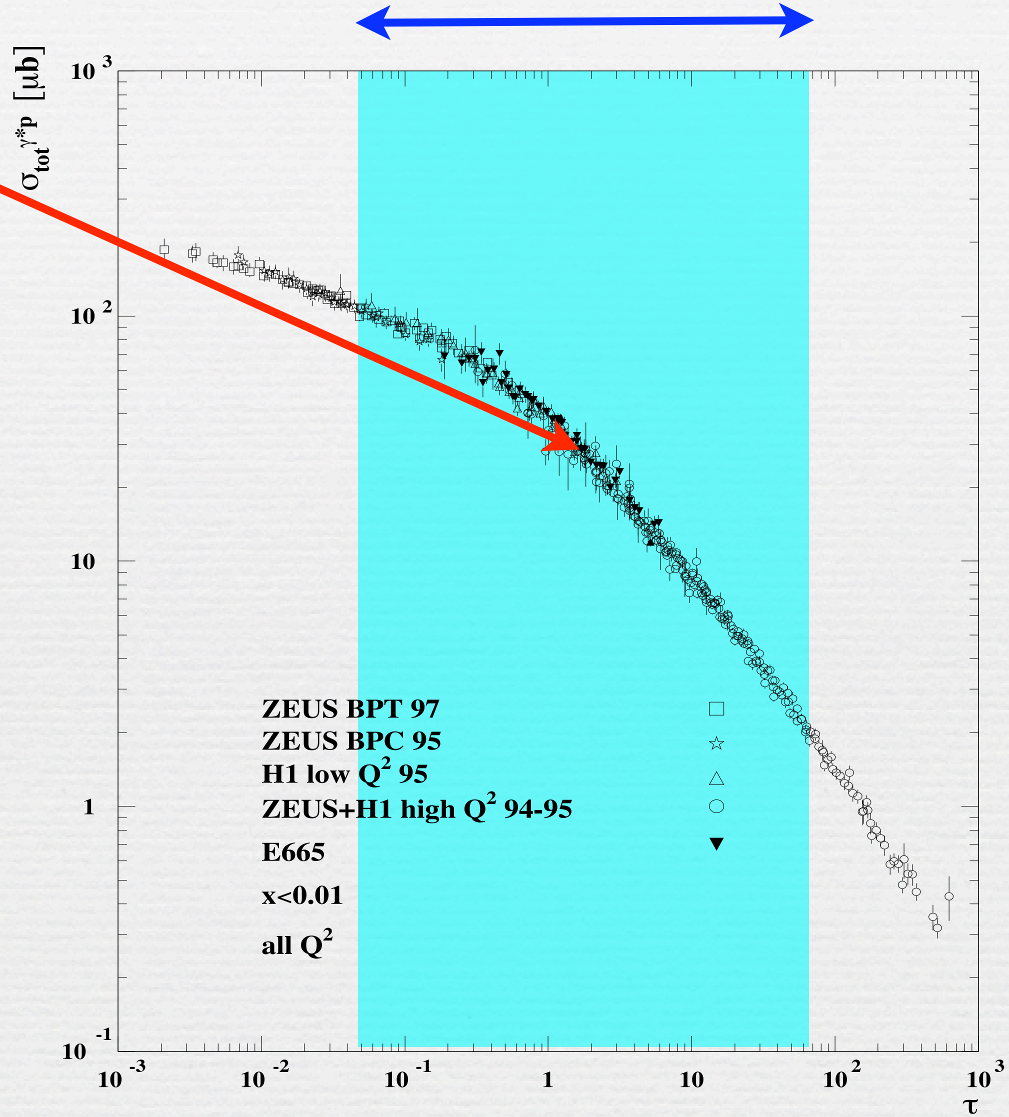


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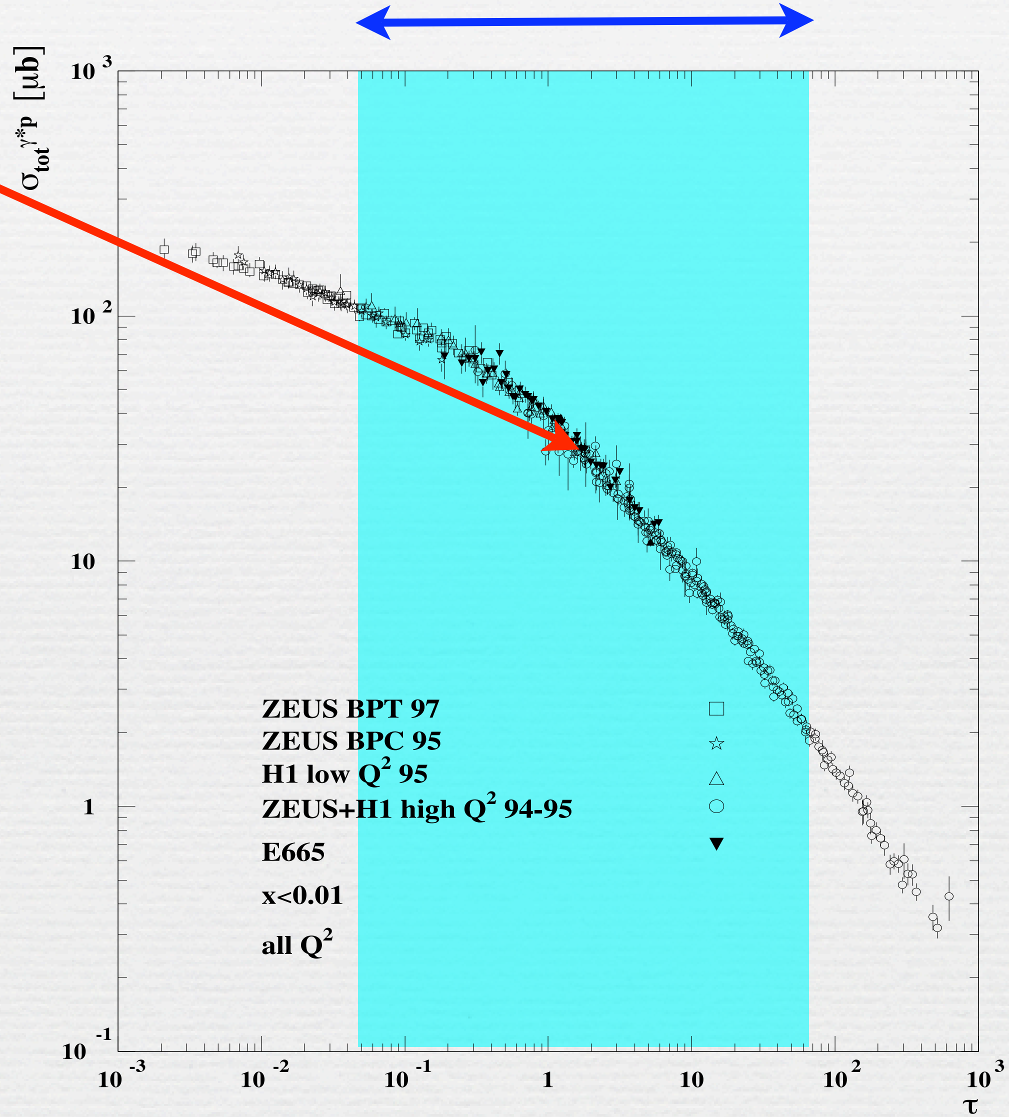


Scaling motivated by the GBW model + dipole picture. Regularity observed in the data independently of the model.

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No scaling at large x, as expected.

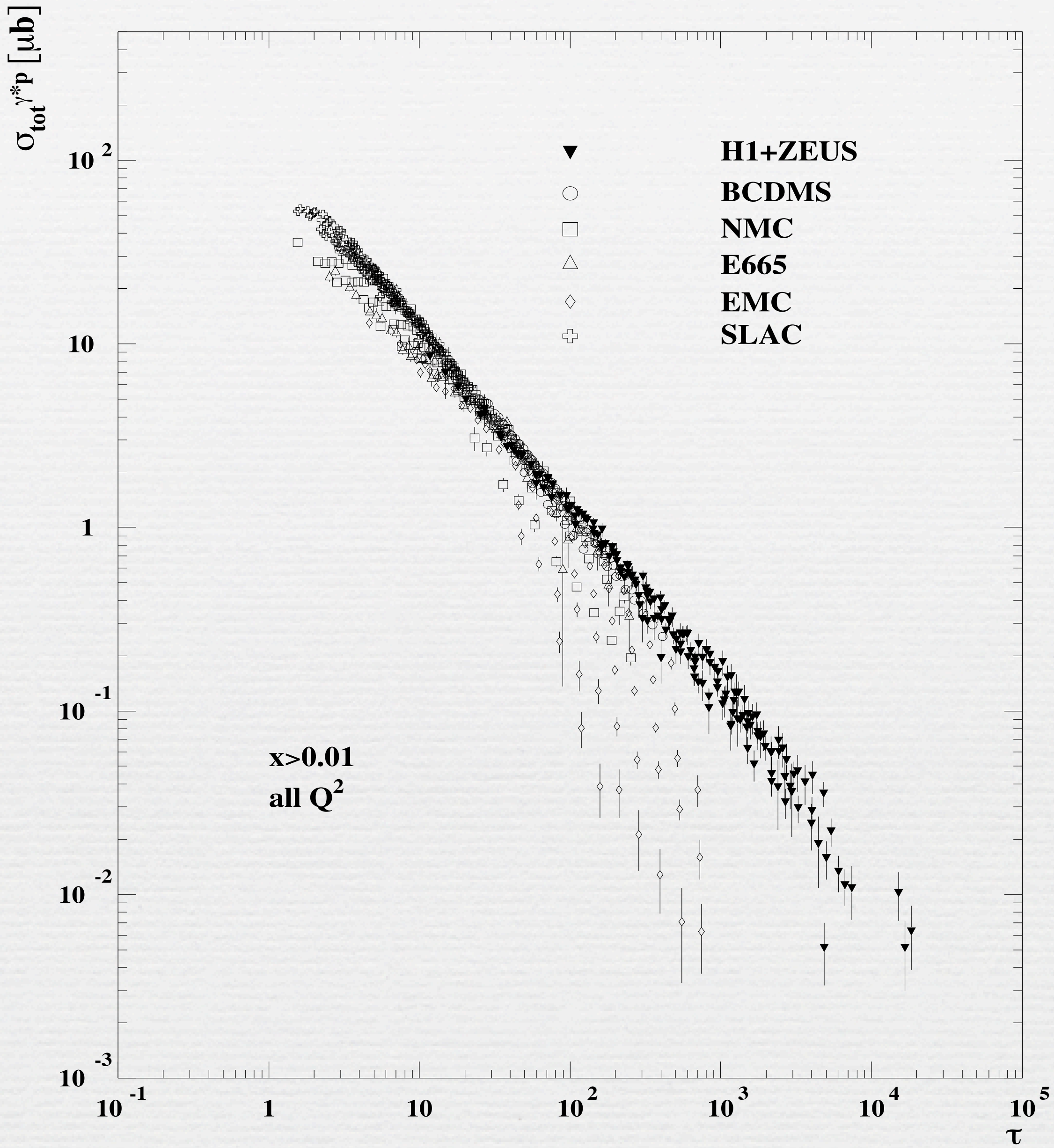


Figure 4: Experimental data on $\sigma_{\gamma^* p}$ from the region $x > 0.01$ plotted versus the scaling variable $\tau = Q^2 R_0^2(x)$.

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$$\hat{\sigma}(x,r) = \sigma_0\, g(r\, Q_s(x))$$

If $r\, Q_s(x)$ is not too large though, i.e. close to the saturation regime

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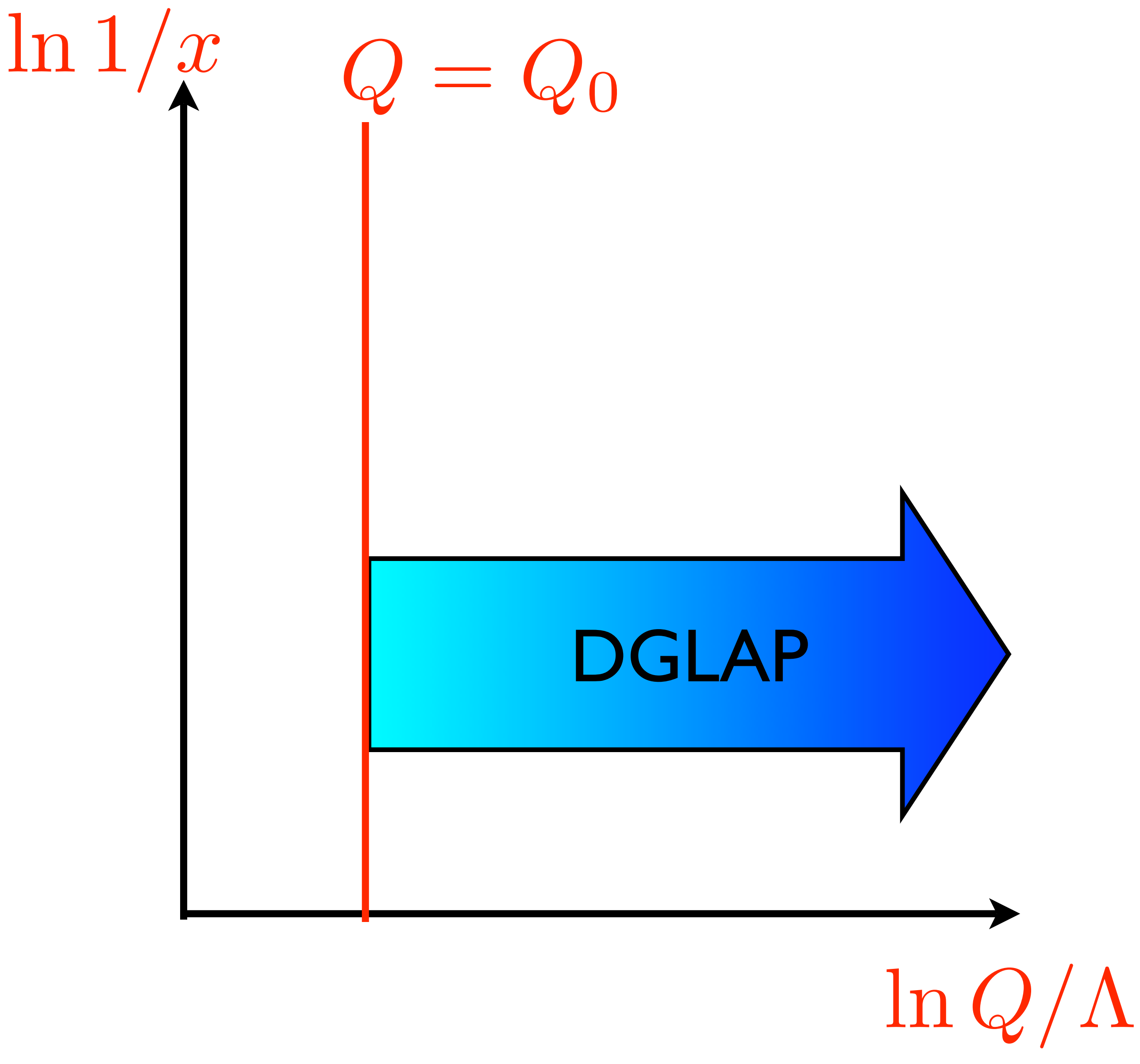
- Saturation scale is relatively low.
- Why scaling works outside the regime of very low Q ?

$$Q^2 > Q_s^2(x)$$

# Geometric scaling vs DGLAP



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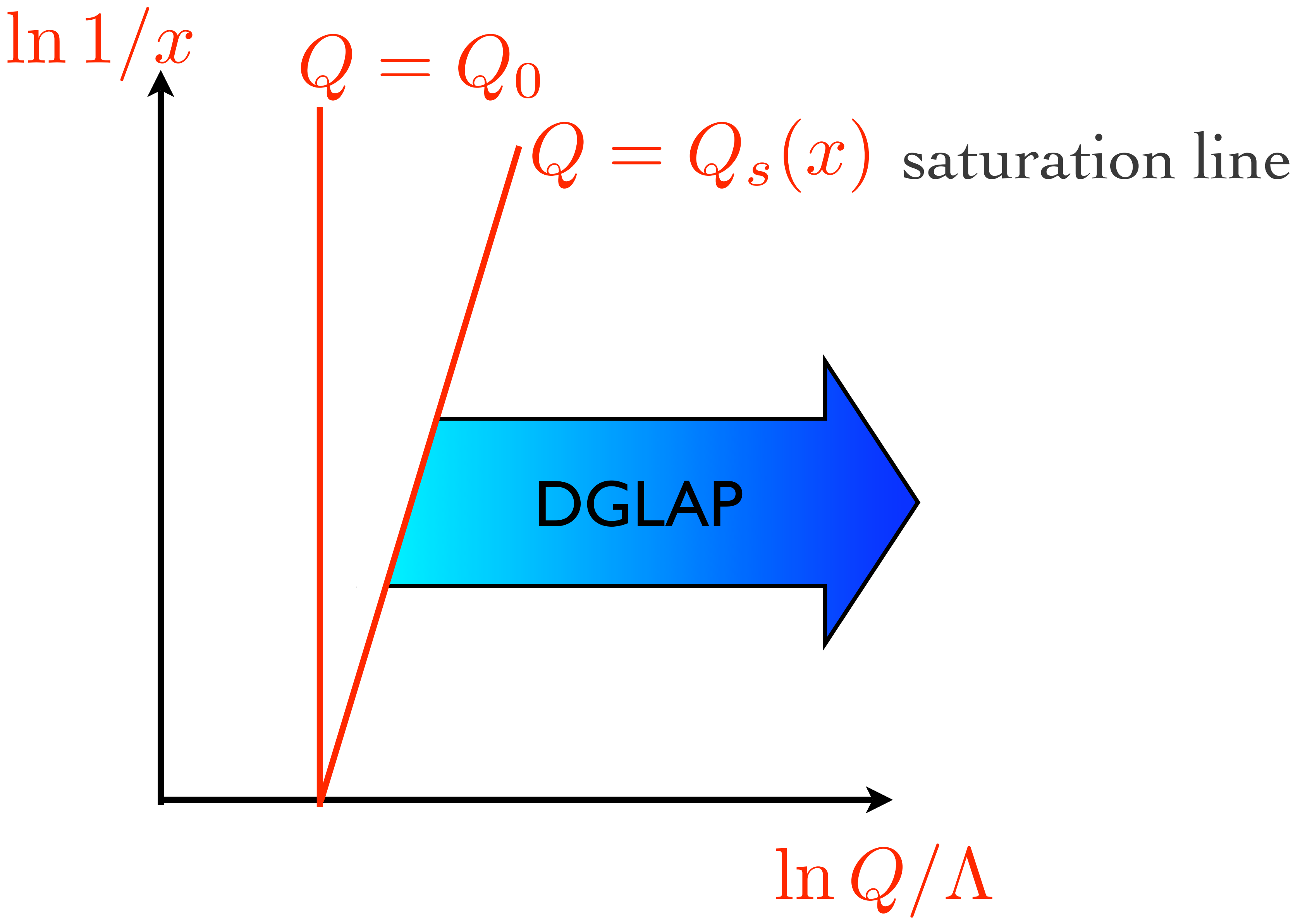

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Scaling condition for the gluon density (at fixed coupling first) at the boundary given by the saturation line

$$\frac{\alpha_s}{2\pi} x g(x, Q^2 = Q_s^2(x)) = \frac{\alpha_s}{2\pi} r^0 x^{-\lambda},$$

## Mellin tr. definitions

$$xg(x,Q^2) = \frac{1}{2\pi i}\int d\omega x^{-\omega} g_\omega(Q^2)$$

$$g_\omega(Q^2) = \int_0^1 dx x^\omega g(x,Q^2),$$

## DGLAP in Mellin space

$$\frac{\partial g_\omega(Q^2)}{\partial \ln(Q^2/\Lambda^2)} = \frac{\alpha_s}{2\pi}\gamma_{gg}(\omega) g_\omega(Q^2)$$

## Anomalous dimension

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## Scaling condition

$$\frac{1}{2\pi i}\int d\omega g_0(\omega) x^{-\omega-\lambda(\alpha_s/2\pi)\gamma_{gg}(\omega)} = r^0 x^{-\lambda}.$$

## Equation for the function $g_0(\omega)$

# Solution for the gluon density

Solution at small x:

$$\frac{\alpha_s}{2\pi}\frac{xg(x,Q^2)}{Q^2} \simeq \frac{r^0}{\widetilde{Q}_0^2}\left(\frac{\alpha_s}{2\pi}\right)\left(\frac{Q^2}{Q_s^2(x)}\right)^{(\alpha_s/2\pi)\gamma_{gg}(\omega_0)-1}$$

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Critical value of the saturation exponent:
determines the existence of scaling.

Example:in the DLLA approximation

$$\gamma_{gg}^{DL}(\omega) = \frac{2N_c}{\omega}$$

$\lambda \geqslant 4\bar{\alpha}_s$ scaling

$\lambda < 4\bar{\alpha}_s$ no scaling

# Running coupling case

Mellin representation

$$\frac{\alpha_s(Q^2)}{2\pi} g_\omega(Q^2) = \frac{\alpha_s(Q_0^2)}{2\pi} g_0(\omega) \left( \frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)} \right)^{b\gamma_{gg}(\omega)-1}$$

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Parameter which controls violation of scaling

$$\alpha_s(Q_s^2(x)) \ln(Q^2/Q_s^2(x))$$

# Summary

- Geometric scaling is expected to hold exactly when $Q^2 \leq Q_s^2(x)$.
- For $Q^2 > Q_s^2(x)$ the non-linear effects in the evolution of the gluon density should be small.
- We solved the DGLAP evolution equation for the gluon density with the initial condition provided along the critical line $Q^2 \equiv Q_s^2(x)$.
- For the fixed coupling the geometric scaling is preserved, provided the exponent $\lambda > \lambda_{crit}$. For $\lambda < \lambda_{crit}$ there is no scaling, since the solution is controlled by the other branch point.
- In the running coupling case the scaling is only approximately preserved. The violation can be factored out.
- In general, geometric scaling is expected to hold even in this case provided $\ln Q^2/Q_s^2(x) \ll \ln Q_s^2(x)/\Lambda^2$.