A duality relation between one-loop and phase-space integrals

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ArXiv:0804.3170 [hep-ph]

next-to-leading order (NLO) cross-sections

- LO @ LHC: 100% uncertainty typically
- NLO @ LHC necessary for 2→3 (many recent results) and 2→4 (not yet a cross section)

(see Zanderighi's talk)

$$\sigma^{NLO} = \int_{m+1} d\sigma^{R} + \int_{m} d\sigma^{V}$$

new feature wrt LO: combine m with m+1

real radiation

virtual contribution

real radiation

kinematics: momentum conservation + observable dependent function

$$\int_{m+1} d\sigma^R = \int d\Phi^{(m+1)}(\{p_i\}) \quad \underbrace{M^{(m+1)}(\{p_i\}) \quad F^{(m+1)}(\{p_i\})}$$

several well known/tested working methods (subtraction, dipole, slicing, mixed, ...)

split phase-space integrand in two parts: $(\ldots)_{\mathrm{fin}} + (\ldots)_{\mathrm{div}}$

IR finite: computable numerically as LO

IR singular: analytically computable up to O(ε)

virtual contribution

$$\int_m d\sigma^V = \int d\Phi^{(m)}(\{p_i\}) \quad \underbrace{\int d^d q \quad M^{(m)}(\{p_i\})} \quad F^{(m)}(\{p_i\})$$

loop integral: in multiparton processes (m≥5) regarded as main practical bottleneck
many new developments in recent years (OPP, generalized Unitarity, ...)

S. Catani @ HP^2 Zürich 2006
GR @ ILC Valencia 2006
T.Gleisberg PhD thesis

general goal

I transform loop integral into customary phase space integral for real radiation (loop ⇔ phase-space duality)

$$\int_{loop} d^d q \;\; M^{(m)}(\{p_i\},q) = \int d\Phi(q) \;\; M^{(m+q)}(\{p_i\},q)$$

$$d\Phi(q) \leftarrow d^d q \;\; \delta_+(q^2)$$

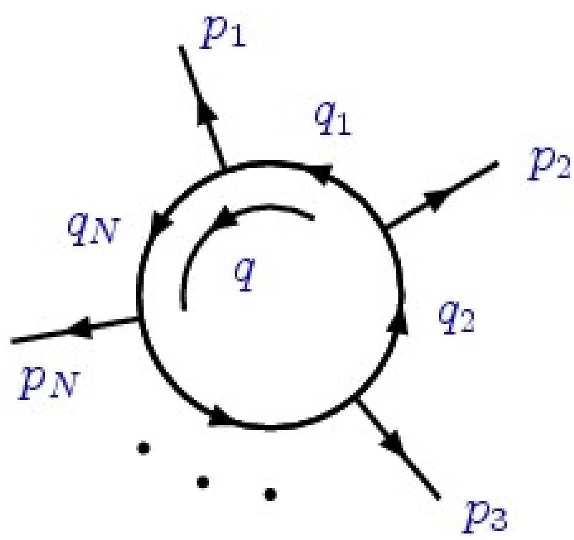
II then treat $\int_{m+q}(\ldots)$
similarly to the real emission contribution $\int_{m+1}(\ldots)$

III Monte Carlo integration

Outline

- The Feynman Tree Theorem
- A duality theorem between one-loop integrals and single-cut phase-space integrals
- Relating the FTT and the duality relation
- Massive integrals and unstable particles
- Gauge poles
- Duality at the amplitude level
- Final remarks

Notation



To simplify the presentation: massless internal lines only (more on massive particles later)

Scalar one-loop integral

$$L^{(N)}(p_1, \dots, p_N) = -i \int \frac{d^d q}{(2\pi)^d} \prod_{i=1}^{N} \frac{1}{q_i^2 + i0}$$

q^μ is the loop momentum (anti-clockwise)

internal lines

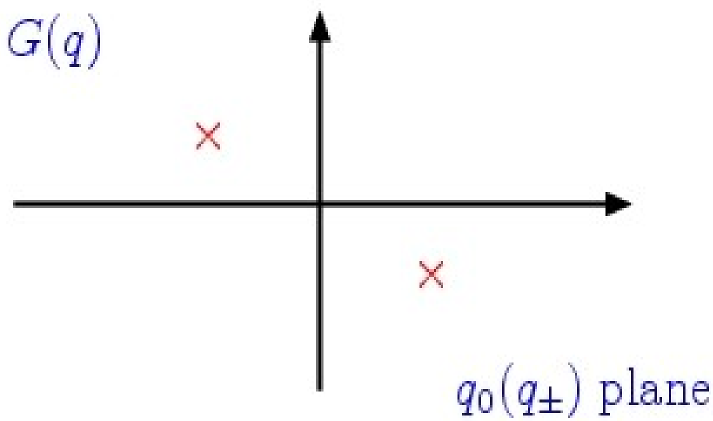
$$q_i = q + \sum_{k=1}^{i} p_k \; , \qquad \sum_{i=1}^{N} p_i = 0 \; , \qquad p_{N+i} = p_i \; .$$

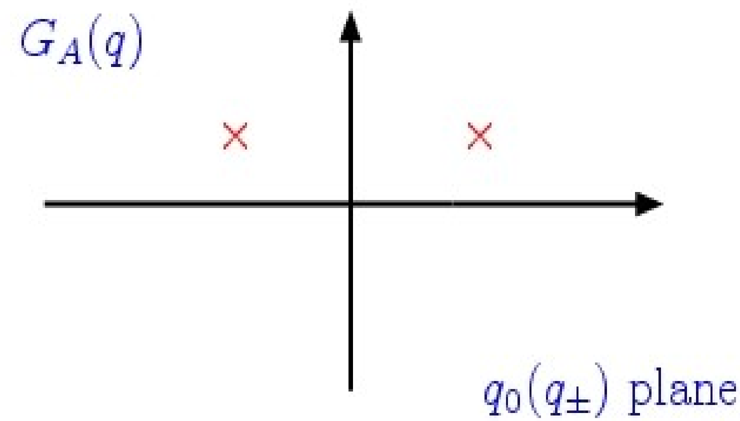
shorthand notation:

$$-i \int \frac{d^d q}{(2\pi)^d} \; \cdots \; \equiv \int_q \cdots \; , \qquad -i \int_{-\infty}^{+\infty} dq_0 \int \frac{d^{d-1} \boldsymbol{q}}{(2\pi)^{d-1}} \; \cdots \; \equiv \int dq_0 \int_{\boldsymbol{q}} \cdots$$

$$\tilde{\delta}(q) \equiv 2\pi \; i \; \delta_+(q^2)$$

Feynman and Advanced propagators





Feynman propagator

$$G(q) \equiv \frac{1}{q^2+i0}$$

+i0: positive frequencies are propagated forward in time, and negative frequencies backward

Advanced propagator

$$G_A(q) \equiv \frac{1}{q^2-i0\,q_0}$$

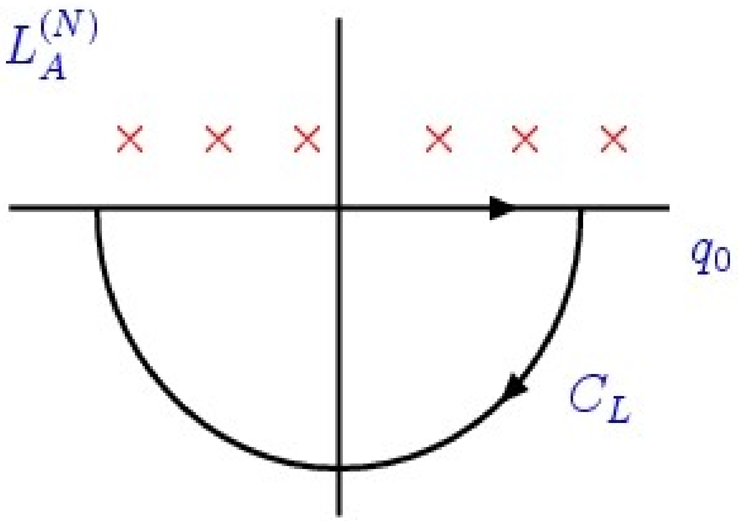
both poles displaced above the real axis (independently of the sign of the energy)

and are related by

$$\frac{1}{x \pm i0} = PV\left(\frac{1}{x}\right) \mp i\pi\,\delta(x)$$

$$G_A(q) = G(q) + \tilde{\delta}(q)$$

Feynman Tree Theorem



Advanced one-loop integral:
Feynman propagators replaced by advanced propagators

$$L_A^{(N)}(p_1, \dots, p_N) = \int_q \prod_{i=1}^{N} G_A(q_i)$$

Cauchy residue theorem

$$L_A^{(N)}(p_1, \dots, p_N) = 0$$

then

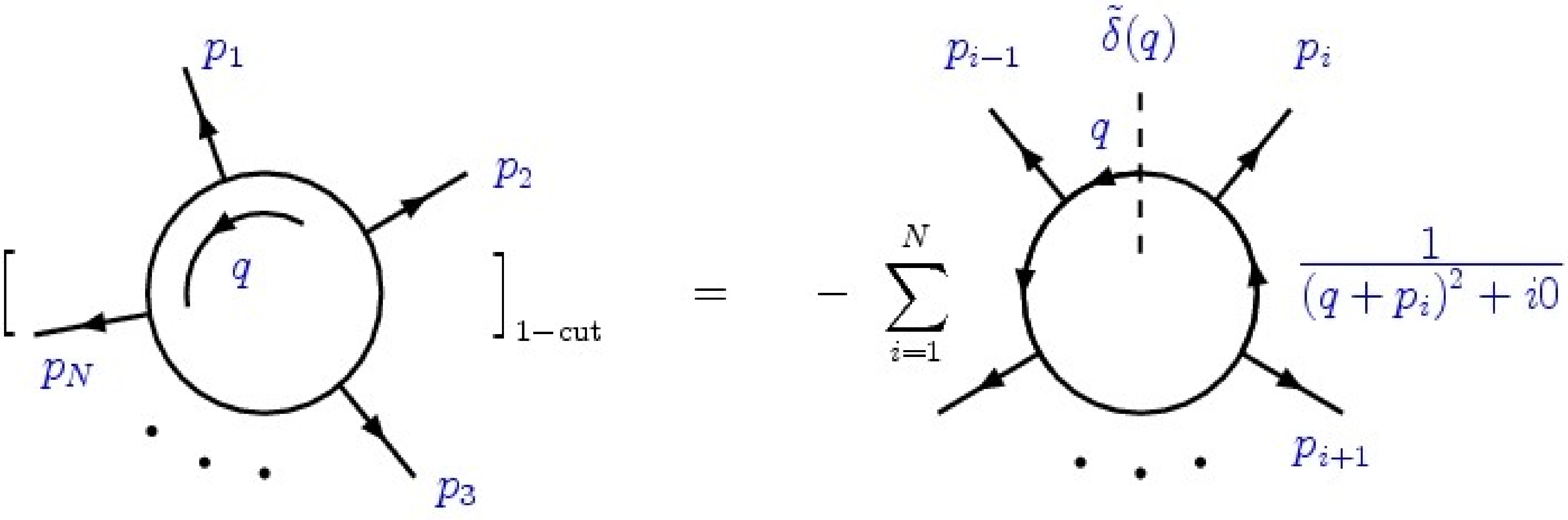
$$L_A^{(N)}(p_1, \dots, p_N) = \int_q \prod_{i=1}^{N} [G(q_i) + \tilde{\delta}(q_i)]$$
$$= L^{(N)} + L_{1-cut}^{(N)} + L_{2-cut}^{(N)} + \dots + L_{N-cut}^{(N)}$$

in four-dimensions, 4-cut maximum

Feynman Tree Theorem

$$L^{(N)}(p_1,\dots,p_N) = -\left[L^{(N)}_{1-cut}(p_1,\dots,p_N) + \dots + L^{(N)}_{N-cut}(p_1,\dots,p_N)\right]$$

The single-cut contribution



FTT for scattering amplitudes

For **relativistic, local and unitary** quantum field theories

$$\mathcal{A}^{(1-loop)} = - \left[\mathcal{A}^{(1-loop)}_{1-cut} + \mathcal{A}^{(1-loop)}_{2-cut} + \dots \right]$$

$\mathcal{A}^{(1\text{-}loop)}$ is a linear combination of one-loop integrals that differ from $L^{(N)}$ only by the inclusion of interaction vertices and, eventually, particle masses

- **particle masses:** real masses (unitary theories) do not affect the imaginary part of the poles

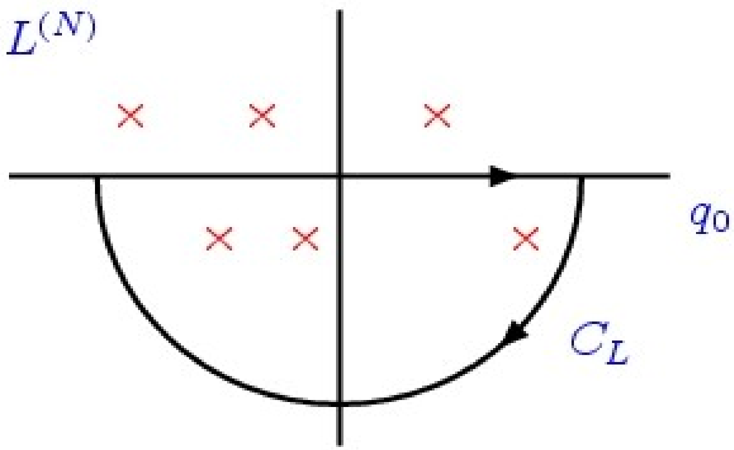
$$\tilde{\delta}(q_i) \rightarrow \tilde{\delta}(q_i, M_i) = 2\pi \ i \ \delta_+(q_i^2 - M_i^2)$$

- **interaction vertices:** introduce numerator factors

  in local theories at worst polynomials in the loop momentum ⇨ no additional singularities (more on gauge poles later)

  unitary constrains the convergence of the q_0 integration at infinity

Duality Theorem



Cauchy residue theorem

close the contour at ∞ on the lower half plane

⇨ select residues with **positive** definite energy

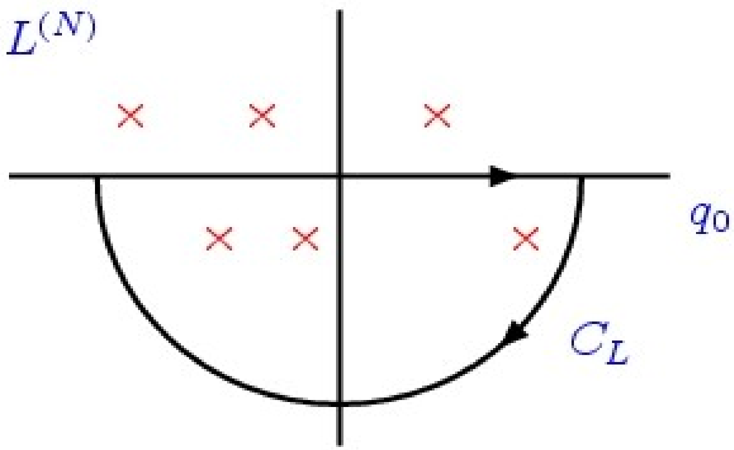
$$L^{(N)}(p_1,\dots,p_N) = -2\pi\, i \int_q \sum \mathrm{Res}_{\mathrm{Im}\, q_0<0} \left[\prod_{i=1}^{N} G(q_i)\right]$$

$$\mathrm{Res}_{ith-pole}\left[\prod_{j=1}^{N} G(q_j)\right] = \left[\mathrm{Res}_{\;ith-pole}\; G(q_i)\right] \left[\prod_{j\neq i}^{N} G(q_j)\right]_{ith-pole}$$

$$\mathrm{Res}_{ith-pole}\frac{1}{q_i^2+i0} = \int dq_0\, \delta_+(q_i^2)$$

- **equivalent to cut that line and set it on-shell**
- **one-loop integral represented as a linear combination of *N* single-cut phase-space integrals**
- **shift $q_i \rightarrow q$ in each term ⇔ single phase-space integral over *N* terms**

Duality Theorem



Cauchy residue theorem

$$L^{(N)}(p_1,\dots,p_N) = -2\pi\, i \int_{\boldsymbol{q}} \sum Res_{\mathrm{Im}\, q_0<0} \left[\prod_{i=1}^{N} G(q_i) \right]$$

$$Res_{ith-pole} \left[\prod_{j=1}^{N} G(q_j) \right] = \left[Res_{\; ith-pole} \; G(q_i) \right] \left[\prod_{j\neq i}^{N} G(q_j) \right]_{ith-pole}$$

$$Res_{ith-pole} \frac{1}{q_i^2+i0} = \int dq_0\, \delta_+(q_i^2)$$

$$\left[\prod_{j\neq i}^{N} \frac{1}{q_j+i0} \right]_{ith-pole} = \prod_{j\neq i}^{N} \frac{1}{q_j^2 - i0\,\eta(q_j-q_i)}$$

- the customary $+i0$ prescription is modified
- ***Lorentz covariant dual prescription***
- ***η is a future-like vector: $\eta_0 > 0$, $\eta^2 \geq 0$***
- analytic continuation: $s_{ij} \rightarrow s_{ij}$ - $i0$ ***wrong***

The calculation is elementary, but involves some **subtle points**

$$\left[\frac{1}{(q+k_j)^2+i0}\right]_{q^2=-i0,\, q_0=q_0^{(+)}} = \frac{1}{2q_0^{(+)}k_{j0}-2\boldsymbol{q}\cdot\boldsymbol{k_j}+k_j^2}$$

where $q_0^{(+)} = \sqrt{\boldsymbol{q}^2-i0}$

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where $$q_0^{(+)} = \sqrt{\boldsymbol{q}^2-i0} \simeq |\boldsymbol{q}|-\frac{i0}{2|\boldsymbol{q}|}+O(i0^2)$$

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$$= \frac{1}{2qk_j+k_j^2-i0\,k_{j0}/|\boldsymbol{q}|}$$

$$q_0^{(+)} = \sqrt{\boldsymbol{q}^2-i0} \simeq |\boldsymbol{q}|-\frac{i0}{2|\boldsymbol{q}|}$$

The calculation is elementary, but involves some **subtle points**

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$$q_0^{(+)} = \sqrt{\boldsymbol{q}^2-i0} \simeq |\boldsymbol{q}|-\frac{i0}{2|\boldsymbol{q}|}$$

- only the sign matters:

  $- i0\; k_{j0}/|\boldsymbol{q}| \rightarrow - i0\; k_{j0} \rightarrow - i0\; \eta k_j$ where $\eta^\mu = (\eta_0, 0)$ with $\eta_0 > 0$

- *different choices of the future-like vector η is equivalent to different choices of the coordinate system*

Duality theorem

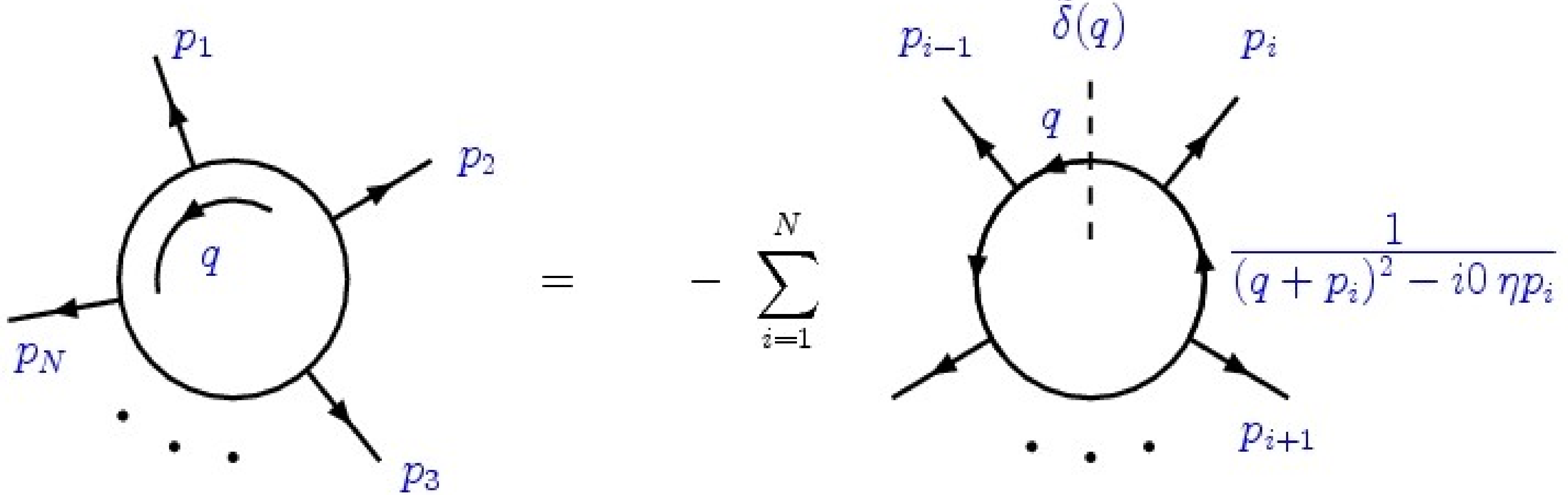
Duality relation between one-loop integrals and phase-space integrals

$$L^{(N)}(p_1,\dots,p_N) = -\,\tilde{L}^{(N)}(p_1,\dots,p_N) = -\left[I^{(N-1)}(p_1,p_{12},\dots,p_{1,N-1}) + cyclic\ perms.\right]$$

where

$$I^{(n)}(k_1,\dots,k_n) = \int_q \tilde{\delta}(q) \prod_{j=1}^{n} \frac{1}{2qk_j + k_j^2 - i0\eta k_j}$$

N one-particle phase-space integrals ⇔ one phase-space integral over *N* tree quantities



FTT-duality relation

● multiple-cut contributions ($m \geq 2$) are absent in the duality relation, only single-cut contributions are involved

● Feynman propagators ($+i0$) replaced by **dual propagators ($-i0\ \eta k_i$)**

● individual cut integrals depend on the future-like vector η^μ (residues are not Lorentz-invariant)
it has to be the same for all, then it cancels

● Single-cut contributions have extra unphysical singularities in the s_{ij} complex plane

η^μ correlates the unphysical single-cut singularities
FTT cancellation among multiple-cut contributions

Relating FTT with duality

an algebraic proof

Feynman and dual propagators are related by

$$\tilde{\delta}(q)\frac{1}{2qk+k^2-i0\eta k} = \tilde{\delta}(q)\left[G(q+k)+\theta(\eta k)\tilde{\delta}(q+k)\right]$$

proof:

$$\frac{1}{x\pm i0}=PV\left(\frac{1}{x}\right)\mp i\pi\delta(x)$$

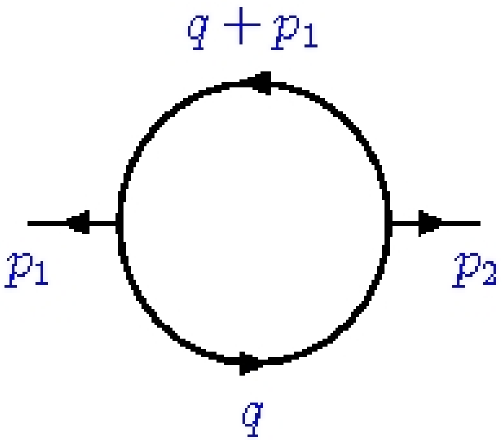
from $q^2=0$ and $q_0>0$ thus $\eta q>0$

plus $\theta(\eta k)$

$\}$ $\eta(q+k)>0$ and $(q+k)^2=0$

$\delta((q+k)^2) \rightarrow \delta_+((q+k)^2)$

Two-point function



$$\tilde{L}^{(2)}(p_1,p_2) = \int_q \tilde{\delta}(q) \left[\left[G(q+p_1) + \theta(\eta\, p_1)\tilde{\delta}(q+p_1) \right] + \left[1 \Leftrightarrow 2 \right] \right]$$

$$= L^{(2)}_{1-cut}(p_1,p_2) + \left[\theta(\eta\, p_1) + \theta(\eta\, p_2) \right] L^{(2)}_{2-cut}(p_1,p_2)$$

momentum conservation

$p_1+p_2=0$ ⇨ **1** ✔

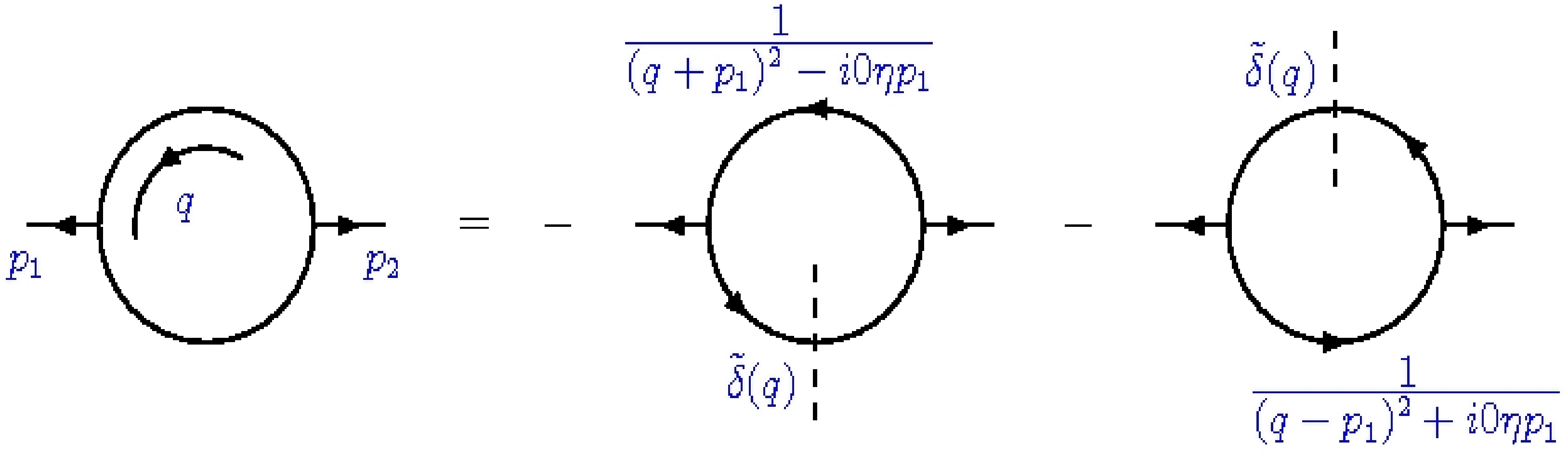
N-point function

The key ingredient is to proof (e.g. by induction) the following algebraic identity

$$\theta(\eta\, p_1)\theta(\eta\, p_{12})\ \cdots\ \theta(\eta\, p_{1,N-1})\ +\ cyclic\ perms.\ =\ 1$$

which follows from momentum conservation

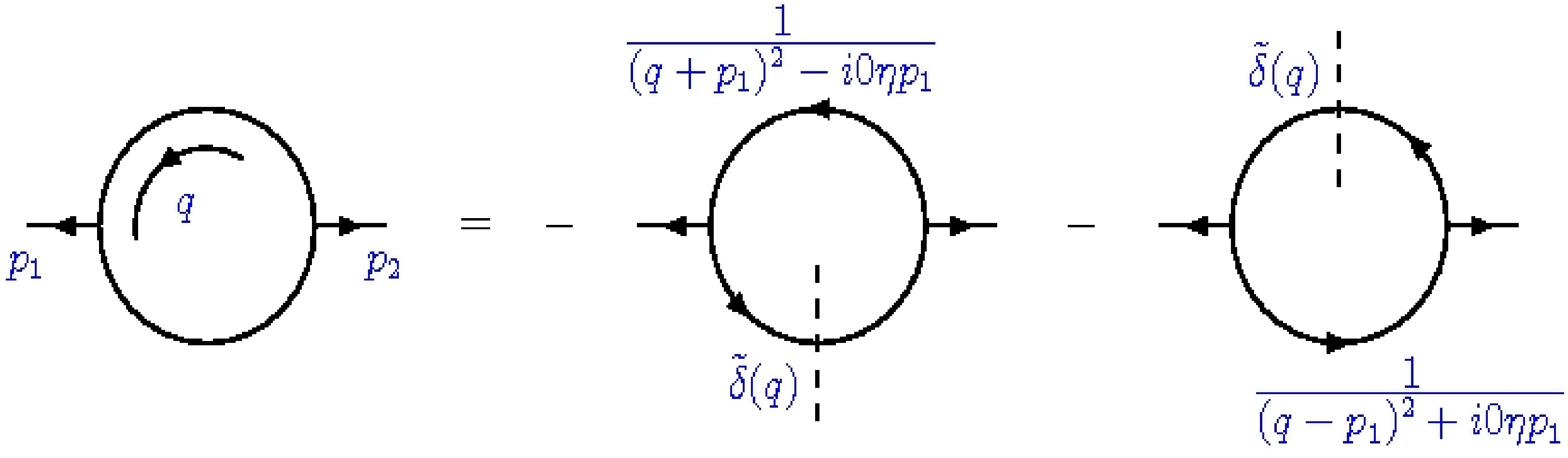
Two-point function from duality



$$\tilde{L}^{(2)}(p_1, p_2) = I^{(1)}(p_1) + (p_1 \Leftrightarrow -p_1)$$

$$I^{(1)}(k) = -\frac{c_\Gamma}{2} \frac{(-k^2 - i0)^{-\epsilon}}{\epsilon(1-2\epsilon)} \left[1 - i \, \frac{\sin(\pi\epsilon)}{\cos(\pi\epsilon)} sign(k^2 \eta k) \right]$$

Two-point function from duality



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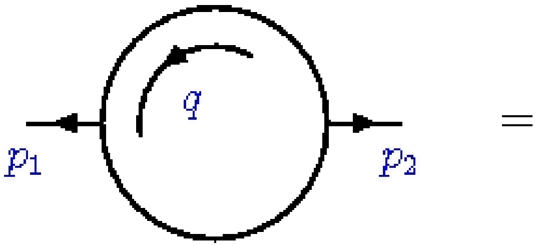
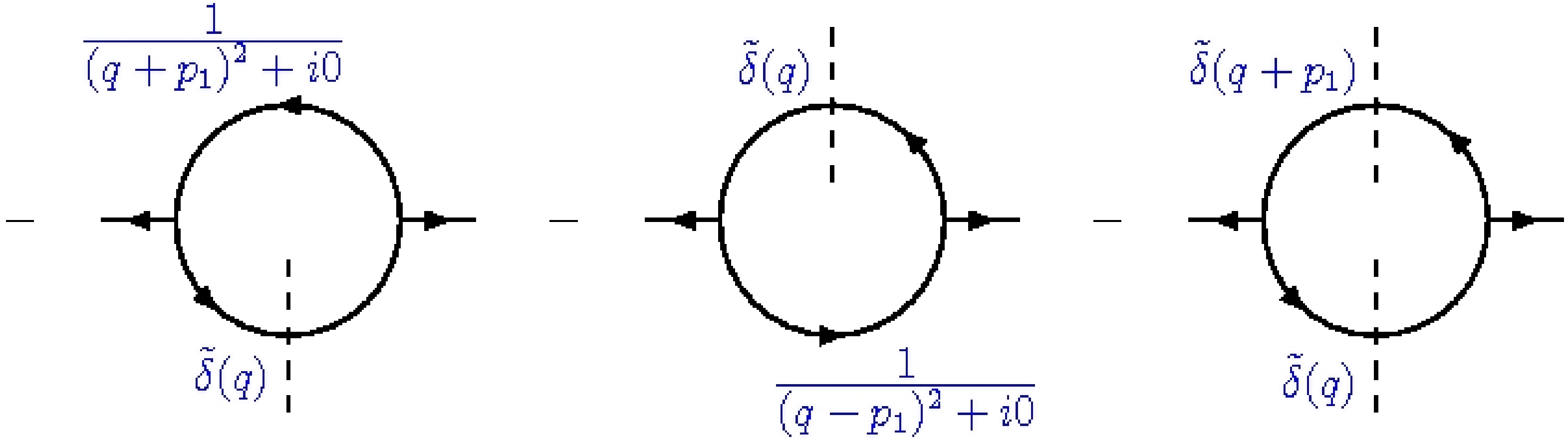
X

Two-point function from FTT

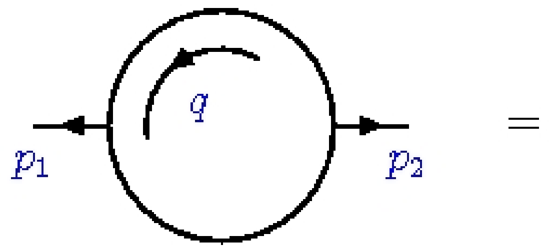


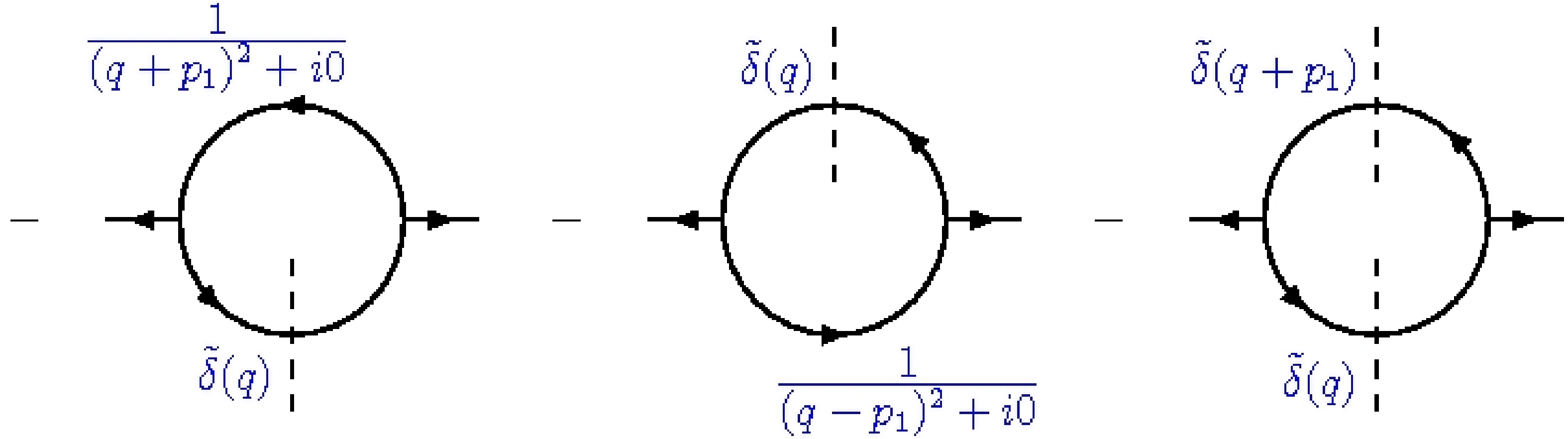


$$L^{(2)}_{1-cut}(p_1, p_2) \;=\; I^{(1)}_{1-cut}(p_1) \;+\; (p_1 \Leftrightarrow -p_1)$$

$$I^{(1)}_{1-cut}(k) \;=\; -\frac{c_\Gamma}{2}\frac{(-k^2-i0)^{-\epsilon}}{\epsilon(1-2\epsilon)}\left[1-i\;\frac{\sin(\pi\epsilon)}{\cos(\pi\epsilon)}\left[\theta(-k^2)+\theta(k^2)\,sign(k_0)\right]\right]$$

Two-point function from FTT





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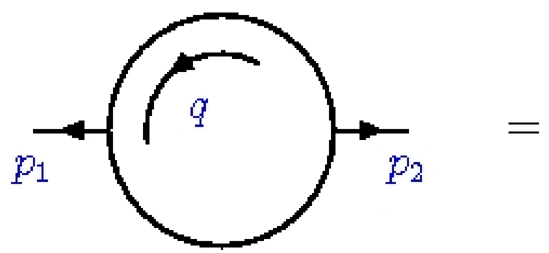
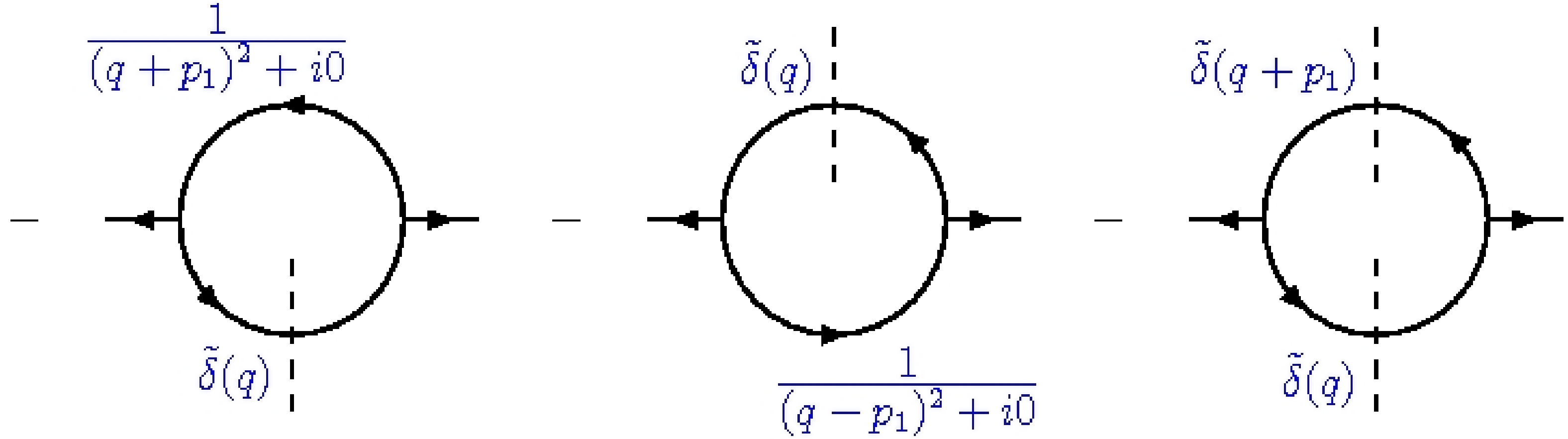
Two-point function from FTT





$$L^{(2)}_{1-cut}(p_1, p_2) \; = \; I^{(1)}_{1-cut}(p_1) \; + \; (p_1 \Leftrightarrow -p_1)$$

$$I^{(1)}_{1-cut}(k) \; = \; -\frac{c_\Gamma}{2} \frac{(-k^2 - i0)^{-\epsilon}}{\epsilon(1-2\epsilon)} \left[1 - i \; \frac{\sin(\pi\epsilon)}{\cos(\pi\epsilon)} \left[\theta(-k^2) + \theta(k^2) \, sign(k_0) \right] \right]$$

$$L^{(2)}_{2-cut}(p_1, p_2) \; = \; -i \, c_\Gamma \frac{(|p_1^2|)^{-\epsilon}}{\epsilon(1-2\epsilon)} \frac{\sin(\pi\epsilon)}{\cos(\pi\epsilon)} \theta(-p_1^2)$$

Massive integrals, complex masses and unstable particles

Real masses: do not affect the dual prescription $\tilde{\delta}(q_i) \rightarrow \tilde{\delta}(q_i, M_i)$

$$\frac{1}{q_j^2 - M_j^2 - i0\,\eta(q_j - q_i)}$$

Massive integrals, complex masses and unstable particles

Real masses: do not affect the dual prescription $\tilde{\delta}(q_i) \rightarrow \tilde{\delta}(q_i, M_i)$

$$\frac{1}{q_j^2 - M_j^2 - i0\,\eta(q_j - q_i)}$$

Unstable particles: Dyson summation produces finite-width effects that lead to the introduction of finite imaginary contributions in the propagators. In the complex mass scheme

$$G_C(q;s) = \frac{1}{q^2 - s}$$

$$s = \mathrm{Re}\ s + i\ \mathrm{Im}\ s \quad \textit{with} \quad \mathrm{Re}\ s > 0 > \mathrm{Im}\ s$$

produces poles in the q_0 plane that are located far from the real axis

Unstable particles

Duality relation

$$\tilde{L}^{(N)}(p_1,\dots,p_N) \rightarrow \tilde{L}^{(N)}(p_1,\dots,p_N) + \underbrace{\tilde{L}_C^{(N)}(p_1,\dots,p_N)}$$

From the poles of the complex-mass propagators

where

$$\tilde{L}_C^{(N)}(p_1,\dots,p_N) = \int_q \sum_{i\in C} \tilde{\delta}(q_i,s_i) \left[\prod_{j\neq i} \dots\right]$$
$$= \int \frac{d^{d-1}\boldsymbol{q}}{(2\pi)^{d-1}} \sum_{i\in C} \frac{1}{2\sqrt{\boldsymbol{q}_i^2+s_i}} \left[\prod_{j\neq i} \dots\right]_{q_{i0}=\sqrt{\boldsymbol{q}_i^2+s_i}}$$

pole has a finite negative imaginary part ⇨ the +i0 prescription of Feynman propagators can be removed

- FTT: modify the 1-cut contribution, but do not produce additional m-cut contributions
- complex mass $s(q^2)$, but always at a finite imaginary distance from real axis

Gauge poles

Quantization of gauge theories requires a gauge-fixing procedure

fictitious particles: Faddeev-Popov ghosts in unbroken non-Abelian gauge theories, or would-be Goldstone bosons in spontaneously broken gauge theories
⇨ ***cut exactly as physical particles***

gauge bosons: polarization tensor 'Hooft-Feynman gauge ✔

$$d^{\mu\nu} = -g^{\mu\nu} + (\xi - 1)\; l^{\mu\nu}(q)\; G_G(q)$$

$l^{\mu\nu}(q)$ propagates longitudinal polarizations, harmless polynomial dependence on q

- ***Spontaneously-broken gauge theories***

$$G_G(q) = \frac{1}{\xi(q^2 + i0) - M^2}$$

unitary gauge (ξ=0) ✔

- ***Un-broken gauge theories***

covariant gauge $G_G(q) = \frac{1}{q^2 + i0}$ second order pole ✘

physical gauge $G_G(q) = \frac{1}{(n \cdot q)^k}, \quad k = 1,2$ if $n \cdot \eta = 0$ ✔

Loop-tree duality for amplitudes

In analogy with the FTT, in unitary and local field theories

$$\mathcal{A}^{(1-loop)} = - \tilde{\mathcal{A}}^{(1-loop)}$$

- starting from $\mathcal{A}^{(1\text{-loop})}$, consider all single cuts
- replace uncut propagators by dual propagators

$$\mathcal{A}^{(1-loop)} \simeq - \int_q \sum_P \tilde{\delta}(q; M_P) \sum_{dof(P)} \mathcal{A}_P^{(tree)}$$

Green's functions

Off-shell Green's function with *N* external legs

$$\mathcal{A}_N^{(1-loop)}(\ldots)=+\frac{1}{2}\int\frac{d^d q}{(2\pi)^{d-1}}\sum_P \delta_+(q^2-M_P^2)\,\sigma(P)\,\tilde{\mathcal{A}}_{N+2}^{(tree)}(P(q)\leftarrow P(q),\ldots)$$

- $\sigma(P)=\pm1$ Bose-Fermi statistics factor
- Σ_P sums over particles and antiparticles

Tree-level amplitude for the forward scattering process $P(q)\rightarrow P(q)$ in the field of N external legs

(**tadpoles** cancel: summing over color,
in QED summing over particles and antiparticles)

e.g.

$$\mathcal{A}_{N+2}^{(tree)}(g(q)\leftarrow g(q),\ldots)=\sum_\lambda\sum_{a,b}\left(\epsilon_{a,\mu}^{(\lambda)}(q)\right)^*\left[\mathcal{A}_{N+2}(g(q),g(-q),\ldots)\right]_{ab}^{\mu\nu}\epsilon_{b,\nu}^{(\lambda)}(q)$$

Scattering amplitudes: only relevant point is the on-shell limit of the corresponding Green's function (wave function factors of the external lines)

Summary

- Derived a duality relation between one-loop integrals and **single-cut** phase space integrals.

- Duality relation realized by a modification of the customary +i0 prescription of the Feynman propagators.

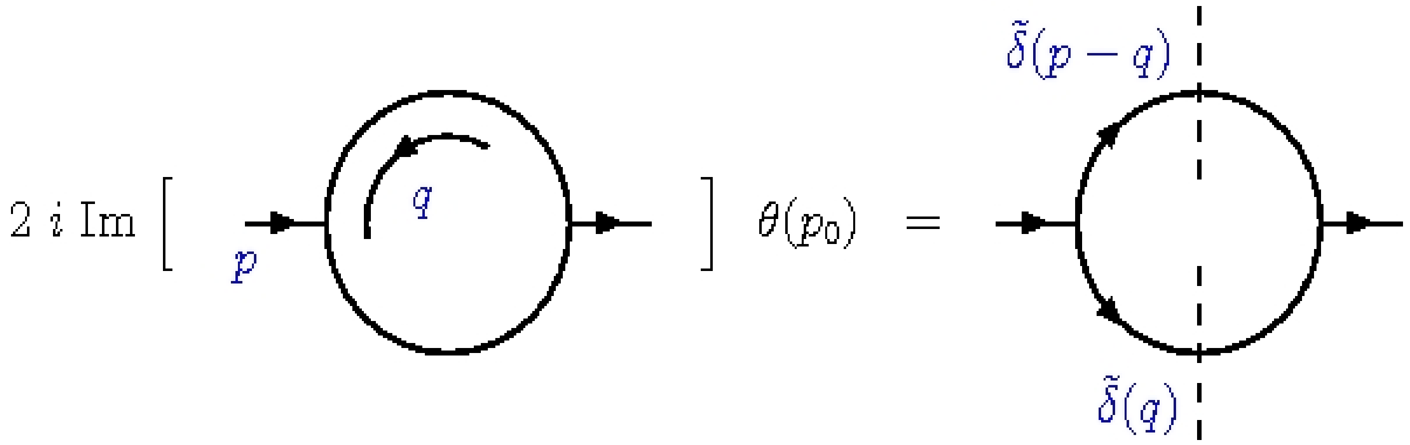
- The new prescription, written in a Lorentz covariant form, compensates for the absence of multiple-cut contributions that appear in the FTT.

- Valid for any relativistic, local and unitary field theory, in arbitrary space-time dimensions.

- Suitable for analytical calculations of one-loop scattering amplitudes, and for numerical evaluation of cross-sections at NLO (on-going implementation)

- natural extension to two-loops, under investigation

Two-point function: Cutkosky



The double-cut contribution from the FTT is different from the **unitarity cut** contribution that gives the imaginary part

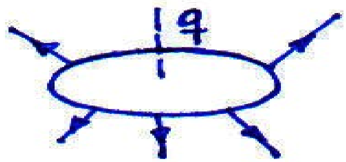
$$2\,i\,\mathrm{Im}[Bub(p^2)]\theta(p_0)=\int_q \tilde\delta(q)\tilde\delta(p-q)=i\,c_\Gamma\frac{(|p^2|)^{-\epsilon}}{\epsilon(1-2\epsilon)}2\sin(\pi\epsilon)\theta(p^2)\theta(p_0)$$

due to the different positive-energy flow in the internal lines

IR classification of dual integrals

IR behaviour depends on the two external momenta joined by the cut line

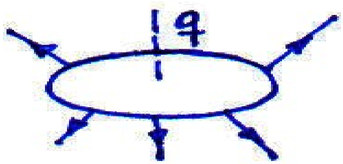
$p_{i-1}^2 \neq 0$ $p_i^2 \neq 0$



$$I_{finite}^{(N-1)} = \int_q \delta_+(q^2) \left[\prod_{j=1}^{N-1} \frac{1}{2\, q \cdot k_j + k_j^2} \right]$$

IR finite (numerically integrable in d=4)

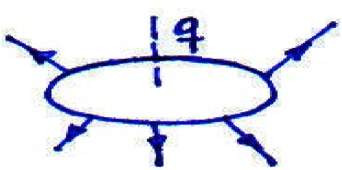
$p_{i-1}^2 \neq 0$ $p_i^2 = p^2 = 0$



$$I_{collinear}^{(N-2)} = \int_q \delta_+(q^2) \frac{1}{p \cdot q} \left[\prod_{j=1}^{N-2} \frac{1}{2\, q \cdot k_j + k_j^2} \right]$$

IR divergent in collinear region *q // p*,
single 1/ε poles

$p_{i-1}^2 = \overline{p}^2 = 0$ $p_i^2 = p^2 = 0$



$$I_{soft}^{(N-3)} = \int_q \delta_+(q^2) \frac{p \cdot \overline{p}}{p \cdot q\, \overline{p} \cdot q} \left[\prod_{j=1}^{N-3} \frac{1}{2\, q \cdot k_j + k_j^2} \right]$$

IR divergent in collinear regions *q // p and q // p*
double $1/\varepsilon^2$ and single 1/ε poles