Prescription dependence of soft gluon resummation

4th HERA and the LHC Workshop
CERN, May 29th, 2008

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Results obtained in collaboration with Stefano Forte and some of our students:

- **SF+GR+J. Rojo+M.Ubiali, PLB635(2006)313, hep-ph/0601048**
- **SF+GR+R. Abbate, PLB657(2007)55, arXiv:0707.2452**
- **SF+GR+A.Vicini, in preparation**
- **SF+GR+M. Bonvini, in preparation**

Sudakov logarithms are the finite left-over of real-virtual cancellation of soft divergences. Can become large if the tagged final state carries most of the available energy (soft emission): Examples:

1. **lepton-nucleon scattering in the quasi-elastic limit:**

$$x = x_{\mathrm{Bj}} = \frac{Q^2}{2p \cdot q}, \qquad x_{\mathrm{Bj}} \to 1$$

2. **production of heavy systems close to threshold:**

$$x = \tau = \frac{Q^2}{s}, \qquad s \gtrsim Q^2$$

3. **transverse momentum spectrum in the small-q_T region:**

$$1 - x = \hat{q}_{\mathrm{T}}^2 = \frac{q_{\mathrm{T}}^2}{Q^2}, \qquad q_{\mathrm{T}}^2 \ll Q^2$$

Typical expression of an observable in QCD:

$$\sigma(x, Q^2) = \int_x^1 \frac{dy}{y} \, \mathcal{L}(y, Q^2) \, \hat{\sigma} \left(\frac{x}{y}, \alpha_{\rm S}(Q^2) \right)$$

Order $\alpha_{\rm S}^n$ coefficient in the expansion of $\hat{\sigma}(z, \alpha_{\rm S})$:

$$\sim \left[\frac{P_{2n-1}(\log(1-z))}{1-z} \right]_+ + \mathrm{non-logarithmic\ terms}$$

Resummation needed

- **if x is close to 1**
- **if $\mathcal{L}(y, Q^2)$ is peaked at small y**

A useful technique: Mellin transformation

$$f(N)=\int_0^1 dx\, x^{N-1}\, f(x); \qquad f(x)=\frac{1}{2\pi i}\int_{\bar{N}-i\infty}^{\bar{N}+i\infty} dN\, x^{-N}\, f(N)$$

(a Laplace transform with $x=e^{-t}$).

- **well defined and analytic in the half-plane $\mathrm{Re\,N} > N_0$ if $f(x)$ is at most as singular as x^{-N_0}**
- **turns convolution products into ordinary products:**

$$\sigma(N,Q^2)=\mathcal{L}(N,Q^2)\,\hat{\sigma}(N,\alpha_{\mathrm{S}}(Q^2))$$

- **phase space factorization: $\delta(z-\prod z_i)\to\prod z_i^{N-1}$**

The region $x\to 1$ is mapped in the region $N\to\infty$:

$$\int_0^1 dx\, x^{N-1}\left[\frac{\log^k(1-x)}{1-x}\right]_+ = \frac{1}{k+1}\,\log^{k+1}\frac{1}{N}+\ldots$$

Ambiguities in resummed results

In QCD resummation is determined by exponentiation and running of the coupling constant:

$$\log \hat{\sigma}(\alpha_s(Q^2), N) = g(\alpha_{\rm S}(Q^2/N))$$

Expanding $\alpha_{\rm S}(Q^2/N)$ **in powers of** $\alpha_s(Q^2)$

$$\alpha_{\rm S}(Q^2/N) = \frac{\alpha_s(Q^2)}{1+\alpha_s(Q^2)\beta_0 \log\frac{1}{N}} = \alpha_s(Q^2)\sum_{n=0}^{\infty}(-\alpha_s(Q^2)\beta_0)^n \log^n \frac{1}{N}$$

one gets

$$\hat{\sigma}(N,\alpha_{\rm S}) = \exp[\log N\, g_1(\alpha_{\rm S}\log N) + g_2(\alpha_{\rm S}\log N) + \alpha_{\rm S}\, g_3(\alpha_{\rm S}\log N) + \ldots]$$

which defines an improved expansion (in powers of $\alpha_{\rm S}$ **with** $\alpha_{\rm S}\log N$ **fixed) for** $\log\hat{\sigma}(N,\alpha_{\rm S})$**.**

A difficulty immediately arises: the resummed cross section has a branch cut on the real positive axis for

$$N \geq N_L \equiv e^{\frac{1}{\beta_0 \alpha_s(Q^2)}}.$$

because of the Landau singularity.

Its inverse Mellin transform does not exist.

Otherwise stated: **expand $\hat{\sigma}(\alpha_{\rm S}(Q^2), N)$ in powers of $\log N$ and take the term-by-term inverse Mellin transform:**

$$\hat{\sigma}(\alpha_{\rm S}(Q^2), z) = \sum_{k=1}^{\infty} c_k \, \frac{1}{2\pi i} \int_{\bar{N}-i\infty}^{\bar{N}+i\infty} dN \, z^{-N} \, \log^k \frac{1}{N}$$

but the series is divergent! **(otherwise, we could interchange the sum over k and the integral over N, but**

$$\sum_{k=1}^{\infty} c_k \, \log^k \frac{1}{N}$$

is only convergent for

$$\left| \beta_0 \alpha_{\rm S}(Q^2) \log \frac{1}{N} \right| < 1,$$

while the integral in N on the path $\mathrm{Re}\, N = \bar{N}$ involves values of N outside this range).

Taking the inverse Mellin transform of each $\log^k N$ term at the relevant (leading, next-to-leading...) logarithmic level the perturbative series converges, but turns out to be a function of $\alpha_S(Q^2(1-z))$: well defined only for $z < z_L = 1 - e^{-\frac{1}{\beta_0 \alpha_S(Q^2)}}$.

Not acceptable: convolution with luminosity involves integration up to $z = 1$!

Completely analogous considerations hold in the case of the resummation of large logarithms of q_T^2/Q^2 in the small-q_T region of the spectrum.

In this case

$$\text{Mellin tr.} \quad \int_0^1 dz\, z^{N-1}\, f(z) \to \text{Fourier tr.} \quad \frac{1}{2\pi}\int d\vec{b}\, e^{-i\vec{b}\cdot\vec{q}_T}\, f(\vec{q}_T)$$

In the space of the Fourier-conjugate variable $\vec{b}$, the $\vec{q}_T$ conservation δ function factorizes:

$$\int d^2 q_T\, \delta(\vec{q}_T - \vec{q}_{T1} - \ldots - \vec{q}_{Tn})\, e^{i\vec{b}\cdot\vec{q}_T} = \prod_{i=1}^{n} \exp(i\vec{b}\cdot\vec{q}_{Ti})$$

The resummed cross section in $\vec{b}$ space has no inverse Fourier transform, again because of the Landau pole of the running coupling.

Rapidity distributions:

$$\sigma(x,Y)=\int_{x_1^0}^{1}\frac{dx_1}{x_1}\int_{x_2^0}^{1}\frac{dx_2}{x_2}\,F_1(x_1)\,F_2(x_2)\,C\left(\frac{x}{x_1x_2},y\right)$$

where

$$x=\frac{Q^2}{S};\quad x_1^0=\sqrt{x}\,e^{Y};\quad x_2^0=\sqrt{x}\,e^{-Y}y=Y-\frac{1}{2}\log\frac{x_1}{x_2};$$

Not a convolution: $y=y(x_1,x_2)$**. Fourier transform wrt** y**:**

$$\sigma(x,Y)=\int_{x}^{1}\frac{dx_1}{x_1}\,F_1(x_1)\int_{x/x_1}^{1}\frac{dx_2}{x_2}\,F_2(x_2)\int_{-\infty}^{+\infty}\frac{dM}{2\pi}\,e^{-iMy}C\left(\frac{x}{x_1x_2},M\right),$$

$$
\begin{aligned}
C\,(z,M) = \int_{-\infty}^{+\infty} dy\, e^{iMy}\, C\,(z,y) \quad &= \quad \int_{\log\sqrt{z}}^{-\log\sqrt{z}} dy\, e^{iMy}\, C\,(z,y) \\
&= \quad \int_{\log\sqrt{z}}^{-\log\sqrt{z}} dy\, C\,(z,y)\; [1 + O(y)] \\
&= \quad C(z) + \ldots
\end{aligned}
$$

because $\log\sqrt{z} \sim -(1-z)/2$. $C(z)$ **is the total cross section, which can be resummed in** N **space as usual. We end up with**

$$
\sigma(N,Y) = L(N,Y)\, C(N)
$$

where

$$
L(N,Y) = \int_0^1 d\tau\, \tau^{N-1}\, F_1(\sqrt{\tau}e^{Y})\, F_2(\sqrt{\tau}e^{-Y}).
$$

Again problems with the Mellin inversion.

A possible solution: the minimal prescription*

$$\sigma(x,Q^2) = \frac{1}{2\pi i}\int_{N_{\rm MP}-i\infty}^{N_{\rm MP}+i\infty} dN\, x^{-N}\, \mathcal{L}(N,Q^2)\, \hat{\sigma}(N,\alpha_{\rm S}(Q^2))$$

with $0 < N_{\rm MP} < N_L$**.**

Not a true inverse Mellin: the integrand is not analytical in any right half-plane, because of the branch cut due to the Landau pole.

***Catani, Mangano, Nason, Trentadue, [NPB 478(1996)273, hep-ph/9604351]**

However:

- **it is well defined for all values of** x
- **it is an asymptotic sum of the original, divergent perturbative expansion**
- **the difference between the original series, truncated at the best-approximation term, and the minimal prescription, is suppressed more strongly than any power of** Λ^2/Q^2**.**

A closer look at the minimal prescription:

$$\sigma(x, Q^2) = \int_0^1 \frac{dy}{y}\, \mathcal{L}(y, Q^2)\, \hat{\sigma}\left(\frac{x}{y}, \alpha_{\rm S}(Q^2)\right)$$

Looks like a convolution, but the integration region $0 \le y \le x$ **cannot be excluded, because**

$$\hat{\sigma}(z, \alpha_{\rm S}(Q^2)) = \frac{1}{2\pi i} \int_{N_{\rm MP} - i\infty}^{N_{\rm MP} + i\infty} dN\, z^{-N}\, \hat{\sigma}(N, \alpha_{\rm S}(Q^2))$$

does not vanish for $z > 1$ **because of the Landau cut.**

One may prefer working with a resummed parton cross section which respects kinematical constraints:

- **the numerical implementation of the minimal prescription formula is not straightforward: $\hat{\sigma}(x/y, \alpha_{\rm S})$ oscillates in the region $y \sim x$, where the luminosity is smooth, and large cancellations take place.**
- **On the other hand, $\mathcal{L}(N, Q^2)$ is typically not available.**

All these problems can be overcome by suitable techniques, but it is interesting to explore different possibilities.

A different approach

Consider a generic quantity Σ, resummed in N space and expanded in powers of $\log N$:

$$\Sigma(\alpha_s(Q^2), L) = \sum_{k=1}^{\infty} h_k \, L^k; \qquad L \equiv \bar{\alpha} \log \frac{1}{N}; \quad \bar{\alpha} = a\beta_0 \alpha_s(Q^2); \quad a = 1, 2$$

The series is convergent for $|L| < 1$, because of the Landau pole at $L = 1$. To log accuracy,

$$\frac{1}{2\pi i} \int_{\overline{N}-i\infty}^{\overline{N}+i\infty} dN x^{-N} \ln^k \frac{1}{N} = \left[\frac{P_{k-1}(\ell)}{1-z} \right]_+$$

($P_{k-1}(\ell)$ a polynomial of degree $k-1$ in $\ell \equiv \log(1-z)$). Thus

$$\Sigma(\alpha_s(Q^2), z) = \left[\frac{R(z)}{1-z} \right]_+ ; \qquad R(z) = \sum_{k=1}^{\infty} h_k \, \bar{\alpha}^k \, P_{k-1}(\ell)$$

which is divergent. Is it Borel summable?

A reminder: Given a generic power series, not necessarily convergent in the Cauchy sense:

$$f(\bar{\alpha}) = \sum_{k=1}^{\infty} f_k \, \bar{\alpha}^k$$

we define its Borel transform

$$\hat{f}(w) = \sum_{k=1}^{\infty} f_k \, \frac{w^{k-1}}{(k-1)!}$$

Because of the factor $(k-1)!$, the Borel transformed series $\hat{f}(w)$ has much better convergence properties. An inverse transformation exists, since

$$\int_0^{+\infty} dw \, e^{-\frac{w}{\bar{\alpha}}} \, w^{k-1} = (k-1)! \, \bar{\alpha}^k \rightarrow f_B(\bar{\alpha}) = \int_0^{+\infty} dw \, e^{-\frac{w}{\bar{\alpha}}} \, \hat{f}(w)$$

The Borel tranform of $R(z)$ with respect to $\bar{\alpha}$,

$$\hat{R}(w,z) = \sum_{k=1}^{\infty} h_k \, \frac{w^{k-1}}{(k-1)!} \, P_{k-1}(\ell)$$

can be shown to be convergent. The inversion integral however,

$$R_B(z) = \int_0^{+\infty} dw \, e^{-\frac{w}{\bar{\alpha}}} \, \hat{R}(z,w)$$

is divergent at $+\infty$.

We cut off the integral at $w = C$:

$$\begin{aligned} R_B(z) &= \int_0^C dw\, e^{-\frac{w}{\bar\alpha}}\, \hat{R}(z,w) \\ &= \sum_{k=1}^{\infty} h_k\, \bar\alpha^k\, P_{k-1}(\ell)\, \frac{\gamma(k; C/\bar\alpha)}{(k-1)!} \end{aligned}$$

where γ is the truncated Γ function

$$\gamma(k;u) = \int_0^u dt\, e^{-t}\, t^{k-1} = (k-1)! \left(1 - e^{-u} \sum_{m=0}^{k-1} \frac{u^m}{m!}\right)$$

Therefore

$$R_B(z) = R(z) - R_{ht}(z)$$

where

$$R_{ht}(z) = e^{-\frac{C}{\bar\alpha}} \sum_{k=1}^{\infty} h_k\, \bar\alpha^k \sum_{m=0}^{k-1} \frac{1}{m!} \left(\frac{C}{\bar\alpha}\right)^m$$

Remarks:

- $R(z)$ **is an asymptotic expansion of** $R_B(z)$**. Indeed,** $R_{ht} \sim e^{-\frac{1}{\bar{\alpha}}}$ **vanishes faster than any power of** $\alpha_s(Q^2)$ **as** $\alpha_s(Q^2) \to 0$**; therefore** $R_B(z) - R(z)|_N$ **is of order** $\alpha_s(Q^2)^{N+1}$**.**

- **Using the leading log expression of** $\alpha_s(Q^2)$

$$\alpha_s(Q^2) \simeq \frac{1}{\beta_0 \ln \frac{Q^2}{\Lambda^2}} \to e^{-\frac{C}{\bar{\alpha}}} \simeq \left(\frac{\Lambda^2}{Q^2}\right)^{C/a} ; \quad a = 1, 2$$

Cutting off the Borel inversion integral at $w = C$ **amounts to including a twist-**t **contribution** $R_{ht}(z)$**, with**

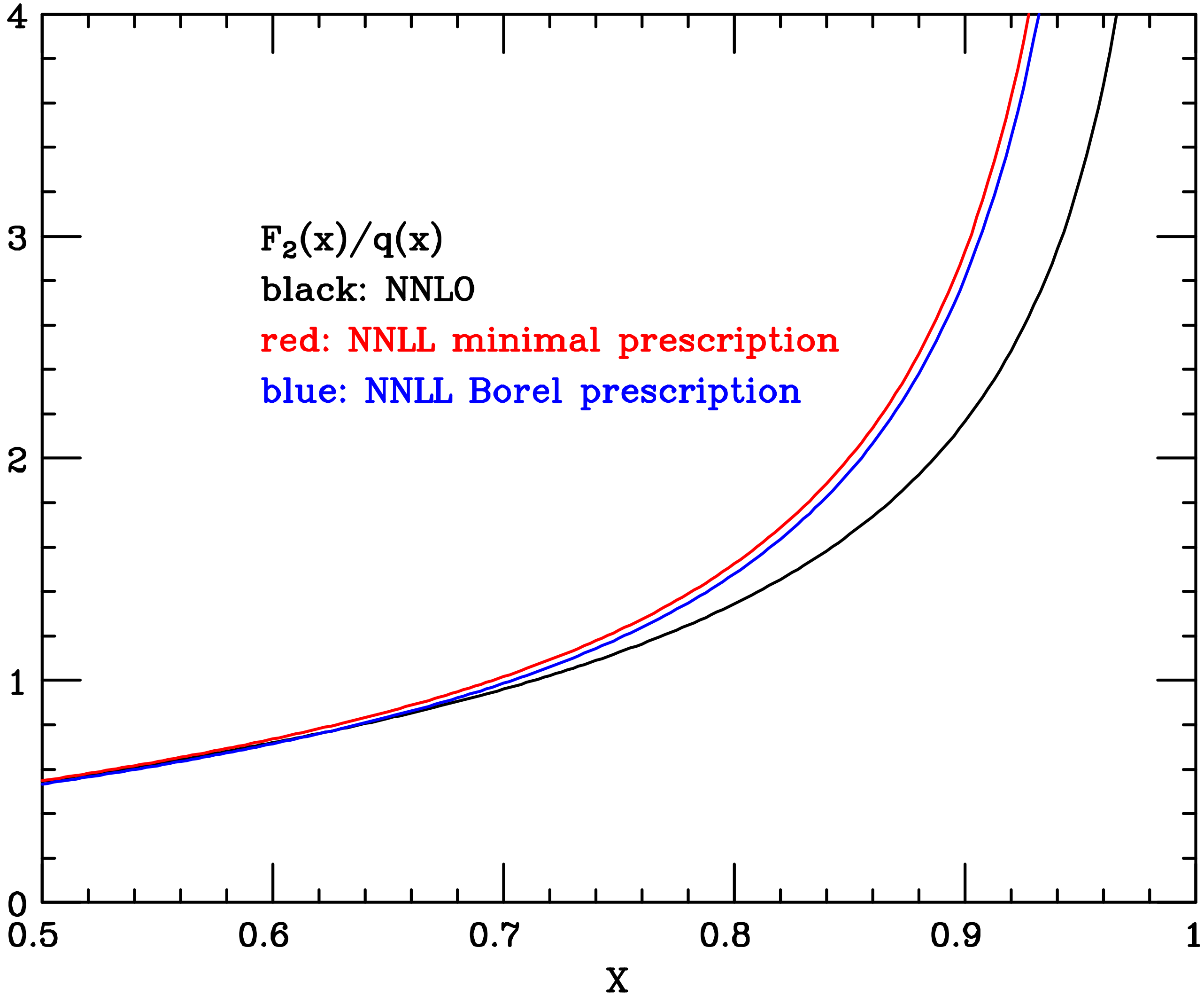
$$t = 2 + \frac{2C}{a}.$$

The divergence of the higher twist term then cancels that of the divergent series, leading to a finite result.

- **The value of C is arbitrary. Dependence on C very mild below the Landau pole.**
- **The choice $C = a$ is minimal: it corresponds to the inclusion of a twist-four term, i.e. a term of the first subleading twist.**
- **Arbitrarily large values of C can be chosen (with some care in the numerical implementation).**
- **The series that defines $R(z)$ can be summed:**

$$\begin{aligned} R(z) &= \int_0^C dw\, e^{-\frac{w}{\bar{\alpha}}}\, \hat{R}(z,w) \\ &= \frac{1}{2\pi i}\oint_H d\xi\, e^{\ell\xi}\, \Delta(1+\xi)\left[e^{-\frac{C}{\bar{\alpha}}}\,\Sigma\left(\frac{C}{\xi}\right) + \frac{1}{\bar{\alpha}}\int_0^C dw\, e^{-\frac{w}{\bar{\alpha}}}\,\Sigma\left(\frac{w}{\xi}\right)\right] \end{aligned}$$

H **any closed path that encloses the branch cut** $-w < \mathrm{Re}\,\xi < 0$**.**



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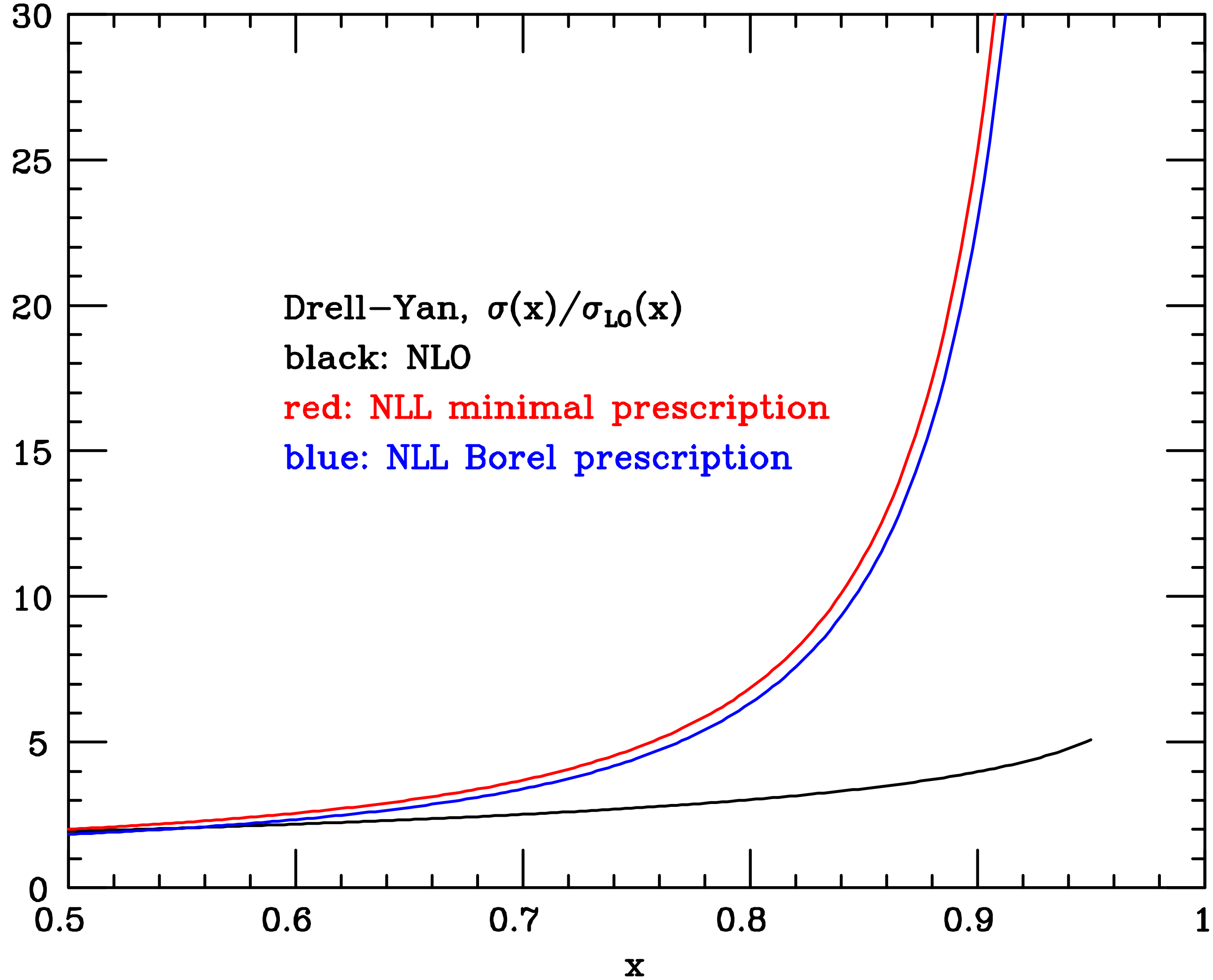
R. Abbate, S. Forte, GR, PLB657(2007)55, arXiv:0707.2452



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S. Forte, GR, in preparation

Outlook

- **There are ambiguities in the computation of observables from resummed quantities in QCD, to be ascribed to the presence of a Landau singularity in the running coupling.**
- **A prescription based on Borel sum and twist expansion can be given, with some advantages on other, widely employed techniques.**
- **Future work: more realistic studies; extension to less inclusive observables, such as transverse momentum spectrum* or rapidity distributions†.**

* **M. Bonvini, S. Forte, GR, in preparation**

† **S. Forte, A. Vicini, GR, in preparation**

Divergence of $R(z)$ and convergence of $\hat{R}(w,z)$

Consider

$$R(z) = \sum_{k=1}^{\infty} h_k \, \bar{\alpha}^k \, P_{k-1}(\ell)$$

The explicit form of $P_{k-1}(\ell)$ is

$$P_{k-1}(\ell) = \sum_{n=1}^{k} c_{kn} \, \ell^{k-n}$$

$$c_{kn} = \frac{\Delta^{(n-1)}(1)}{(n-1)!} \, \frac{k!}{(k-n)!}; \qquad \Delta(z) = \frac{1}{\Gamma(z)}$$

Keeping only the first $1, 2, \ldots$ terms in the sum corresponds to the leading $\log(1-x)$, next-to-leading $\log(1-x), \ldots$ approximation.

Hence,

$$R_K(z) = \sum_{k=1}^{K} h_k\,\bar{\alpha}^k \sum_{n=1}^{k} \frac{\Delta^{(n-1)}(1)}{(n-1)!}\,\frac{k!}{(k-n)!}\,\ell^{k-n}$$

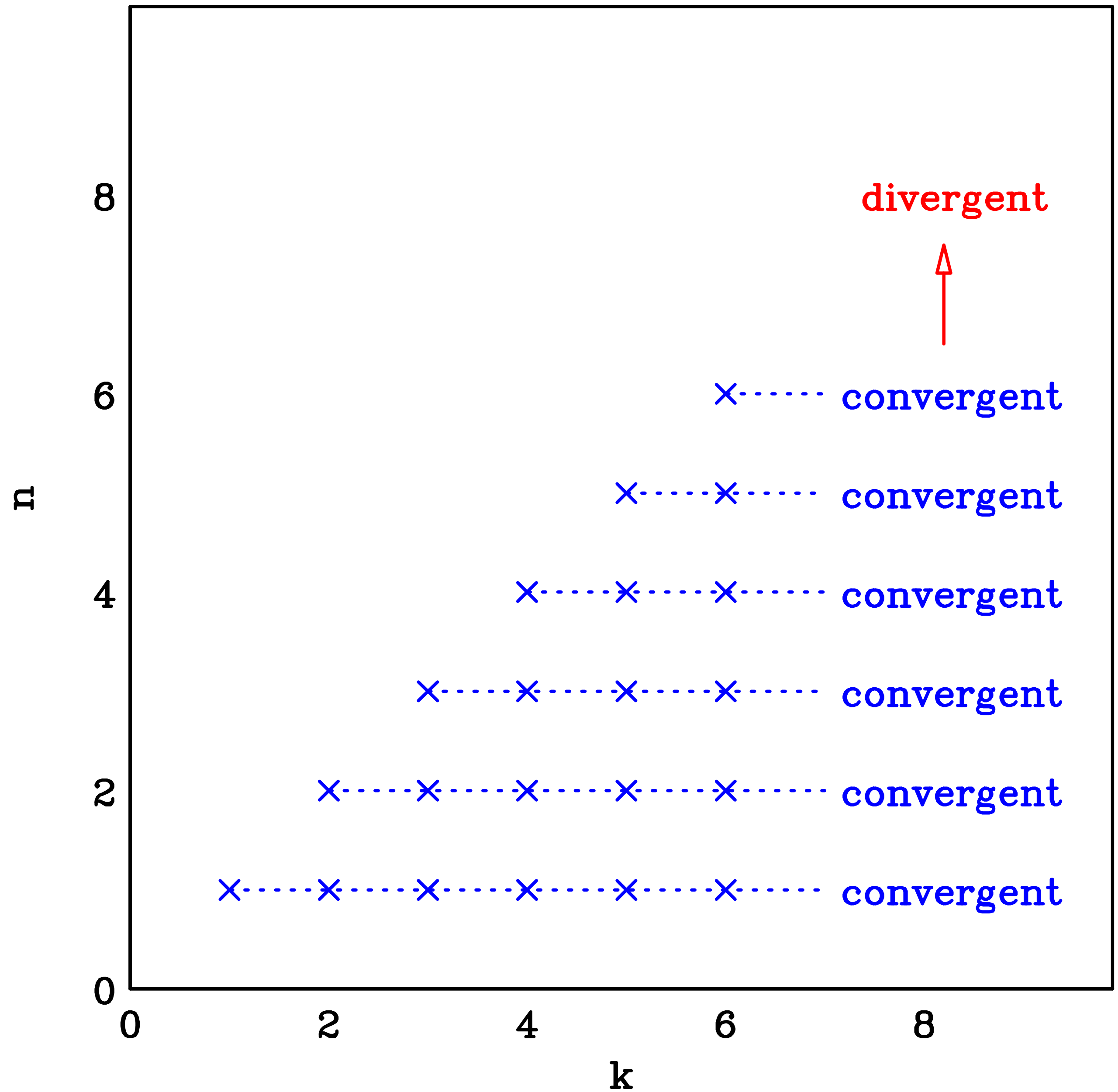
If the sum over n is truncated at $n = \bar{n}$, we get a convergent result. This is because

$$\frac{k!}{(k-n)!}\,\ell^{k-n} = \frac{d^n \ell^k}{d\ell^n}$$

and thus

$$R_K(z) = \sum_{n=1}^{\bar{n}} \frac{\Delta^{(n-1)}(1)}{(n-1)!}\,\frac{d^n}{d\ell^n} \sum_{k=1}^{K} h_k\,\bar{\alpha}^k\,\ell^{k-n} \to \sum_{k=n}^{\bar{n}} \frac{\Delta^{(n-1)}(1)}{(n-1)!}\,\frac{d^n}{d\ell^n}\,\Sigma(\bar{\alpha}\ell)$$

which is convergent for $|\bar{\alpha}\ell| < 1$, because of the Landau pole at $\bar{\alpha}\ell = 1$.



Terms in the expansion of $R(z)$ in powers $\bar{\alpha}^k \log^{k-n}(1-x)$.

The full expansion

$$R_K(z) = \sum_{n=1}^{K} \frac{\Delta^{(n-1)}(1)}{(n-1)!} \sum_{k=n}^{K} \frac{k!}{(k-n)!} \, h_k \, \bar{\alpha}^k \, \ell^{k-n}$$

is instead divergent. To see this, replace

$$\frac{1}{(k-n)!} = \frac{1}{2\pi i} \oint_H d\xi \, e^{\xi} \, \xi^{-(k-n)-1}$$

(H any closed contour around the origin $\xi = 0$) we get

$$R_K(z) = \frac{1}{2\pi i} \oint_H \frac{d\xi}{\xi} \, e^{\xi} \sum_{n=1}^{K} \frac{\Delta^{(n-1)}(1)}{(n-1)!} \left(\frac{\xi}{\ell}\right)^n \sum_{k=n}^{K} k! \, h_k \left(\frac{\bar{\alpha}\ell}{\xi}\right)^k$$

Since $\sum_k h_k \, L^k$ has convergence radius 1, $\sum_k k! \, h_k \, L^k$ has convergence radius 0.

By a similar manipulation, we can show that the Borel transform of $R(z)$ is a convergent series:

$$\hat{R}(w,z) = \sum_{n=1}^{\infty} \frac{\Delta^{(n-1)}(1)}{(n-1)!} \sum_{k=n}^{\infty} k\, h_k\, w^{k-1}\, \frac{\ell^{k-n}}{(k-n)!}$$

Using again

$$\frac{1}{(k-n)!} = \frac{1}{2\pi i} \oint_H d\xi\, e^{\xi}\, \xi^{-(k-n)-1}$$

and shifting $\xi \to \ell\xi$ we get

$$\hat{R}(z,w) = \frac{1}{2\pi i} \oint_H \frac{d\xi}{\xi}\, e^{\ell\xi} \sum_{n=1}^{\infty} \frac{\Delta^{(n-1)}(1)}{(n-1)!} \xi^{n-1} \sum_{k=n}^{\infty} k\, h_k\, \left(\frac{w}{\xi}\right)^{k-1}$$

Now,

- **terms with $k = 1, \ldots, n-1$ would give zero after ξ integration;**
- **the Taylor expansion of Δ has convergence radius ∞;**
- **the Taylor expansion of Σ has convergence radius 1**

So finally

$$
\begin{aligned}
\hat{R}(z,w) &= \frac{1}{2\pi i}\oint_H d\xi\, e^{\ell\xi}\,\Delta(1+\xi)\,\frac{d}{dw}\Sigma\left(\frac{w}{\xi}\right); \quad |\xi| > w \text{ on } H \\
R(z) &= \int_0^C dw\, e^{-\frac{w}{\bar{\alpha}}}\,\hat{R}(z,w) \\
&= \frac{1}{2\pi i}\oint_H d\xi\, e^{\ell\xi}\,\Delta(1+\xi)\left[e^{-\frac{C}{\bar{\alpha}}}\,\Sigma\left(\frac{C}{\xi}\right) + \frac{1}{\bar{\alpha}}\int_0^C dw\, e^{-\frac{w}{\bar{\alpha}}}\,\Sigma\left(\frac{w}{\xi}\right)\right]
\end{aligned}
$$

which is explicitly written in terms of the function Σ.

The resummed (Mellin-transformed) cross section $\Sigma(\alpha_{\mathrm{S}}, L)$ has a branch cut in the complex plane L in

$$-\infty < \mathrm{Re\,L} \leq -1; \qquad \mathrm{Im}\, L = 0$$

which is mapped into

$$-w \leq \mathrm{Re}\, \xi \leq 0; \qquad \mathrm{Im}\, \xi = 0$$

for $\Sigma(\alpha_{\mathrm{S}}, w/\xi)$. The contour H must be chosen so that it encloses the cut, and therefore is pushed to large negative values of $\mathrm{Re}\, \xi$ as $w \to +\infty$. In that region, $\Delta(1+\xi)$ oscillates with factorially growing amplitude, and the w integral does not converge.