Evolution with Subleading Color Contributions

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Motivations of Shower Development

Actually I have heard two of them...

✗ “Earn as many citations as PYTHIA does ...”

✓ Improving classical shower approaches based on 20-25 years of experience. HERWIG++, ARIADNE / VINCIA, PYTHIA / DIPOLE SHOWER (by Mainz group)

- ➡ *Matching* at Born level (CKKW, MLM, ...)
- ➡ *Matching* at NLO level (MC@NLO, ...)

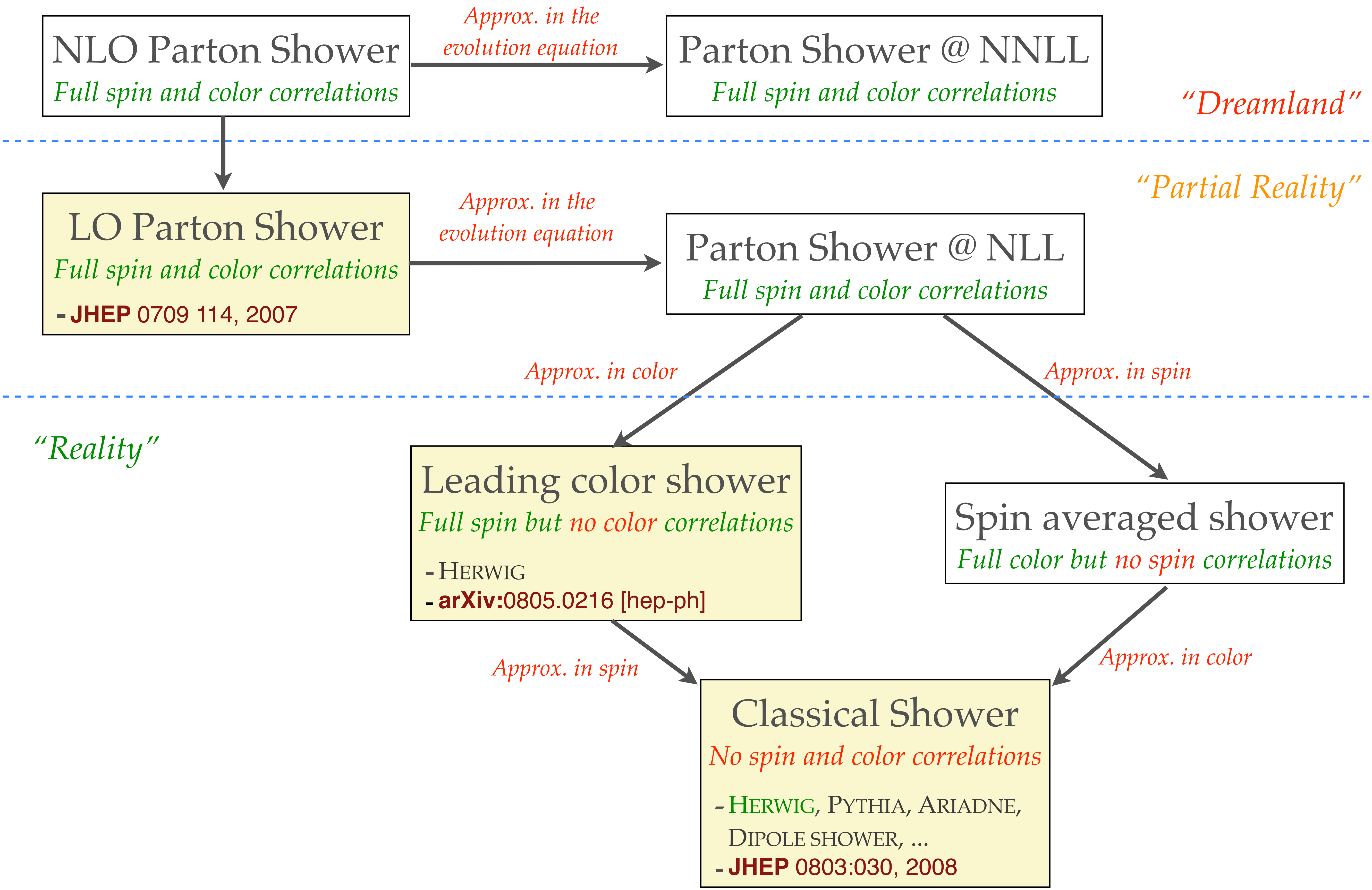
Non-systematic approximations ⟹ *Not predictive!*

... and the one what we try to follow

✓ Making parton shower predictive (will go into NLOJET++)

- ➡ The *bottleneck* is the color treatment.
- ➡ .....

Shower Family Tree



Do we need subleading color?

Matrix element square is

$$|\mathcal{M}(\{p,f\}_m)|^2 = N_c^n \sum_{\{c\}_m} |A(\{p,f,c\}_m)|^2 + \mathcal{O}\left(\frac{1}{N_c^2}\right)$$

where $A(\{p,f,c\}_m)$ is the color subamplitudes of the color configuration $\{c\}_m$

Cross sections at $\sqrt{s} = 1960$ GeV, with structure functions, in nanobarns, $p_T > 10\,\text{GeV}$ $|\eta| < 2.0$.

Process	σ_0 : Normal	σ_1 : Large Nc component	$\frac{\sigma_1 - \sigma_0}{\sigma_0}$
ud→W+g	0.1029(5)D+01	0.1158(5)D+01	13%
ud→W+gg	0.1018(8)D+00	0.1283(10)D+00	26%
ud→W+ggg	0.1119(17)D-01	0.1564(22)D-01	40%
ud→W+gggg	0.1339(36)D-02	0.2838(71)D-02	120%

Results were calculated by HELAC

Yes, we need.

Do we need subleading color?

Parton shower starts from the tree level exact matrix elements

$$|\mathcal{M}(\{p,f\}_m)|^2 = N_c^n \sum_{\{c\}_m} \Bigg\{ |A(\{p,f,c\}_m)|^2 + \sum_{\{c'\}_m} \frac{1}{N_c^\sigma} A(\{p,f,c\}_m) A(\{p,f,c'\}_m)^* \Bigg\}$$

How to assign color in the shower when the evolution starts from the interference contributions? *In classical shower there is no way*. With a simple trick we can get the normalization right.

$$|\mathcal{M}(\{p,f,c\}_m)|^2 = \underbrace{\frac{|A(\{p,f,c\}_m)|^2}{\sum_{\{c\}_m} |A(\{p,f,c\}_m)|^2}}_{\text{Probability of } \{c\}_m} |\mathcal{M}(\{p,f\}_m)|^2$$

Yes, we need.

Note this is just the standard K-factor trick.

Do we need subleading color?

Some people are thinking about NLO level shower. I think it is too early but who knows they might be right. It is clear that there is no way to go higher order with leading color approximation.

$$\alpha_s \approx \frac{1}{N_c^2} \approx 0.1$$

There are two perturbative parameters. The *formal expansion* of the splitting operator is

$$\mathcal{H}_\mathrm{I} = \frac{\alpha_\mathrm{s}}{2\pi}\mathcal{H}_\mathrm{I}^{(0,0)} + \frac{\alpha_\mathrm{s}}{2\pi}\frac{1}{N_c^2}\mathcal{H}_\mathrm{I}^{(0,1)} + \left(\frac{\alpha_\mathrm{s}}{2\pi}\right)^2\mathcal{H}_\mathrm{I}^{(1,0)} + \cdots$$

Furthermore we *need two color indices* to represent a partonic states (interference terms).

Note that $\mathcal{H}_I$ is an operator and it is impossible to do this expansion in practice.

Yes, we need.

Density Operator

The physical cross section is

$$\sigma[F] = \sum_m \int [d\{p,f\}_m]\, \mathrm{Tr}\{\underbrace{\rho(\{p,f\}_m)}_{\text{density operator in color} \otimes \text{spin space}} F(\{p,f\}_m)\}$$

The density operator is

$$\rho(\{p,f\}_m) = \big|\mathcal{M}(\{p,f\}_m)\big\rangle \frac{f_{a/A}(\eta_{\mathrm{a}},\mu_F^2)\, f_{b/B}(\eta_{\mathrm{b}},\mu_F^2)}{2\eta_{\mathrm{a}}\eta_{\mathrm{b}}\, p_A\cdot p_B} \big\langle \mathcal{M}(\{p,f\}_m)\big|$$

or expanding it on a color and spin basis

$$\rho(\{p,f\}_m) = \sum_{s,c}\sum_{s',c'} \big|\{s,c\}_m\big\rangle\, \rho(\{p,f,s',c',s,c\}_m)\, \big\langle\{s',c'\}_m\big|$$

Statistical States

The set of functions $\rho(\{p, f, s', c', s, c\}_m)$ forms a vector space.

Basis: $\big|\{p, f, s', c', s, c\}_m\big)$

Completeness relation :

$$1 = \sum_m \int [d\{p, f, s', c', s, c\}_m] \, \big|\{p, f, s', c', s, c\}_m\big)\big(\{p, f, s', c', s, c\}_m\big|$$

Inner product of the basis states:

$$\big(\{p, f, s', c', s, c\}_m\big|\{\tilde{p}, \tilde{f}, \tilde{s}', \tilde{c}', \tilde{s}, \tilde{c}\}_{\tilde{m}}\big) = \delta_{m,\tilde{m}} \; \delta(\{p, f, s', c', s, c\}_m; \{\tilde{p}, \tilde{f}, \tilde{s}', \tilde{c}', \tilde{s}, \tilde{c}\}_{\tilde{m}})$$

A physical state which is related to the density matrix:

$$\boxed{\big|\rho\big) = \int [d\{p, f, s', c', s, c\}_m] \, \rho(\{p, f, s', c', s, c\}_m) \big|\{p, f, s', c', s, c\}_m\big)}$$

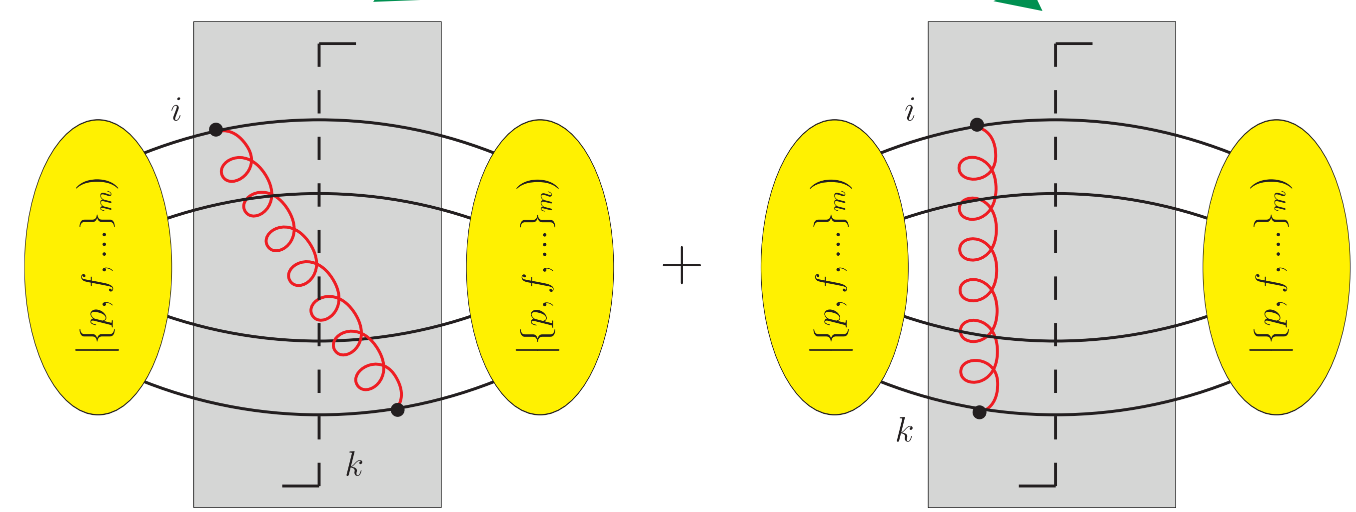
Shower Evolution

Using the factorization properties of the QCD the approximated order by order calculation can be organized according to

$$\mathcal{U}(t,t') = 1 + \int_{t'}^{t} d\tau\, \mathcal{U}(t,\tau)\big[\mathcal{H}_I(\tau) - \mathcal{V}(\tau)\big]$$

From the unitary condition:

$$\big(1\big|\mathcal{V}(t) = \big(1\big|\mathcal{H}_{\mathrm{I}}(t)$$



resolved radiations *unresolved radiation*

The shower form of the solution is

$$\mathcal{U}(t,t') = \mathcal{N}(t,t') + \int_{t'}^{t} d\tau\, \mathcal{U}(t,\tau)\, \mathcal{H}_I(\tau)\, \mathcal{N}(\tau,t')$$

and the no-splitting operator is

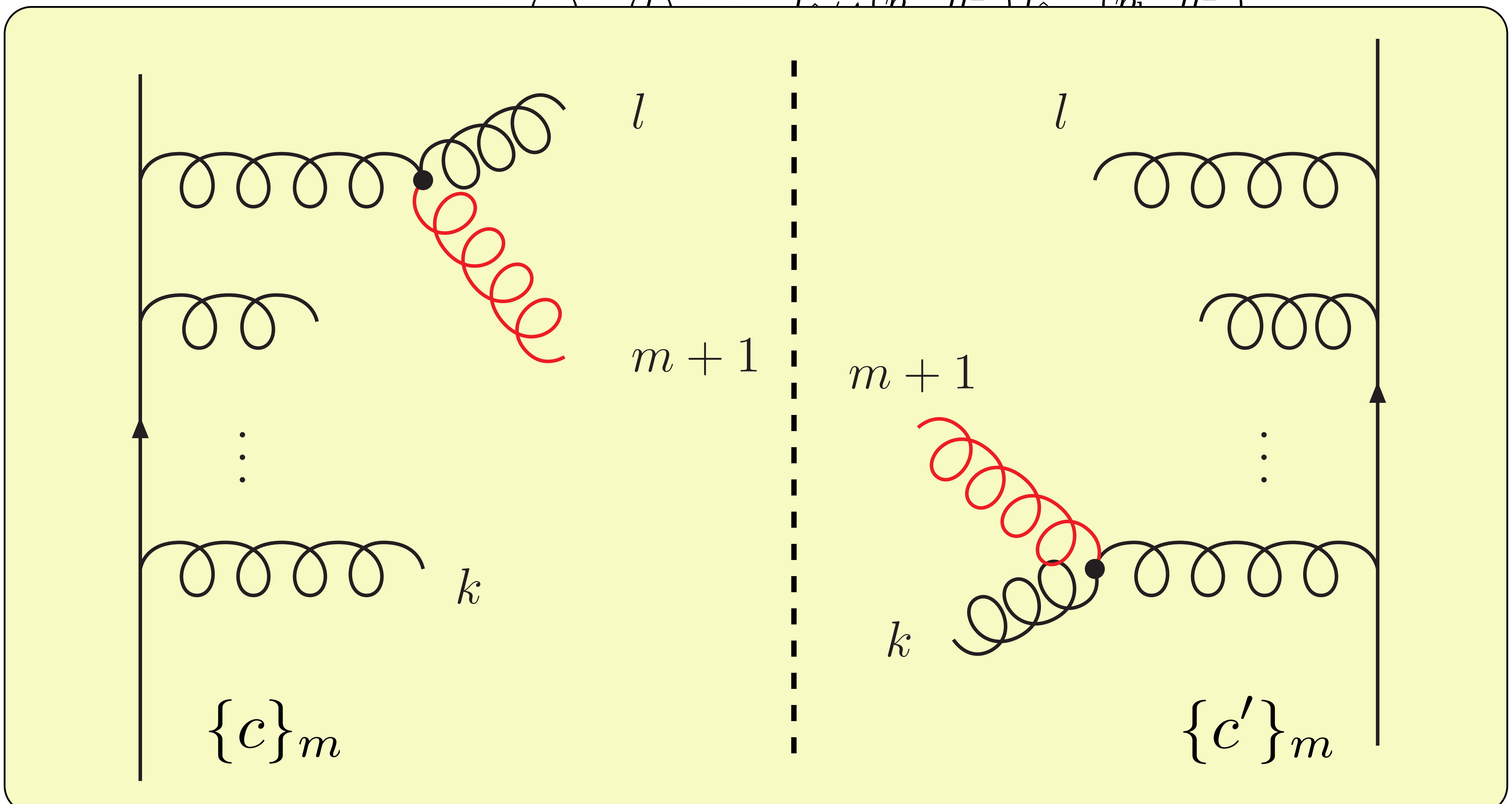
$$\mathcal{N}(t,t') = \mathbb{T}\exp\left(-\int_{t'}^{t} d\tau\, \mathcal{V}(\tau)\right)$$

Full Splitting Operator

$$
\begin{aligned}
&\big(\{\hat{p},\hat{f},\hat{s}',\hat{c}',\hat{s},\hat{c}\}_{m+1}\big|\mathcal{H}_{\mathrm{I}}(t)\big|\{p,f,s',c',s,c\}_m\big)\\
&= \sum_{l\in\{\mathrm{a},\mathrm{b},1,\dots,m\}} (m+1)\frac{n_{\mathrm{c}}(a)n_{\mathrm{c}}(b)\,\eta_{\mathrm{a}}\eta_{\mathrm{b}}}{n_{\mathrm{c}}(\hat{a})n_{\mathrm{c}}(\hat{b})\,\hat{\eta}_{\mathrm{a}}\hat{\eta}_{\mathrm{b}}}\,\frac{f_{\hat{a}/A}(\hat{\eta}_{\mathrm{a}},\mu_F^2)f_{\hat{b}/B}(\hat{\eta}_{\mathrm{b}},\mu_F^2)}{f_{a/A}(\eta_{\mathrm{a}},\mu_F^2)f_{b/B}(\eta_{\mathrm{b}},\mu_F^2)}\\
&\quad\times\big(\{\hat{p},\hat{f}\}_{m+1}\big|\mathcal{P}_l\big|\{p,f\}_m\big)\,\delta(t-T_l(\{\hat{p},\hat{f},\hat{s}',\hat{c}',\hat{s},\hat{c}\}_{m+1}))\\
&\quad\times\Bigg[\theta(\hat{f}_{m+1}=\mathrm{g})\sum_{\substack{k\in\{\mathrm{a},\mathrm{b},1,\dots,m\}\\ k\neq l}}\Bigg\{\big(\{\hat{c}',\hat{c}\}_{m+1}\big|\mathcal{G}(l,k;\{\hat{f}\}_{m+1})\big|\{c',c\}_m\big)\\
&\qquad\qquad\times\Big[A_{lk}(\{\hat{p}\}_{m+1})\big(\{\hat{s}',\hat{s}\}_{m+1}\big|\mathcal{W}(l,k;\{\hat{f},\hat{p}\}_{m+1})\big|\{s',s\}_m\big)\\
&\qquad\qquad\quad-\frac{1}{2}\big(\{\hat{s}',\hat{s}\}_{m+1}\big|\mathcal{W}(l,l;\{\hat{f},\hat{p}\}_{m+1})\big|\{s',s\}_m\big)\Big]\\
&\qquad\quad+\big(\{\hat{c}',\hat{c}\}_{m+1}\big|\mathcal{G}(k,l;\{\hat{f}\}_{m+1})\big|\{c',c\}_m\big)\\
&\qquad\qquad\times\Big[A_{lk}(\{\hat{p}\}_{m+1})\big(\{\hat{s}',\hat{s}\}_{m+1}\big|\mathcal{W}(k,l;\{\hat{f},\hat{p}\}_{m+1})\big|\{s',s\}_m\big)\\
&\qquad\qquad\quad-\frac{1}{2}\big(\{\hat{s}',\hat{s}\}_{m+1}\big|\mathcal{W}(l,l;\{\hat{f},\hat{p}\}_{m+1})\big|\{s',s\}_m\big)\Big]\Bigg\}\\
&\qquad+\theta(\hat{f}_{m+1}\neq\mathrm{g})\,\big(\{\hat{c}',\hat{c}\}_{m+1}\big|\mathcal{G}(l,l;\{\hat{f}\}_{m+1})\big|\{c',c\}_m\big)\mathcal{W}(l,l;\{\hat{f},\hat{p}\}_{m+1})\Bigg]
\end{aligned}
$$

Full Splitting Operator

$$\big(\{\hat p,\hat f,\hat s',\hat c',\hat s,\hat c\}_{m+1}\big|\mathcal{H}_{\rm I}(t)\big|\{p,f,s',c',s,c\}_m\big)$$



$$+\,\theta(\hat f_{m+1}\neq {\rm g})\ \big(\{\hat c',\hat c\}_{m+1}\big|\mathcal{G}(l,l;\{\hat f\}_{m+1})\big|\{c',c\}_m\big)\mathcal{W}(l,l;\{\hat f,\hat p\}_{m+1})\Big]$$

Solution of the Evolution Equation

The idea is split the splitting operator "*good*" and "*bad*" part and expand the evolution operator in the "bad" splitting operator.

$$\mathcal{H}_I(t) = \underbrace{\mathcal{H}_I^{(J)}(t)}_{\textit{Fully exponentiated}} + \underbrace{\mathcal{H}_I^{(S)}(t)}_{\textit{Subtracted}}$$

The inclusive splitting operators are

$$\big(1\big|\mathcal{V}^{(J)}(t) = \big(1\big|\mathcal{H}_I^{(J)}(t) \qquad \text{and} \qquad \big(1\big|\mathcal{V}^{(S)}(t) = \big(1\big|\mathcal{H}_I^{(S)}(t)$$

Now the *good part* of the evolution operator is

$$\mathcal{U}^{(J)}(t,t') = \mathcal{N}^{(J)}(t,t') + \int_{t'}^{t} d\tau\, \mathcal{U}^{(J)}(t,\tau)\mathcal{H}_I^{(J)}(\tau)\mathcal{N}^{(J)}(\tau,t')$$

The full evolution operator is given by

$$\mathcal{U}(t,t') = \mathcal{U}^{(J)}(t,t') + \int_{t'}^{t} d\tau\, \mathcal{U}(t,\tau)\left[\mathcal{H}_I^{(S)}(\tau) - \mathcal{V}^{(S)}(\tau)\right]\mathcal{U}^{(J)}(\tau,t')$$

Solution of the Evolution Equation

The idea is split the splitting operator "*good*" and "*bad*" part and expand the evolution operator in the "bad" splitting operator.

$$\mathcal{H}_I(t) = \mathcal{H}_I^{(J)}(t) + \mathcal{H}_I^{(S)}(t)$$

Fully exponentiated *Subtracted*

The

$$\mathcal{U}(t,t') = \mathcal{U}^{(J)}(t,t') + \int_{t'}^{t} d\tau\, \mathcal{U}^{(J)}(t,\tau)\,[\mathcal{H}_I^{(S)}(\tau) - \mathcal{V}^{(S)}(\tau)]\,\mathcal{U}^{(J)}(\tau,t')$$

$$+ \int_{t'}^{t} d\tau_2 \int_{t'}^{\tau_2} d\tau_1\, \mathcal{U}^{(J)}(t,\tau_2)\,[\mathcal{H}_I^{(S)}(\tau_2) - \mathcal{V}^{(S)}(\tau_2)]$$

$$\times\, \mathcal{U}^{(J)}(\tau_2,\tau_1)[\mathcal{H}_I^{(S)}(\tau_2) - \mathcal{V}^{(S)}(\tau_1)]\,\mathcal{U}^{(J)}(\tau_1,t')$$

$$+ \cdots$$

Now

The full evolution operator is given by

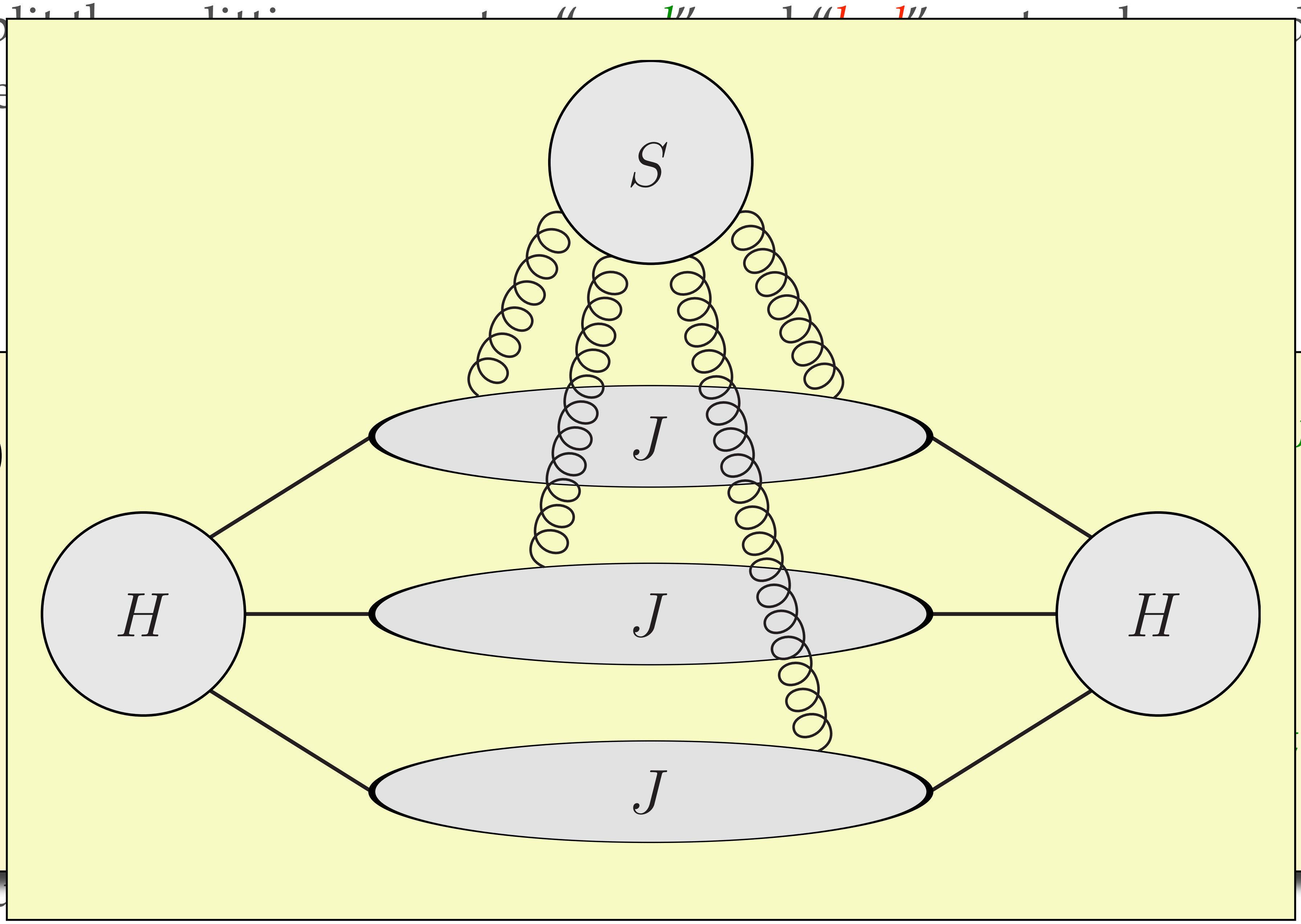
$$\mathcal{U}(t,t') = \mathcal{U}^{(J)}(t,t') + \int_{t'}^{t} d\tau\, \mathcal{U}(t,\tau)\,[\mathcal{H}_I^{(S)}(\tau) - \mathcal{V}^{(S)}(\tau)]\,\mathcal{U}^{(J)}(\tau,t')$$

Solution of the Evolution Equation

The idea is sp... d the
evolution ope...

The ... $\mathcal{U}(t,t')$ ... $^{J)}(\tau,t')$

Now ... $'$)



The full evolu...

$$\mathcal{U}(t,t') = \mathcal{U}^{(J)}(t,t') + \int_{t'}^{t} d\tau\, \mathcal{U}(t,\tau)\, [\mathcal{H}_I^{(S)}(\tau) - \mathcal{V}^{(S)}(\tau)]\, \mathcal{U}^{(J)}(\tau,t')$$

Jet Splitting Operator

We approximate the color operator using a projection

$$\big(\{\hat{c}',\hat{c}\}_{m+1}\big|\mathcal{G}(k,l;\{\hat{f}\}_{m+1})\big|\{c',c\}_m\big) = \big(\{\hat{c}',\hat{c}\}_{m+1}\big|t_k^\dagger \otimes t_l\big|\{c',c\}_m\big)$$

$$\big(\{\hat{c}',\hat{c}\}_{m+1}\big|\mathcal{C}(l,m+1)\,\mathcal{G}(k,l;\{\hat{f}\}_{m+1})\big|\{c',c\}_m\big)$$

The projection keep the color connected part

$$\mathcal{C}(l,m+1)\big|\{c',c\}_{m+1}\big) = \begin{cases} \big|\{c',c\}_{m+1}\big) & l \text{ and } m+1 \text{ color connected} \\ & \text{in } \{c'\}_{m+1} \text{ and in } \{c\}_{m+1} \\ 0 & \text{otherwise} \end{cases}$$

The corresponding quantum level operator is

$$\mathcal{C}(l,m+1) = C(l,m+1)^\dagger \otimes C(l,m+1)$$

Jet Splitting Operator

We approximate the color operator using a projection

$(\{\hat{c}$ $\}_m)$

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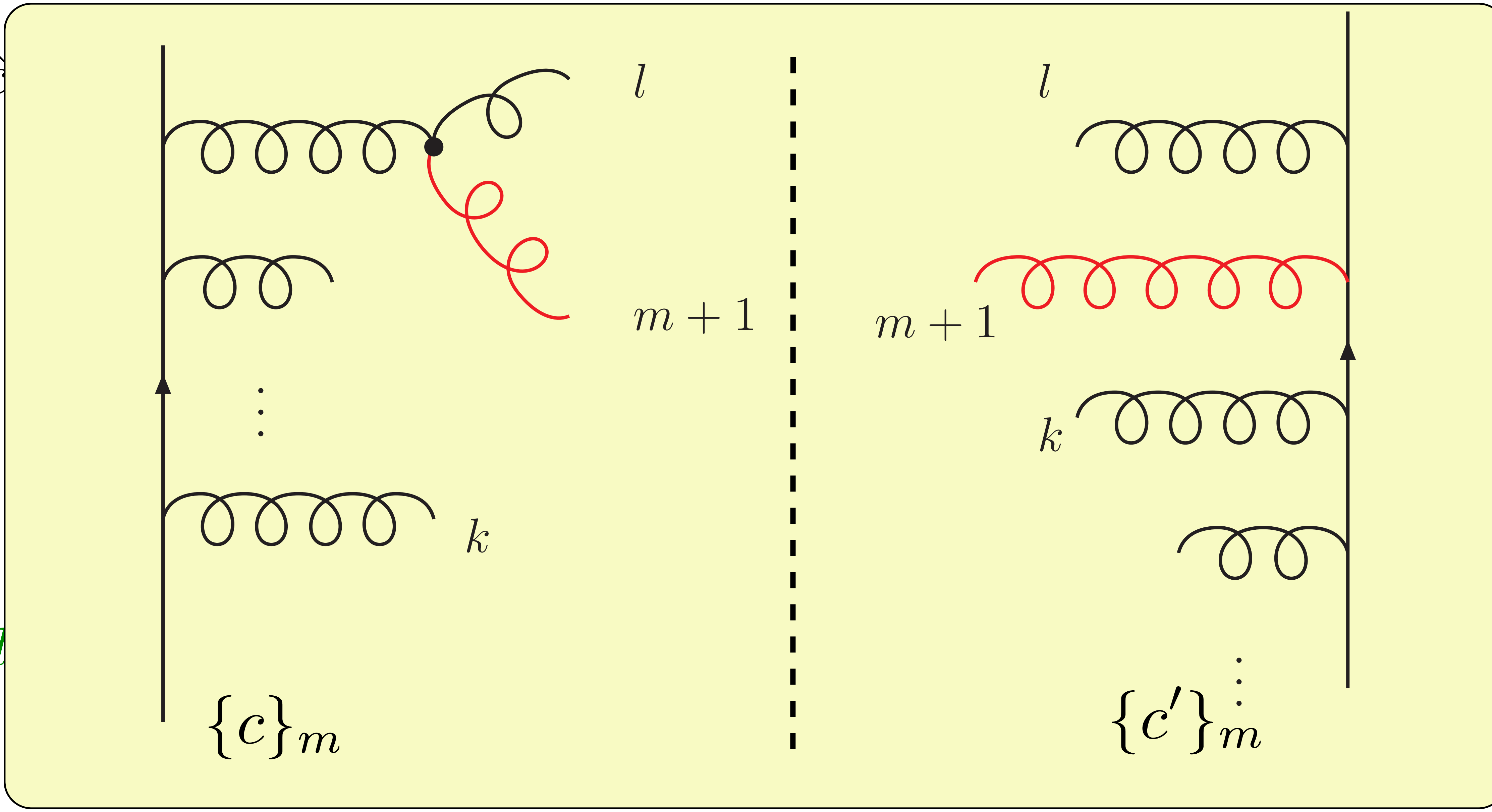
$\mathcal{C}(l$



The corresponding quantum level operator is

$$\color{green}{\mathcal{C}(l,m+1)} = \color{red}{C(l,m+1)^\dagger} \otimes \color{blue}{C(l,m+1)}$$

Jet Splitting Operator

Now the jet splitting operator is

$$
\begin{aligned}
&(\{\hat{p},\hat{f},\hat{s}',\hat{c}',\hat{s},\hat{c}\}_{m+1}|\mathcal{H}_{\rm I}^{(\rm J)}(t)|\{p,f,s',c',s,c\}_m)\\
&\quad = \sum_{l\in\{\rm a,b,1,\dots,m\}}(m+1)\frac{n_{\rm c}(a)n_{\rm c}(b)\,\eta_{\rm a}\eta_{\rm b}}{n_{\rm c}(\hat{a})n_{\rm c}(\hat{b})\,\hat{\eta}_{\rm a}\hat{\eta}_{\rm b}}\,\frac{f_{\hat{a}/A}(\hat{\eta}_{\rm a},\mu_F^2)f_{\hat{b}/B}(\hat{\eta}_{\rm b},\mu_F^2)}{f_{a/A}(\eta_{\rm a},\mu_F^2)f_{b/B}(\eta_{\rm b},\mu_F^2)}\\
&\qquad\times(\{\hat{p},\hat{f}\}_{m+1}|\mathcal{P}_l|\{p,f\}_m)\,\delta(t-T_l(\{\hat{p},\hat{f},\hat{s}',\hat{c}',\hat{s},\hat{c}\}_{m+1}))\\
&\qquad\times\Bigg[\theta(\hat{f}_{m+1}={\rm g})\sum_{\substack{k\in\{\rm a,b,1,\dots,m\}\\ k\neq l}}\\
&\qquad\qquad\Big\{(\{\hat{c}',\hat{c}\}_{m+1}|\mathcal{C}(l,m+1)\mathcal{G}(l,k;\{\hat{f}\}_{m+1})|\{c',c\}_m)\\
&\qquad\qquad\quad\times\Big[A_{lk}(\{\hat{s}',\hat{s}\}_{m+1}|\mathcal{W}(l,k;\{\hat{f},\hat{p}\}_{m+1})|\{s',s\}_m)\\
&\qquad\qquad\qquad-\frac{1}{2}(\{\hat{s}',\hat{s}\}_{m+1}|\mathcal{W}(l,l;\{\hat{f},\hat{p}\}_{m+1})|\{s',s\}_m)\Big]\\
&\qquad\qquad+(\{\hat{c}',\hat{c}\}_{m+1}|\mathcal{C}(l,m+1)\mathcal{G}(k,l;\{\hat{f}\}_{m+1})|\{c',c\}_m)\\
&\qquad\qquad\quad\times\Big[A_{lk}(\{\hat{s}',\hat{s}\}_{m+1}|\mathcal{W}(k,l;\{\hat{f},\hat{p}\}_{m+1})|\{s',s\}_m)\\
&\qquad\qquad\qquad-\frac{1}{2}(\{\hat{s}',\hat{s}\}_{m+1}|\mathcal{W}(l,l;\{\hat{f},\hat{p}\}_{m+1})|\{s',s\}_m)\Big]\Big\}\\
&\qquad\quad+\theta(\hat{f}_{m+1}\neq{\rm g})\;\mathcal{G}(l,l;\{\hat{f}\}_{m+1})\mathcal{W}(l,l;\{\hat{f},\hat{p}\}_{m+1})\Bigg]
\end{aligned}
$$

- This operator can evolve interference contribution.
- Full collinear and soft+collinear contributions included.
- Wide angle pure soft contributions are not fully included. Omitted part is suppressed by $1/N_c^2$. It is treated perturbatively.
- The corresponding inclusive splitting operator can be exponentiated easily.
- Leads to a quasi Markovian process.

Jet Splitting Operator

Now the jet splitting operator is

$$
\begin{aligned}
&(\{\hat p,\hat f,\hat s',\hat c',\hat s,\hat c\}_{m+1}|\mathcal{H}_{\rm I}^{(\rm J)}(t)|\{p,f,s',c',s,c\}_m)\\
&\quad = \sum_{l\in\{\ldots\}} \ldots\\
&\quad \times \ldots\\
&\quad \times \Bigg[\theta(\hat f_{m+1} = \mathrm{g}) \sum_{\substack{k\in\{\mathrm{a},\mathrm{b},1,\dots,m\}\\ k\neq l}}\\
&\qquad \Big\{(\{\hat c',\hat c\}_{m+1}|\mathcal{C}(l,m+1)\mathcal{G}(l,k;\{\hat f\}_{m+1})|\{c',c\}_m)\\
&\qquad\quad \times \Big[A_{lk}(\{\hat s',\hat s\}_{m+1}|\mathcal{W}(l,k;\{\hat f,\hat p\}_{m+1})|\{s',s\}_m)\\
&\qquad\qquad - \frac{1}{2}(\{\hat s',\hat s\}_{m+1}|\mathcal{W}(l,l;\{\hat f,\hat p\}_{m+1})|\{s',s\}_m)\Big]\\
&\qquad + (\{\hat c',\hat c\}_{m+1}|\mathcal{C}(l,m+1)\mathcal{G}(k,l;\{\hat f\}_{m+1})|\{c',c\}_m)\\
&\qquad\quad \times \Big[A_{lk}(\{\hat s',\hat s\}_{m+1}|\mathcal{W}(k,l;\{\hat f,\hat p\}_{m+1})|\{s',s\}_m)\\
&\qquad\qquad - \frac{1}{2}(\{\hat s',\hat s\}_{m+1}|\mathcal{W}(l,l;\{\hat f,\hat p\}_{m+1})|\{s',s\}_m)\Big]\Big\}\\
&\quad + \theta(\hat f_{m+1} \neq \mathrm{g})\ \mathcal{G}(l,l;\{\hat f\}_{m+1})\mathcal{W}(l,l;\{\hat f,\hat p\}_{m+1})\Bigg]
\end{aligned}
$$

$$
\mathcal{N}^{(\rm J)}(t,t')\big|p,f,s',c',s,c\}_m) = \exp\left\{-\int_{t'}^{t} d\tau\,[\lambda_1(\{p,f,c\}_m) + \lambda_2(\{p,f,c'\}_m)]\right\} \times \big|p,f,s',c',s,c\}_m)
$$

- This operator can evolve ... llinear ...
- Wide angle pure soft contributions are not fully included. Omitted part is suppressed by $1/N_c^2$. It is treated perturbatively.
- The corresponding inclusive splitting operator can be exponentiated easily.
- Leads to a quasi Markovian process.

Leading Color Approximation

Recalling the evolution equation of the “jet” evolution operator

$$\mathcal{U}^{(J)}(t,t') = \mathcal{N}^{(J)}(t,t') + \int_{t'}^{t} d\tau\, \mathcal{U}^{(J)}(t,\tau)\mathcal{H}_I^{(J)}(\tau)\mathcal{N}^{(J)}(\tau,t')$$

and from this the evolution equation of the *leading color approximation* can be given by another projection

$$\mathcal{U}^{(LC)}(t,t') = \mathcal{N}^{(J)}(t,t')\mathcal{P}_D + \int_{t'}^{t} d\tau\, \mathcal{U}^{(LC)}(t,\tau)\mathcal{H}_I^{(J)}(\tau)\mathcal{N}^{(J)}(\tau,t')\mathcal{P}_D$$

where the projection $\mathcal{P}_D$ keeps the color diagonal contributions only

$$\mathcal{P}_D\big|\{c',c\}_m\big) = \begin{cases} \big|\{c',c\}_m\big) & \text{if } \{c\}_m = \{c'\}_m \\ 0 & \text{otherwise} \end{cases}$$

Conclusions

- We have a well define formalism to describe parton shower algorithms. *Note we have equation!*
- The formalism itself help us to have better understanding of already know effects and approximations (color coherence, leading color approximation,..)
- We have shown that it is possible to compute parton shower with full color *efficiently* and *systematically* using mainly standard Monte Carlo techniques.
- Implementation .... (*coming soon ....*)