CERN, 27 May '08

Singlet QCD Evolution at Small-x: the ABF Approach

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and
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Based on G.A., R. Ball, S.Forte:

hep-ph/9911273 (NPB <u>575</u>,313)
hep-ph/0001157 (lectures)
hep-ph/0011270 (NPB <u>599</u>,383)
hep-ph/01042
hep-ph/0109178 (NPB <u>621</u>,359)
hep-ph/0306156 (NPB <u>674</u>,459),
hep-ph/0310016

in particular, on our most recent works:
hep-ph/0512237, (NPB <u>742</u>,1,2006), hep-ph/0606323
0802.0032[hep-ph], (NPB <u>799</u>,199,2008),
R. Ball, 0708.1277[hep-ph] (NPB <u>796</u>,137,2008),

- Short summary of our approach
- Final results and work in progress on applications

In QCD only the evolution of structure functions is computable:

our goal is to construct a relatively simple, closed form, improved anom. dimension $\gamma_I(\alpha,N)$ or splitting function $P_I(\alpha,x)$

$P_I(\alpha,x)$ should

- reproduce the perturbative results at large x
- resum BFKL corrections at small x
- properly treat running coupling effects
- be sufficiently simple to be included in fitting codes

Then compute the resummed coefficients and control the scheme dependence --> structure functions

All of this is now completed

The comparison of the result with the data provides a qualitatitavely new test of the theory (in progress)

Moments

$$\xi = \log\frac{1}{x}; \qquad t = \log\frac{Q^2}{\mu^2}$$

$$G(x, Q^2) \equiv G(\xi, t) = x[g(x, Q^2) + k\Sigma(x, Q^2)]$$

Singlet quark

For each moment: singlet eigenvector with largest anomalous dimension eigenvalue

$$G(N, t) = \int_0^1 x^{N-1} G(x, Q^2) dx = \int_0^\infty e^{-N\xi} G(\xi, t) d\xi$$

Mellin transf. (MT)

$$G(\xi, t) = \int_{-i\infty}^{+i\infty} e^{N\xi} G(N, t) \frac{dN}{2\pi i}$$

Inverse MT (ξ>0)

t-evolution eq.n

$$\boxed{\frac{d}{dt} G(N, t) = \gamma(N, \alpha(t)) G(N, t)}$$

γ: anom. dim

$$\gamma(N, \alpha) = \alpha\gamma_{1l}(N) + \alpha^2\gamma_{2l}(N) + \alpha^3\gamma_{3l}(N) + \ldots$$

Pert. Th.: LO NLO NNLO

known Moch, Vermaseren, Vogt '04

Recall:

$$\gamma(N) = \int_0^1 x^N P(x)dx$$

$$P(x) = \frac{\alpha}{x}\left(\alpha \log\frac{1}{x}\right)^n \leftrightarrow \gamma(N) = n!\left(\frac{\alpha}{N}\right)^{n+1}$$

splitting function anomalous dimension

At 1-loop:

$$\alpha \cdot \gamma_{1l}(N) = \alpha \cdot \left[\frac{1}{N} - A(N)\right]$$

This corresponds to the "double scaling" behavior at small x:

$$G(\xi, t) \sim \exp\left[\sqrt{\frac{4n_C}{\pi\beta_0} \cdot \xi \cdot \frac{\log Q^2/\Lambda^2}{\log \mu^2/\Lambda^2}}\right]$$

$$\beta(\alpha) = -\beta_0\alpha^2 + \ldots$$

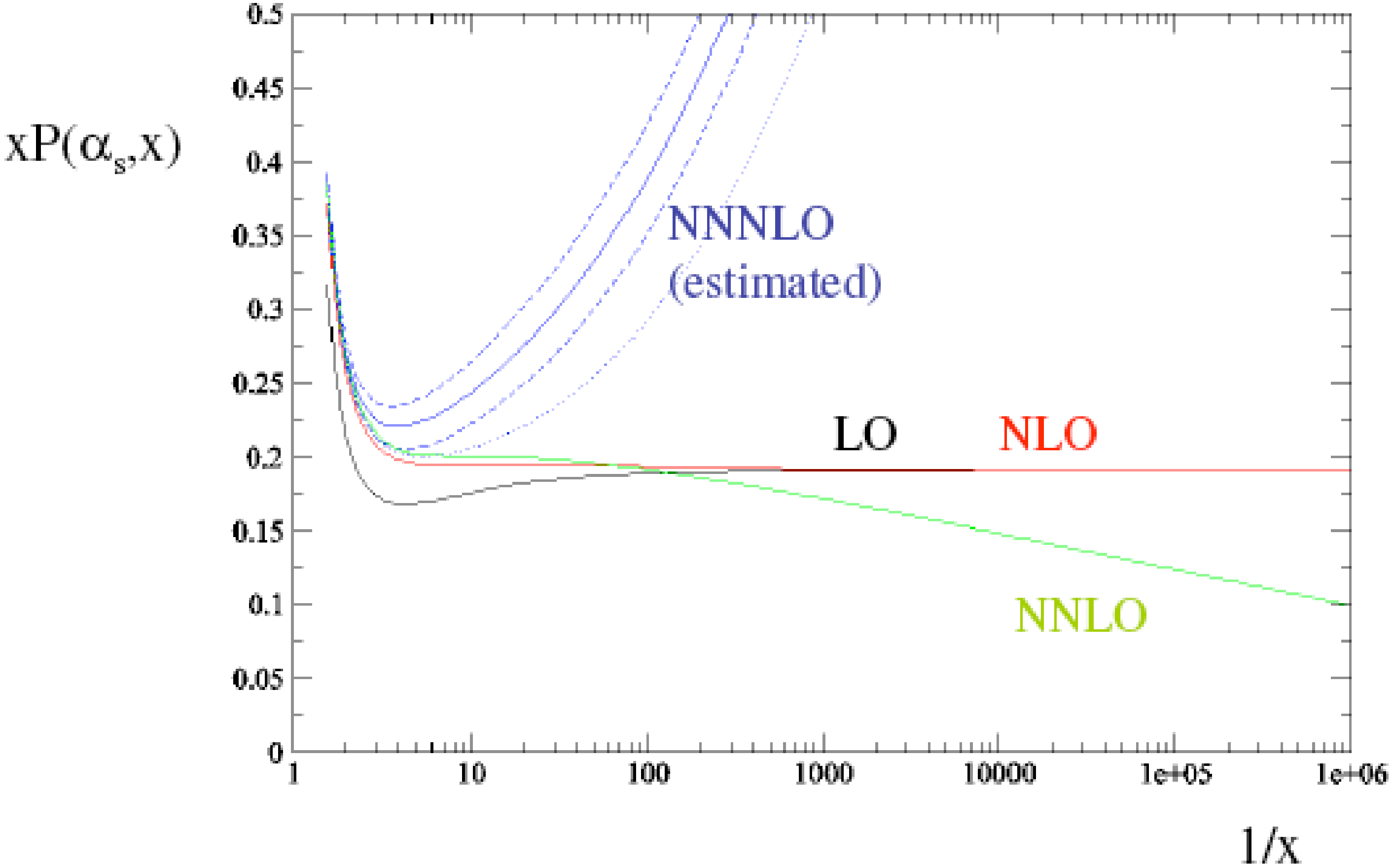
A. De Rujula et al '74/Ball, Forte '94

Amazingly supported by the data

$\alpha_s = 0.2$

$n_f = 0$

The singlet splitting function



$$xP(\alpha_s, x) \sim a_1\alpha_s + a_2\alpha_s^2 + (a_3 \ln x + b_3)\alpha_s^3 + (a_4 \ln^3 x + b_4 \ln^2 x)\alpha_s^4 + \cdots$$

LO NLO NNLO NNNLO

In principle the BFKL approach provides a tool to control $(\alpha/N)^n$ corrections to $\gamma(N,\alpha)$, that is $(\alpha \log 1/x)^n$ to $xP(x,Q^2)/\alpha$

Define t- Mellin transf.:

$$G(\xi, M) = \int_{-\infty}^{+\infty} e^{-Mt} G(\xi, t)\,dt$$

with inverse:

$$G(\xi, t) = \int_{-i\infty}^{+i\infty} e^{Mt} G(\xi, M) \frac{dM}{2\pi i}$$

ξ-evolution eq.n (BFKL) [at fixed α]:

$$\boxed{\frac{d}{d\xi} G(\xi, M) = \chi(M, \alpha) G(\xi, M)}$$

with $\chi(M, \alpha) = \alpha \cdot \chi_0(M) + \alpha^2 \cdot \chi_1(M) + \ldots$

known

χ_0, χ_1 contain all info on $(\alpha \log 1/x)^n$ and $\alpha(\alpha \log 1/x)^n$

Bad behaviour, bad convergence

The minimum value of $\alpha\chi_0$ at M=1/2 is the Lipatov intercept:

$$\lambda_0 = \alpha\chi_0\left(\frac{1}{2}\right) = \frac{\alpha n_C}{\pi} 4\ln 2 = \alpha c_0 \sim 2.65\alpha \sim 0.5$$

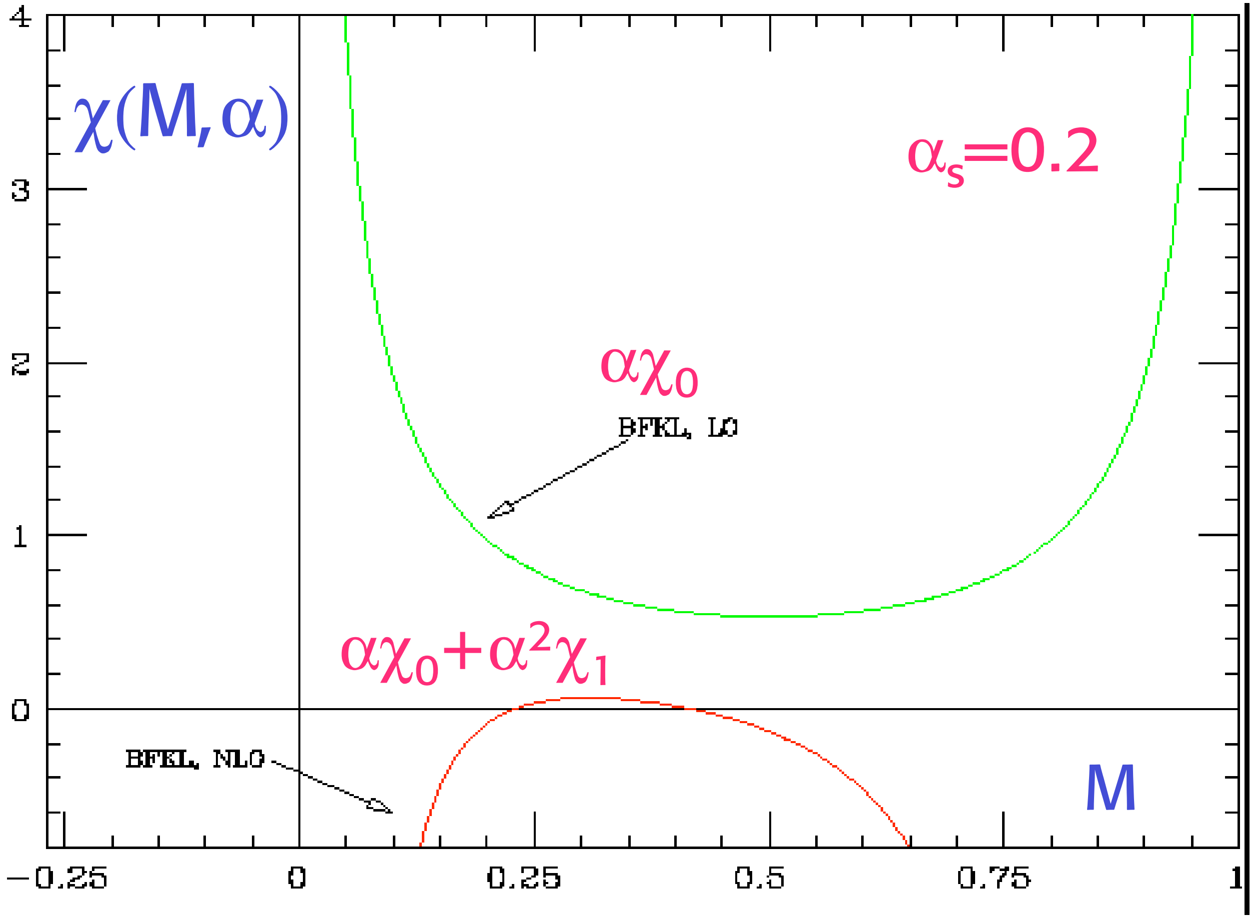
It corresponds to (for x->0, Q^2 fixed):

$$xP(x) \sim \alpha x^{-\lambda_0}$$

Too hard, not supported by data

But the NLO terms are very large

χ_1 totally overwhelms χ_0!!



Basic ingredients of the resummation procedure

Only subleading terms added, double counting avoided

- Duality relation $\chi(\gamma(\alpha, N), \alpha) = N$

  from consistency of 1/x and Q^2 evolutions

- Momentum conservation $\chi(0, \alpha) = 1$

  as $\gamma(\alpha,1) = 0$

  χ finite at M=0!

- Symmetry properties of the BFKL kernel (M <--> 1-M)

  Lipatov-Salam-Ciafaloni

  χ finite at M=1!

  χ has a minimum

- Running coupling effects

  through a quadratic approx. near the minimum of χ.
  Solution: Bateman (or Airy) functions

⊕ We implement the steps on χ and then obtain γ by duality

In the region of t and ξ where both

$$\frac{d}{dt}G(N,t) = \gamma(N,\alpha)G(N,t)$$

$$\frac{d}{d\xi}G(\xi,M) = \chi(M,\alpha)G(\xi,M)$$

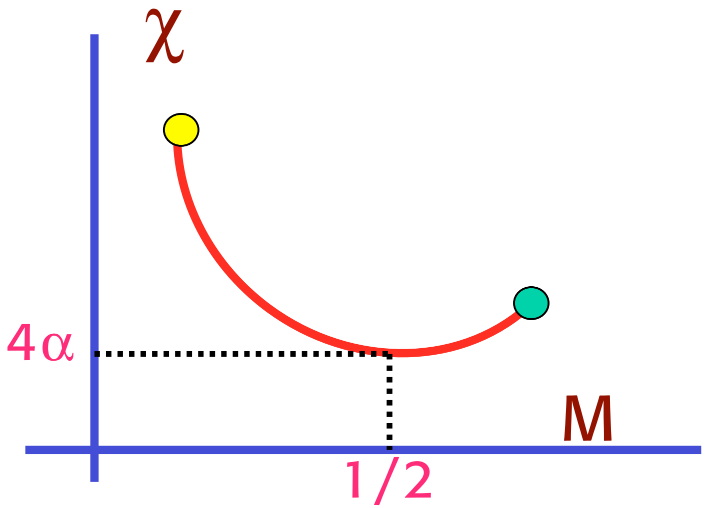
are approximately valid, the "duality" relation holds:

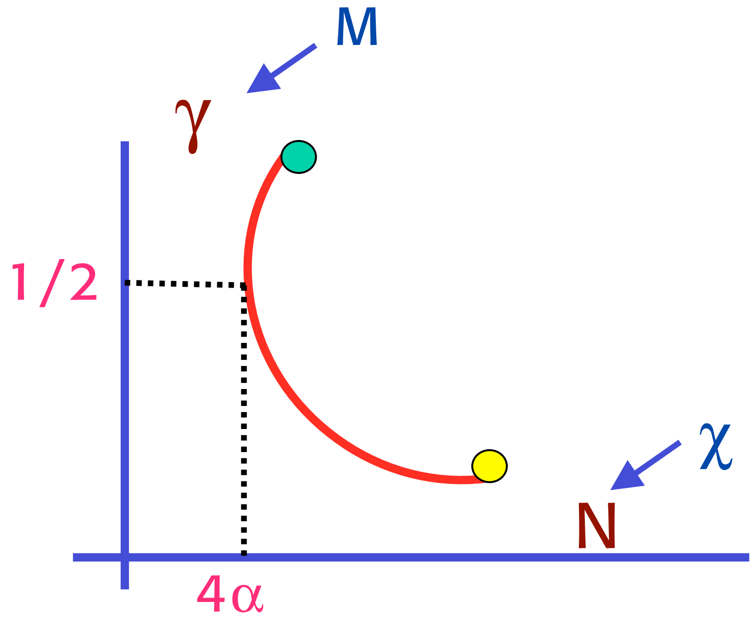
$$\boxed{\chi(\gamma(\alpha,N),\alpha) = N}$$

Note: γ is leading twist while χ is all twist.
Still the two perturbative exp.ns are related and improve each other.
Non perturbative terms in χ correspond to power or exp. suppressed terms in γ.

$\chi(\gamma(N)) = N$

Graphically duality is a reflection





Example: if $\chi(M, \alpha) = \alpha\left[\frac{1}{M} + \frac{1}{1-M}\right] \Longrightarrow$

$$\Longrightarrow \quad \alpha\left[\frac{1}{\gamma} + \frac{1}{1-\gamma}\right] = N \quad \Longrightarrow \quad \gamma = \frac{1}{2}\left[1 \pm \sqrt{1 - \frac{4\alpha}{N}}\right]$$

Note: γ contains $(\alpha/N)^n$ terms and has a cut

For example at 1-loop:

$$\chi_0(\gamma_s(\alpha,N))=N/\alpha$$

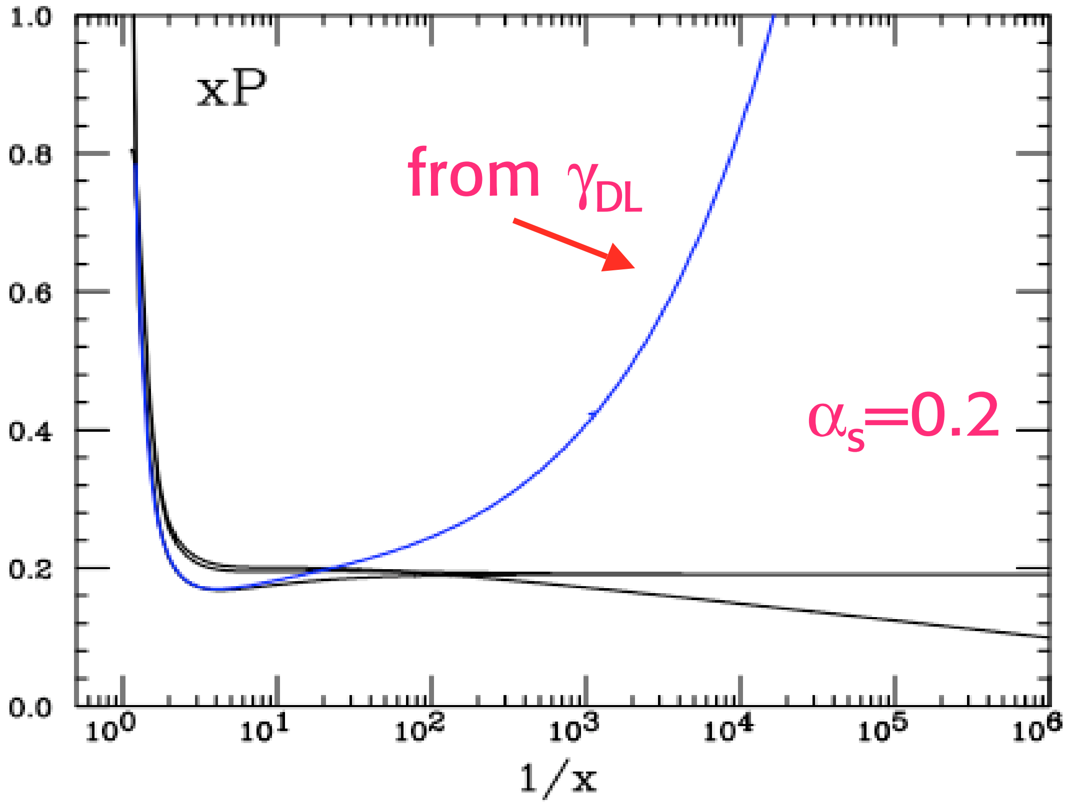
χ_0 improves γ by adding a series of terms in $(\alpha/N)^n$:

$\chi_0 \to \gamma_s\left(\frac{\alpha}{N}\right)$ $\qquad$ $\gamma_s\left(\frac{\alpha}{N}\right) = \sum_k c_k\left(\frac{\alpha}{N}\right)^k$

$$\boxed{\gamma_{DL}(\alpha, N) = \alpha \cdot \gamma_{1l}(N) + \gamma_s\left(\frac{\alpha}{N}\right) + \ldots \text{ -double count.}}$$

γ_{DL} is the naive result from 1-loop+(LO)BFKL

The data discard such a large increase at small x



Similarly it is very important to improve χ by using γ_{1l}

Near M=0, $\chi_0 \sim 1/M$, $\chi_1 \sim -1/M^2$.......

Duality + momentum cons. ($\gamma(\alpha,N=1)=0$)

$$\chi(\gamma(\alpha, N), \alpha) = N \longrightarrow \chi(0, \alpha) = 1$$

$$\lim_{M\to 0} \chi(M, \alpha) \approx \frac{\alpha}{M+\alpha}$$

$$\begin{cases} \gamma(\chi(M)) = M \longrightarrow \gamma_{1l} \Rightarrow \chi_s\left(\frac{\alpha}{M}\right) \\ \chi_s\left(\frac{\alpha}{M}\right) = \sum_k d_k \left(\frac{\alpha}{M}\right)^k \end{cases}$$

$$\boxed{\chi_{DL}(M, \alpha) = \alpha \cdot \chi_0(M) + \chi_s\left(\frac{\alpha}{M}\right) + \ldots \text{ -double count.}}$$

Double Leading Expansion

$$\gamma(N,\alpha) = \alpha\cdot\gamma_{1l}(N) + \ldots \sim \alpha\cdot\left[\frac{1}{N} - A(N)\right]$$

Momentum conservation: $\gamma(1,\alpha)=0$ $\longrightarrow$ $A(1)=1$

Duality: $\gamma(\chi(M)) = M$ $\longrightarrow$ $\alpha\cdot\left[\frac{1}{\chi} - A(\chi)\right] = M$ $\longrightarrow$

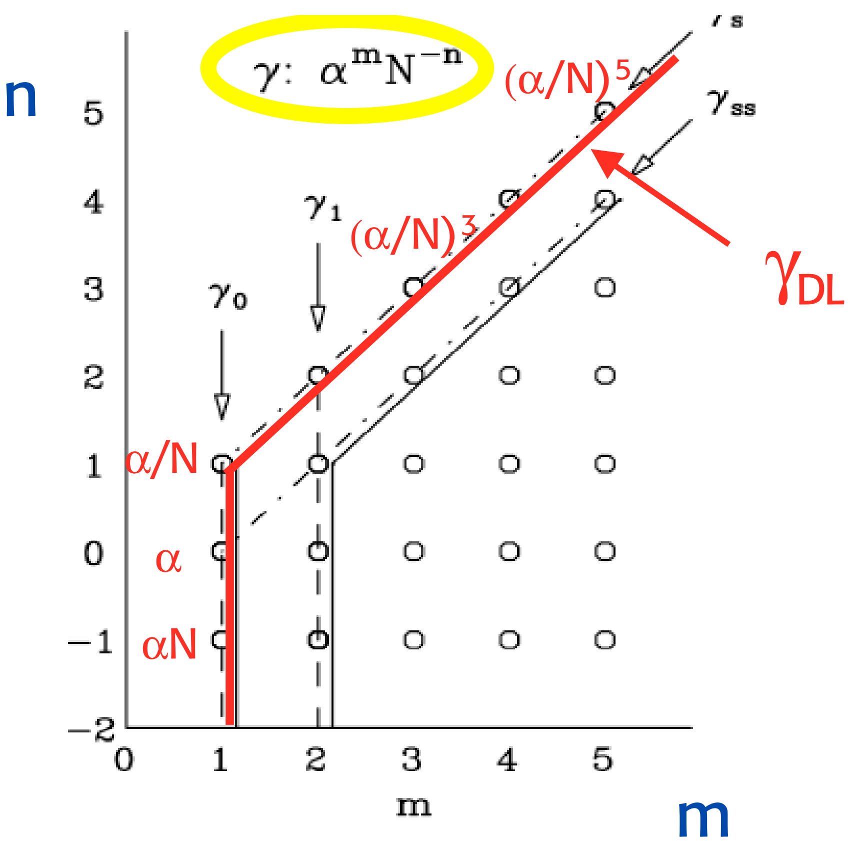
$\longrightarrow$ $\chi = \dfrac{\alpha}{M+\alpha A(\chi)}$ $\longrightarrow$ $\chi(M\sim 0) \sim \dfrac{\alpha}{M+\alpha A(1)} \sim \dfrac{\alpha}{M+\alpha}$

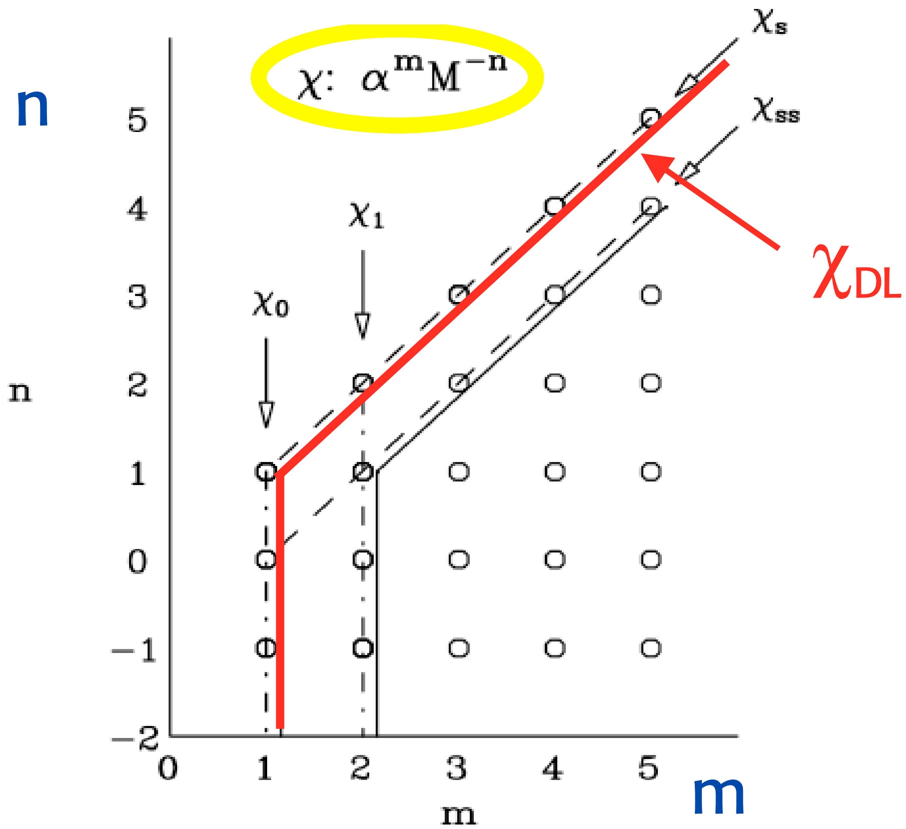
$$\chi_{DL}(M,\alpha) = \alpha\cdot\chi_0(M) + \chi_s\left(\frac{\alpha}{M}\right) + \ldots \text{ -double count.}$$

$$\chi_0(M) = \alpha\cdot\left[\frac{1}{M} + 0(M^2)\right]$$

$$\gamma_{DL}(\alpha, N) = \alpha \cdot \gamma_{1l}(N) + \gamma_s\left(\frac{\alpha}{N}\right) + \ldots \text{ -double count.}$$

$$\chi_{DL}(M, \alpha) = \alpha \cdot \chi_0(M) + \chi_s\left(\frac{\alpha}{M}\right) + \ldots \text{ -double count.}$$

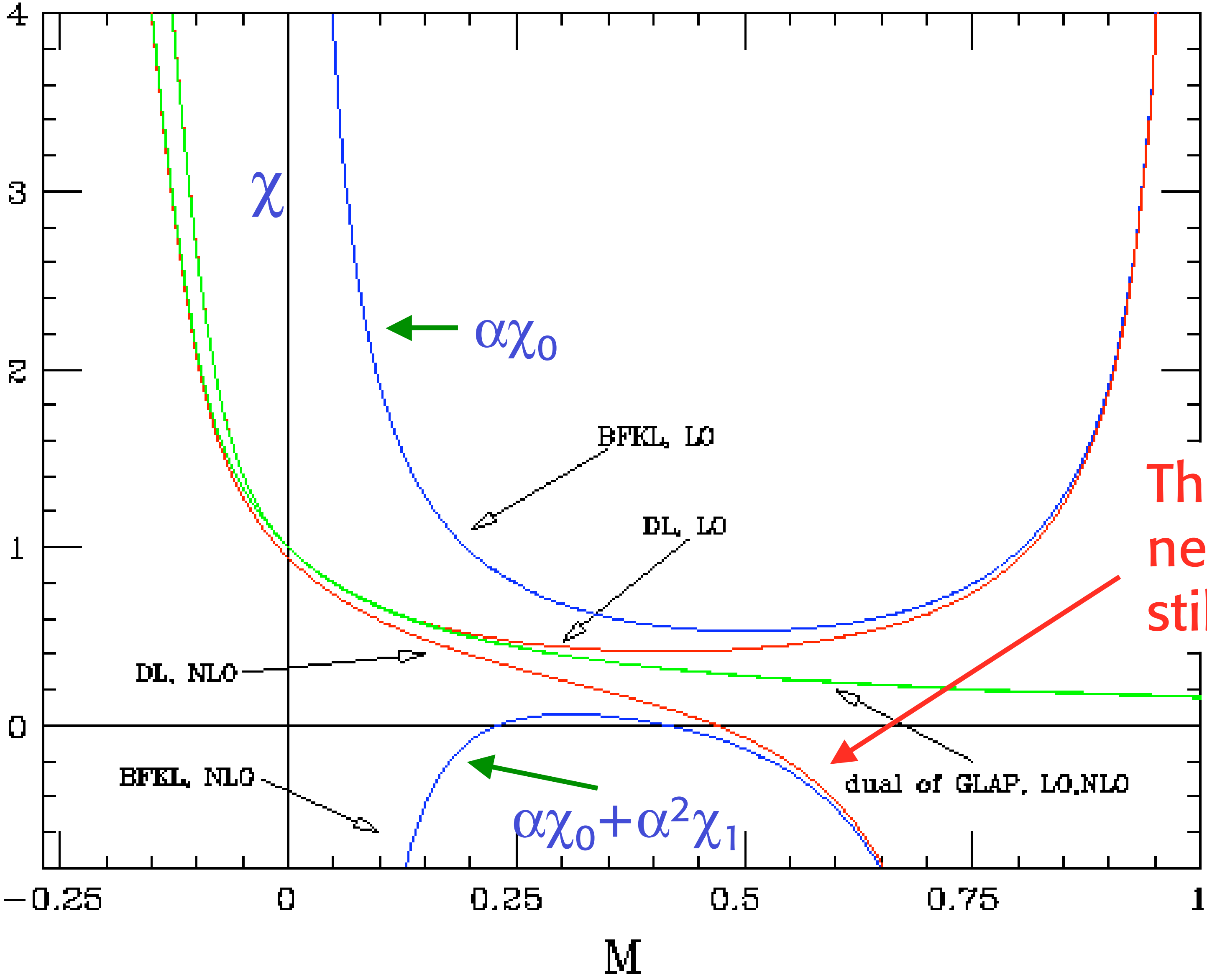




In the DL expansion one sums over "frames" rather than over vertical lines like in ordinary perturb. theory

DL, LO: $\chi_{DL}(M,\alpha) = \alpha \cdot \chi_0(M) + \chi_s\left(\frac{\alpha}{M}\right)$ +... -double count.

BFKL, LO



The NLO-DL is good near M=0, but it is still bad near M=1

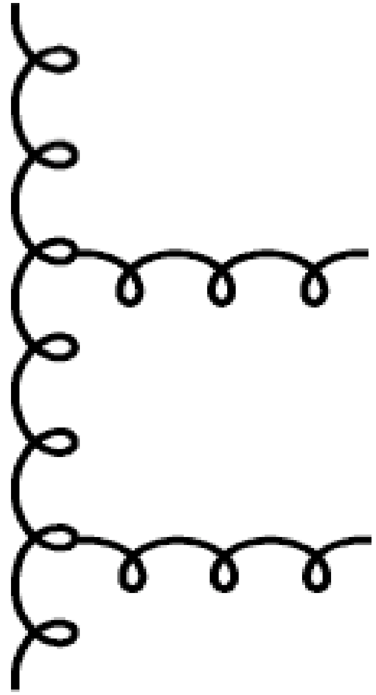
Can be fixed by symmetrization

Symmetrization

Fadin, Lipatov'98, Salam '98,
Ciafaloni, Colferai, Salam '99

The BFKL kernel is symmetric under exchange of the external gluons

$k^2 \Leftrightarrow Q^2$



$Q^2 \Leftrightarrow k^2$

This implies a symmetry under M <--> 1- M for $\chi(\alpha,M)$ broken by two effects:

- Running coupling effects ($\alpha(Q^2)$ breaks the symmetry)
- The change of scale from the BFKL symm. scale $\xi=\ln(s/Qk)$ to the DIS scale $\xi=\ln(s/Q^2)$

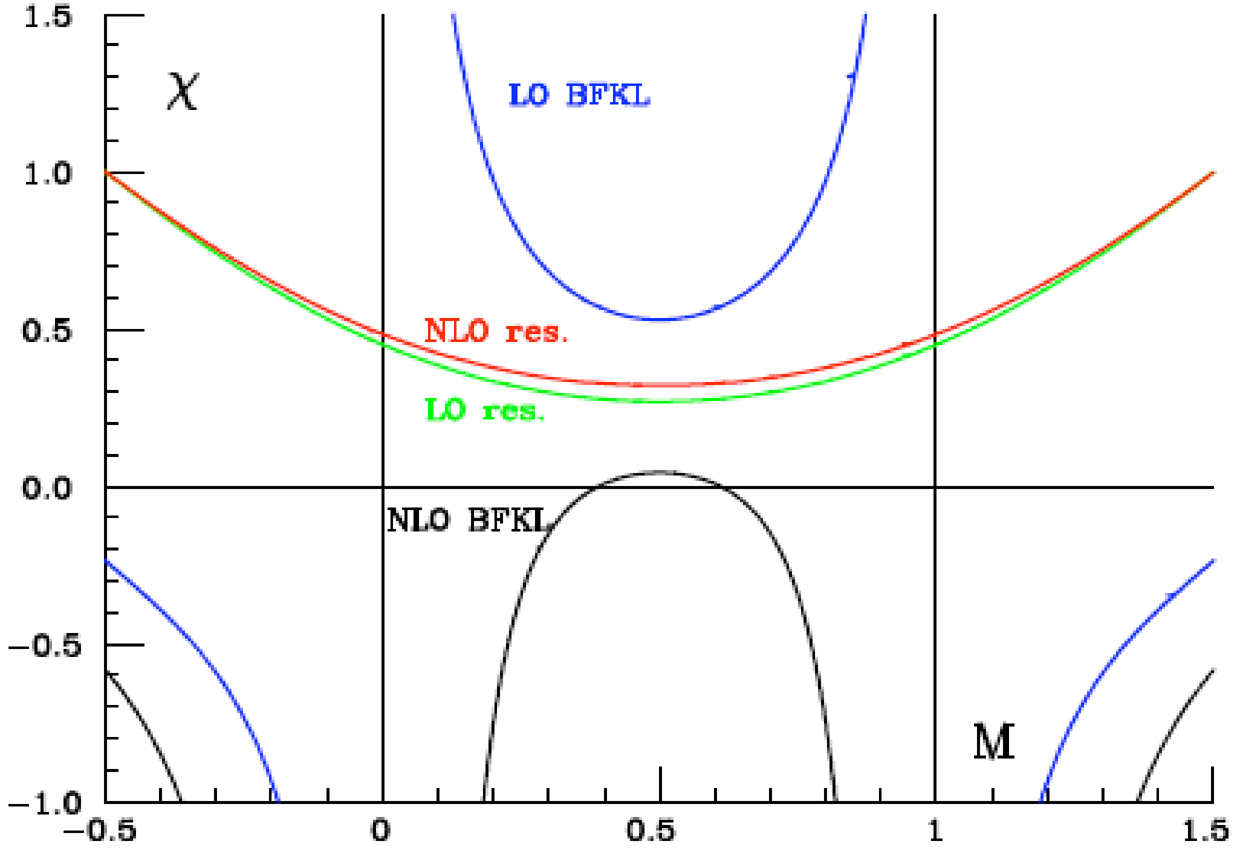
$$\chi_{DIS}\left(M+\frac{\chi_{SYMM}(M)}{2}\right) = \chi_{SYMM}(M)$$

$$\chi = \chi_{DIS}$$

Symmetrization makes χ regular at M=0 AND M=1

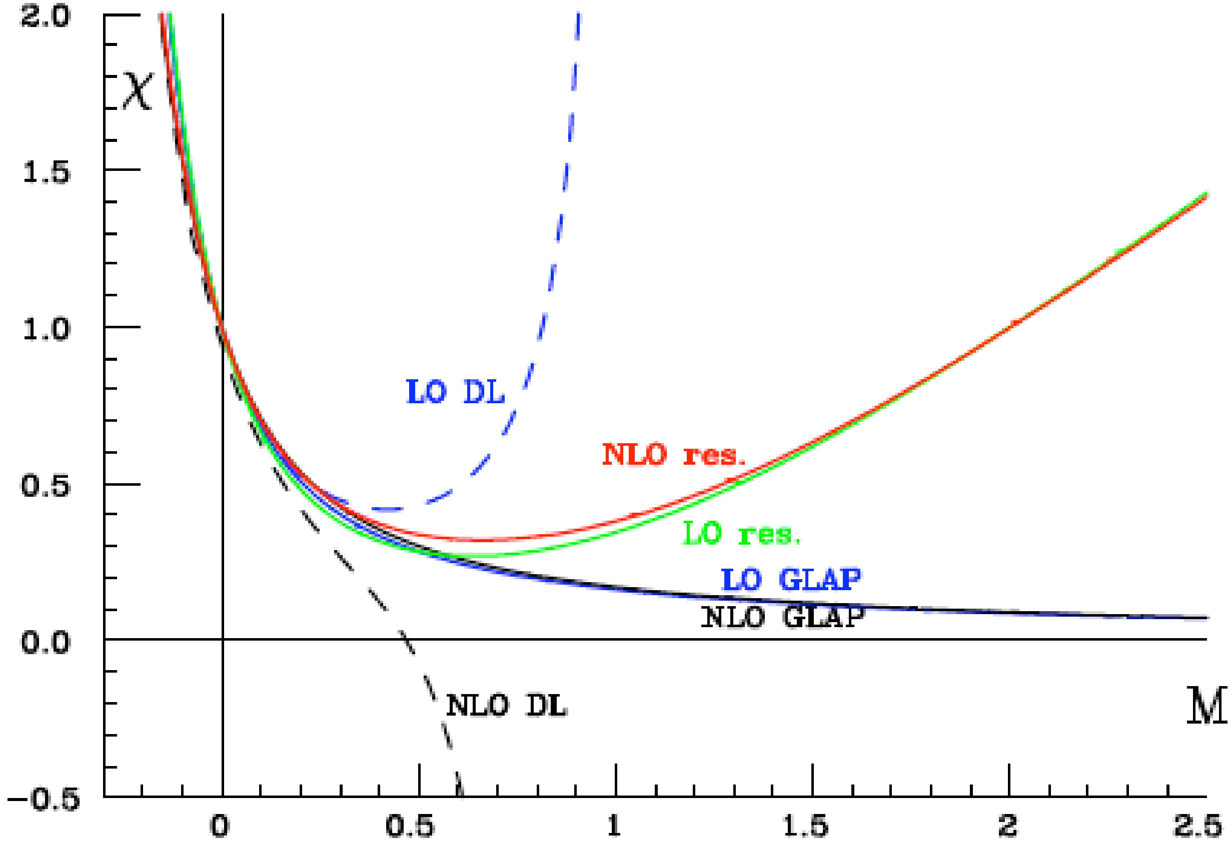
In symmetric variables:

fixed coupling: α=0.2



Note how the symmetrized LO DL and NLO DL are very close!

The same now in DIS variables



All χ curves have a minimum and follow γ closer.
The remaining ingredient is the running of the coupling.

A considerable further improvement is obtained by including running coupling effects

Recall that the x-evolution equation was at fixed α

$$\frac{d}{d\xi}G(\xi, M) = \chi(M, \alpha)G(\xi, M)$$

The implementation of running coupling in BFKL is not simple. In fact in M-space α becomes an operator

$$\alpha(t) = \frac{\alpha}{1+\beta_0\alpha t} \Rightarrow \frac{\alpha}{1-\beta_0\alpha\frac{d}{dM}}$$

In leading approximation:

$$\frac{d}{d\xi}G(\xi, M) = \chi(M, \alpha)G(\xi, M)$$

$$\frac{d}{d\xi}G(\xi, M) = \frac{\alpha}{1-\beta_0\alpha\frac{d}{dM}}\chi_0(M)G(\xi, M)$$

The small x behaviour is controlled by the minimum of $\chi(M)$

We make a quadratic expansion of $\chi(M)$ near the minimum.

$$\chi_q(\hat{\alpha}_s, M) = [c(\hat{\alpha}_s) + \tfrac{1}{2}\kappa(\hat{\alpha}_s)(M - M_s)^2]$$

diff. operator

We can solve the equation exactly:

For c, k proportional to α : the solution is an Airy function

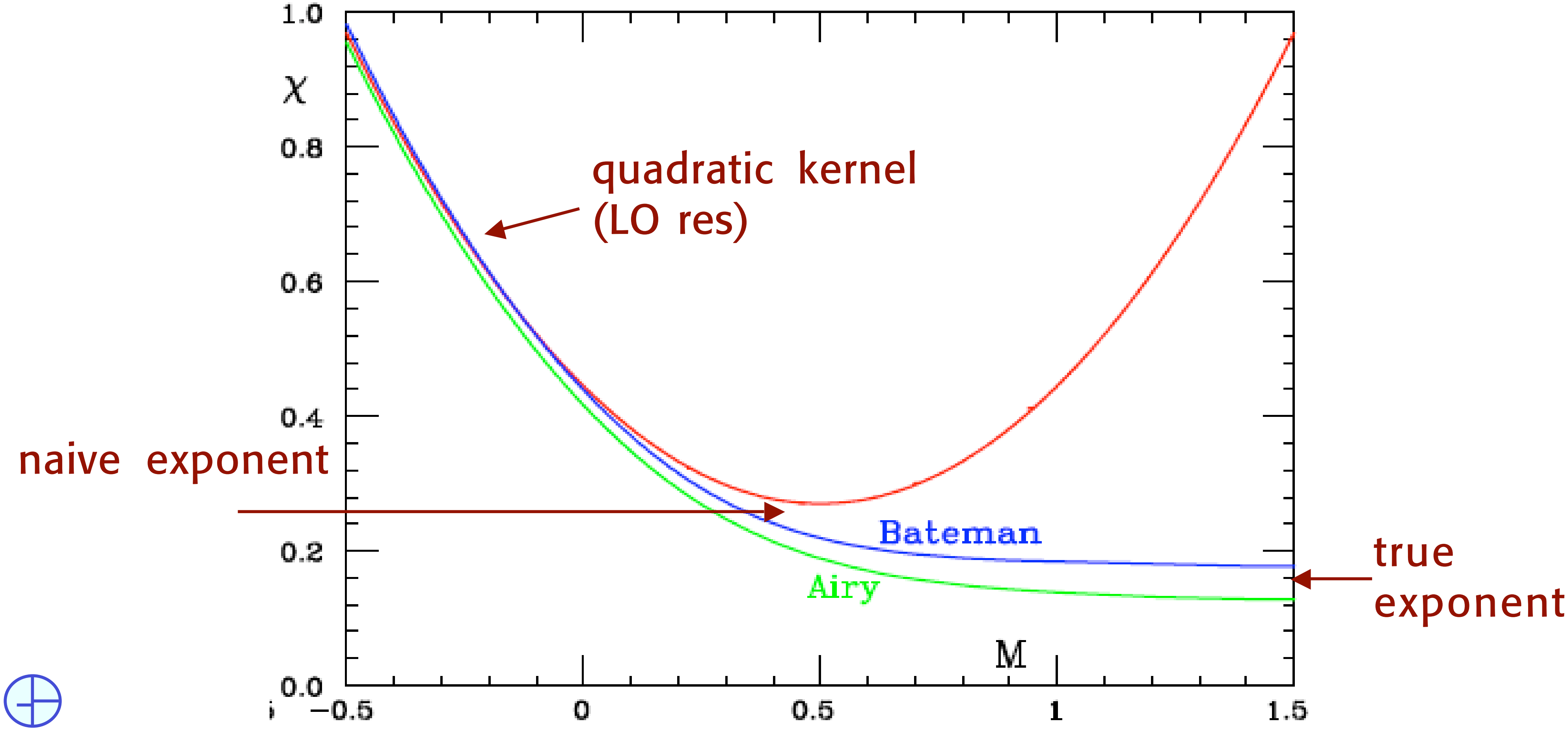
For example, if we take $\chi(\alpha,M) \sim \alpha\, \chi_0(M)$

For general $c(\alpha), \kappa(\alpha)$, to the required accuracy, it is sufficient to make a linear expansion in $\hat{\alpha} - \alpha$

the solution is a Bateman function.

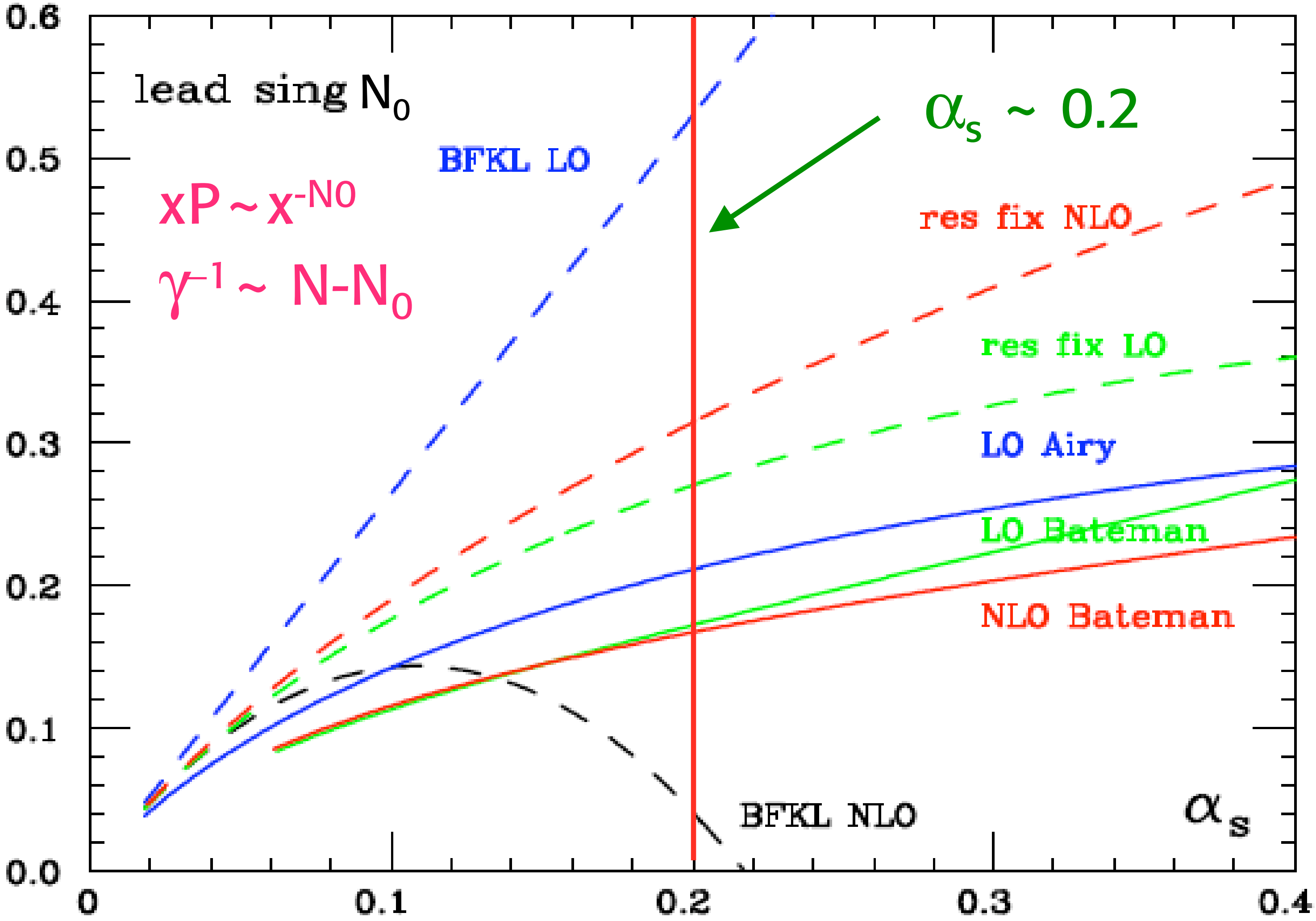
The asymptotic small x behaviour is considerably softened by the running!

Note that the running effect is not replacing $\alpha \to \alpha(Q^2)$ in the naive exponent



DL resummation with symmetrization and running coupling effects progressively soften the small x behaviour

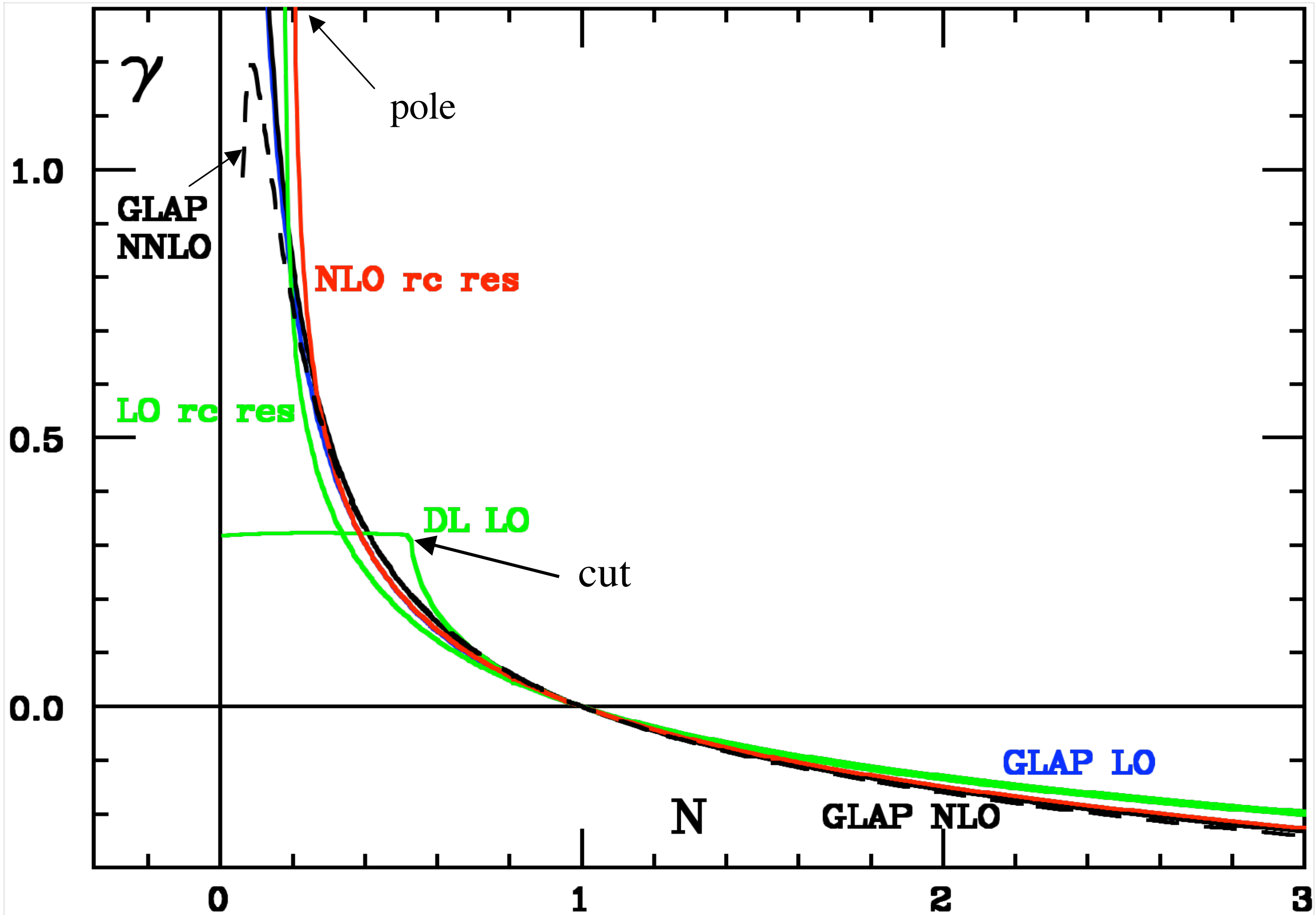
In Airy/Bateman the Lipatov cut is replaced by a pole



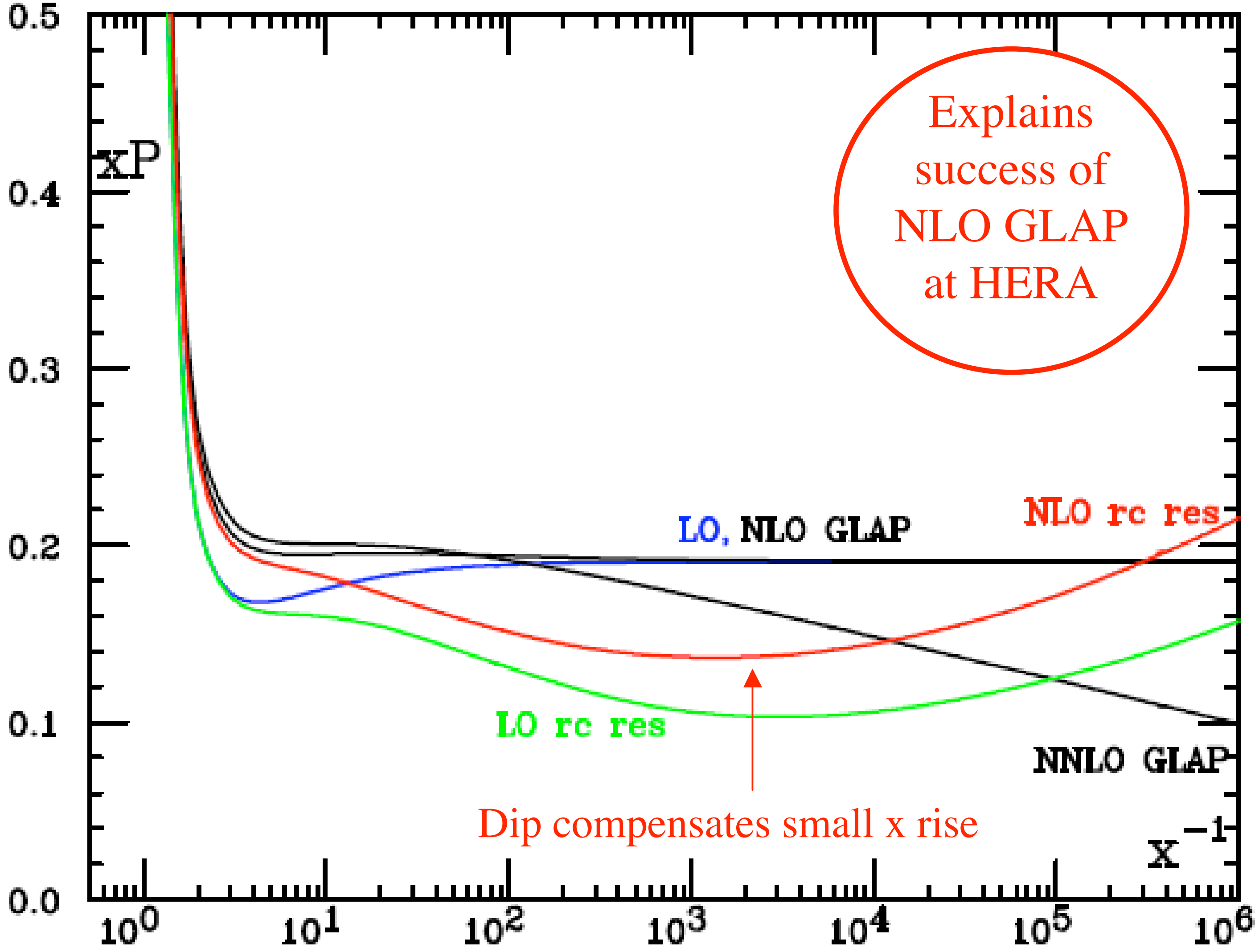
The Resummed Anomalous Dimension: γ^{+}

$\alpha_s = 0.2$

$n_f = 0$



Here are the complete results using the DL resummation, symmetry and running coupling effects at LO and NLO



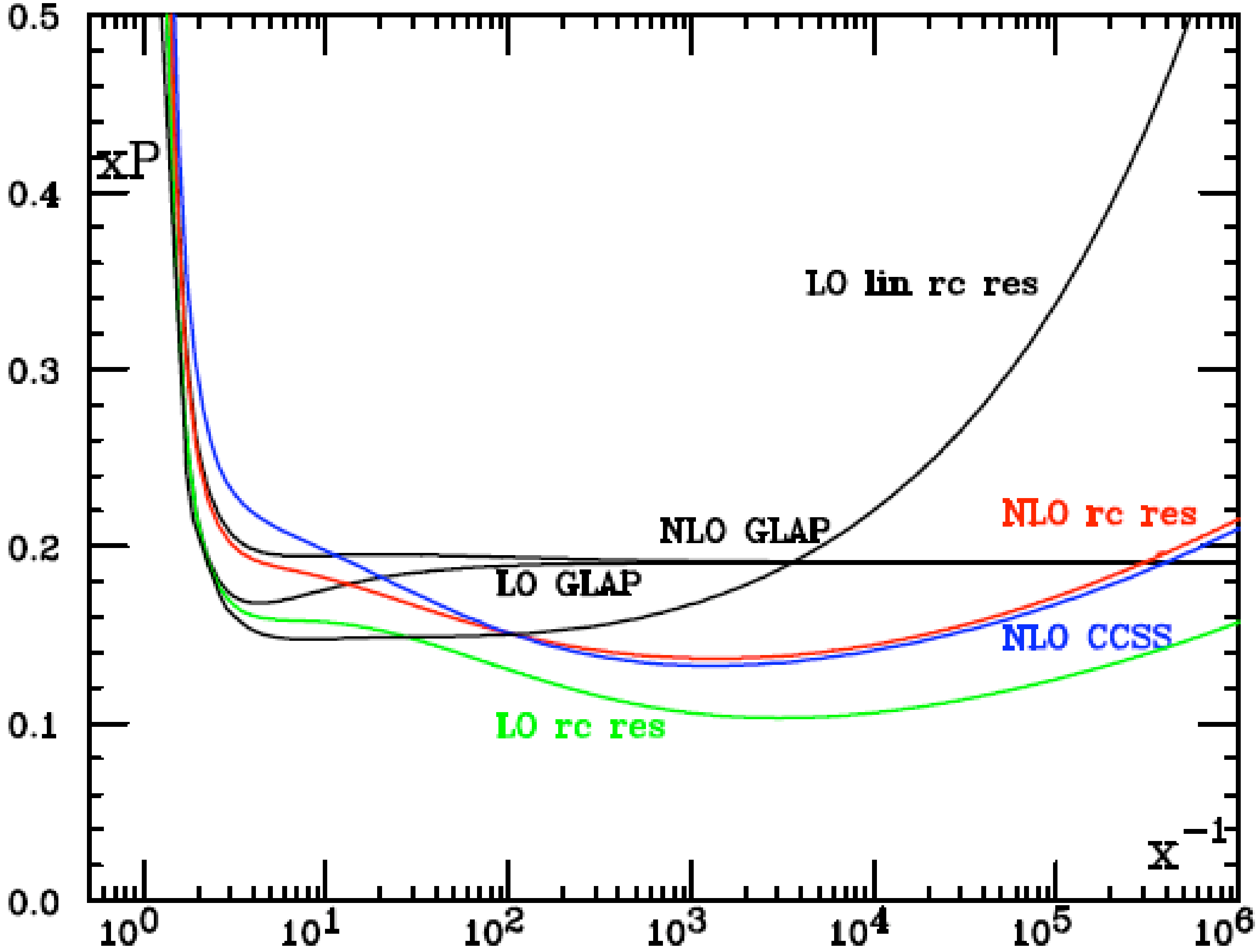
$\alpha_s = 0.2$

$n_f = 0$

The Resummed Splitting Function: P^+

$\alpha_s = 0.2$

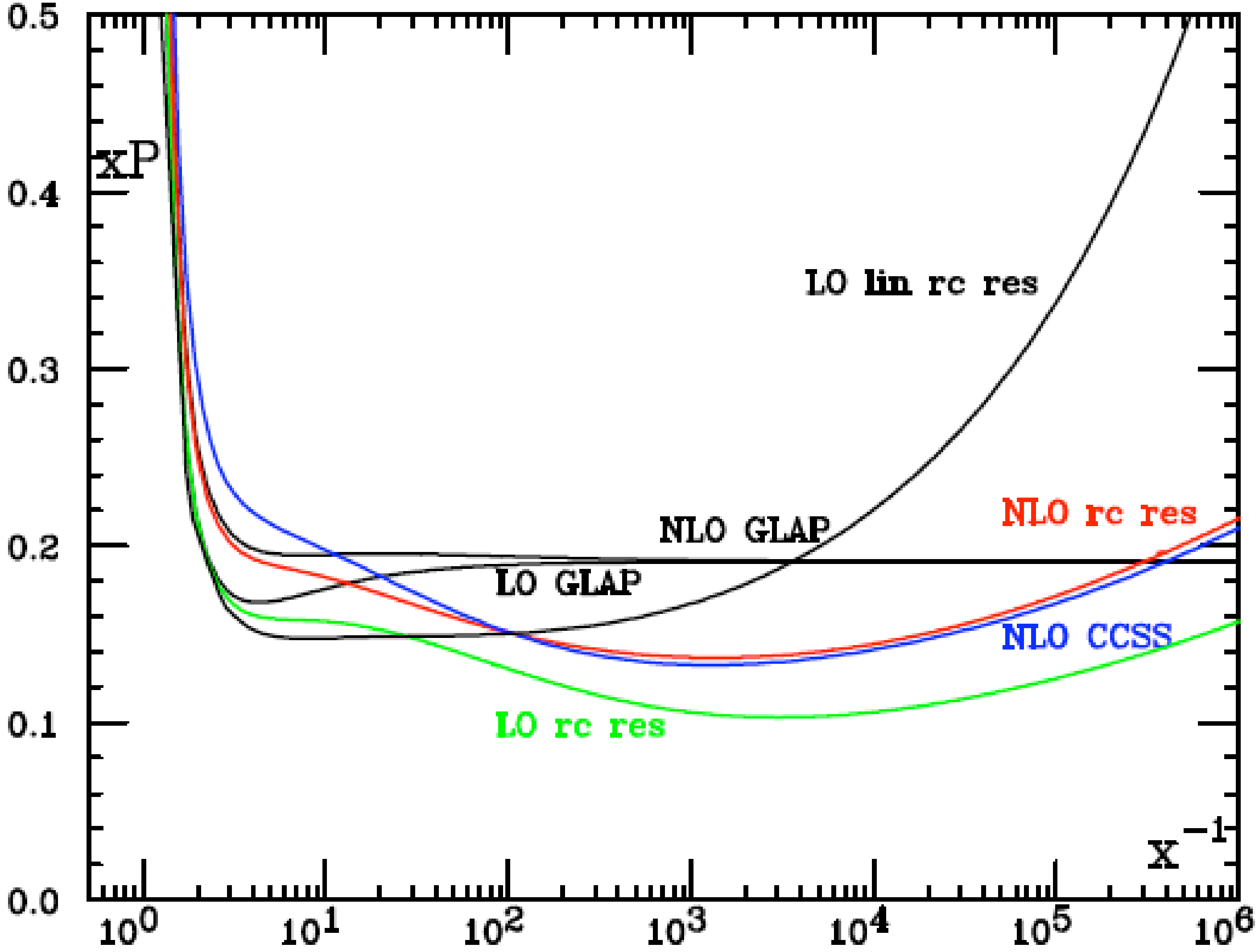
$n_f = 0$



Comparison with Ciafaloni et al

At small x the agreement is too good: largely accidental in view of theoretical ambiguities

The main diff. with CCSS is that they solve the running coupling eqn. numerically (no quadratic expansion near minimum). They do not include NLO GLAP. The matching with PT is not perfect.



Part 2

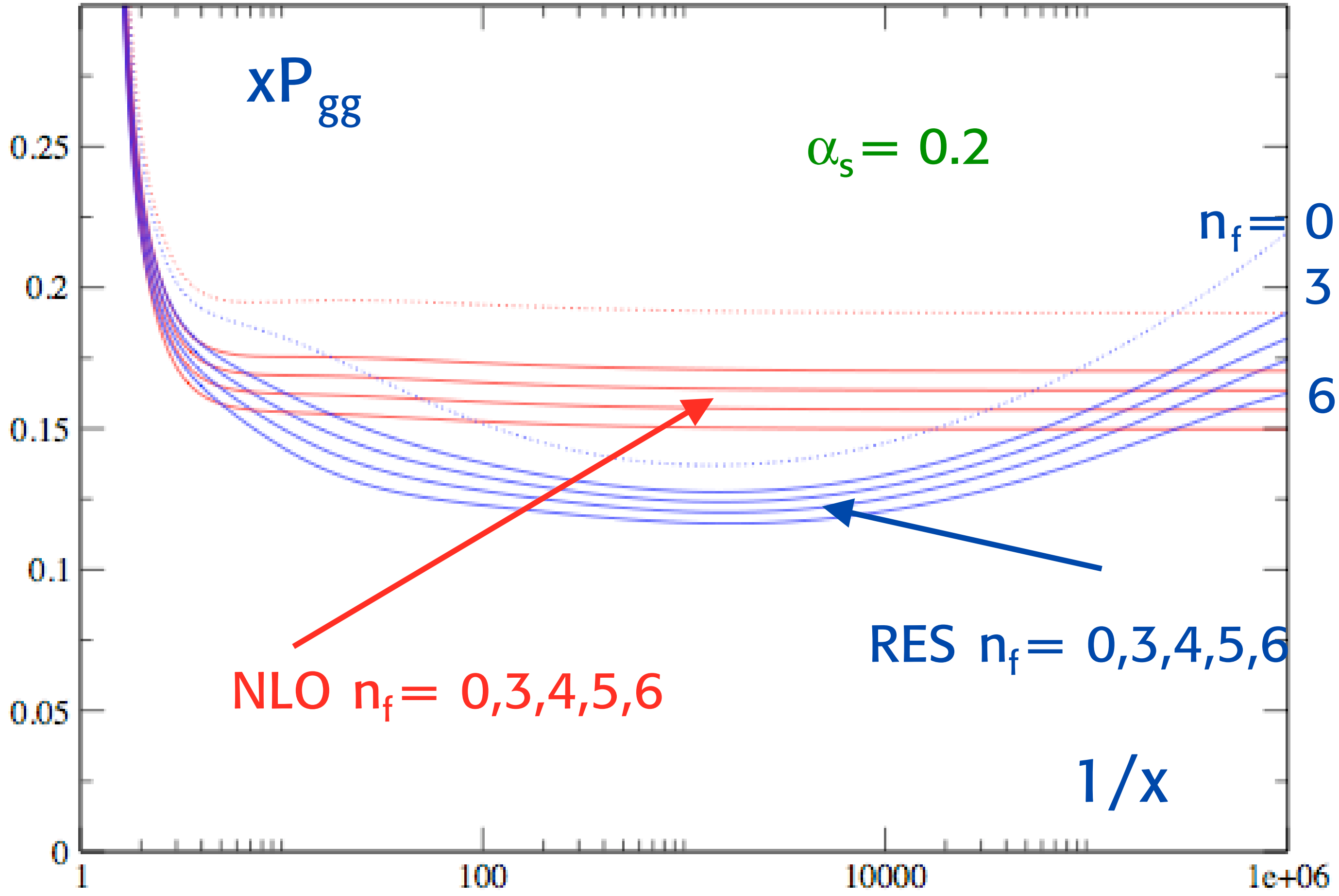
New results

NPB <u>799</u>,199,2008 and 0802.0968 [hep-ph]

The previous curves are for $n_f = 0$

At finite n_f the diagonalization of xP is more complicated

The n_f dependence is not negligible

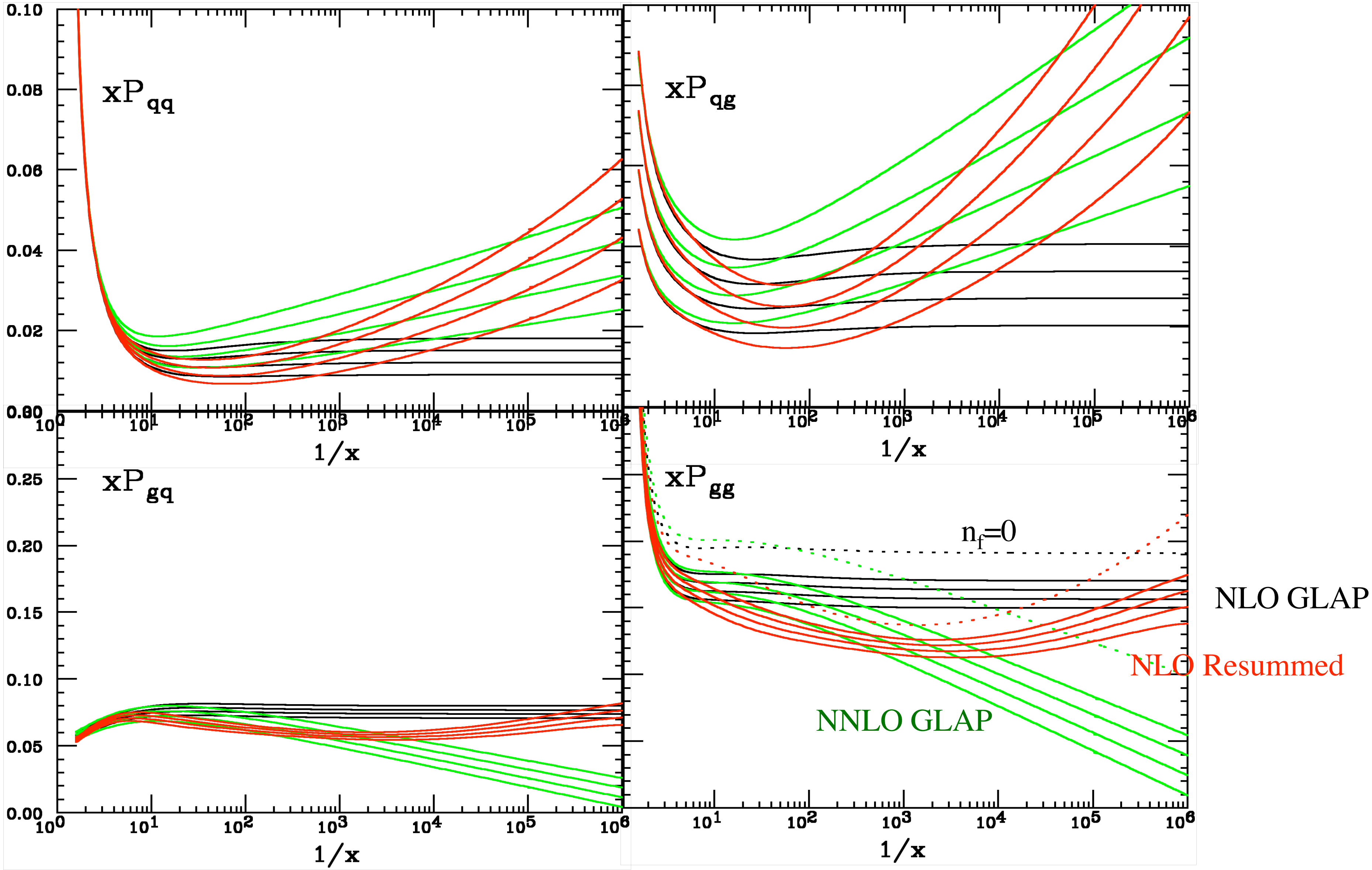


Prel. presented at the HERA-LHC Workshop, DESY, March '07

Q_0MS^{bar}

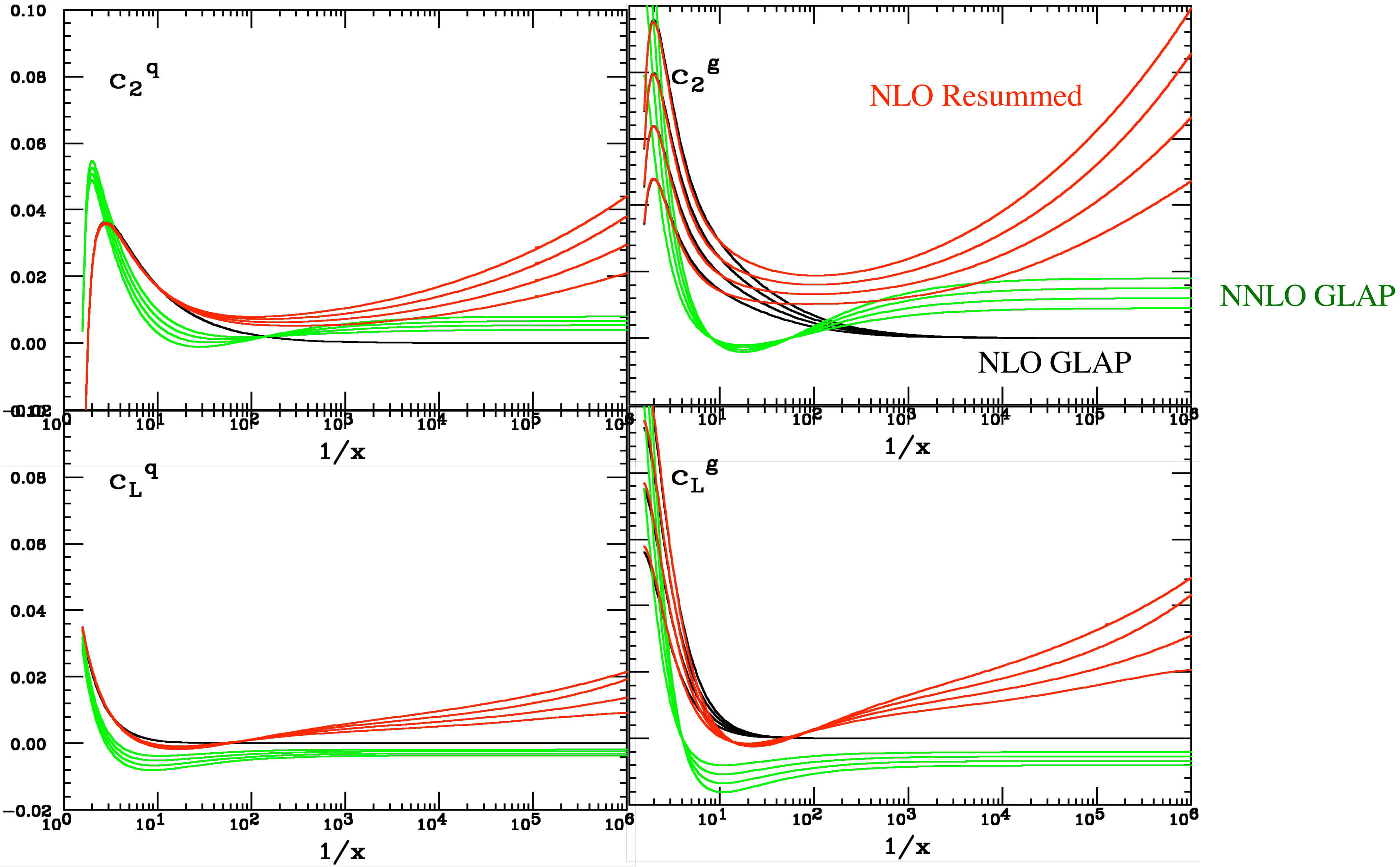
Splitting Functions: $n_f = 3, 4, 5, 6$

$\alpha_s = 0.2$



Q_0MS^{bar} Coefficient Functions: $n_f = 3, 4, 5, 6$

$\alpha_s = 0.2$



The xP_{ab} shown so far are in the $Q_0 MS^{bar}$ scheme because in MS^{bar} there is a singularity in M=1/2 both in the splitting function and the coefficient which in Q_0 MS^{bar} is absorbed in the pdf's (in pert. theory Q_0 and MS^{bar} coincide up to and including NNLO but differ at higher orders)

Catani, Hautmann '93; Ciafaloni '95....;
Ball, Forte '99.......;
Ciafaloni, Colferai, Salam, Stasto '06

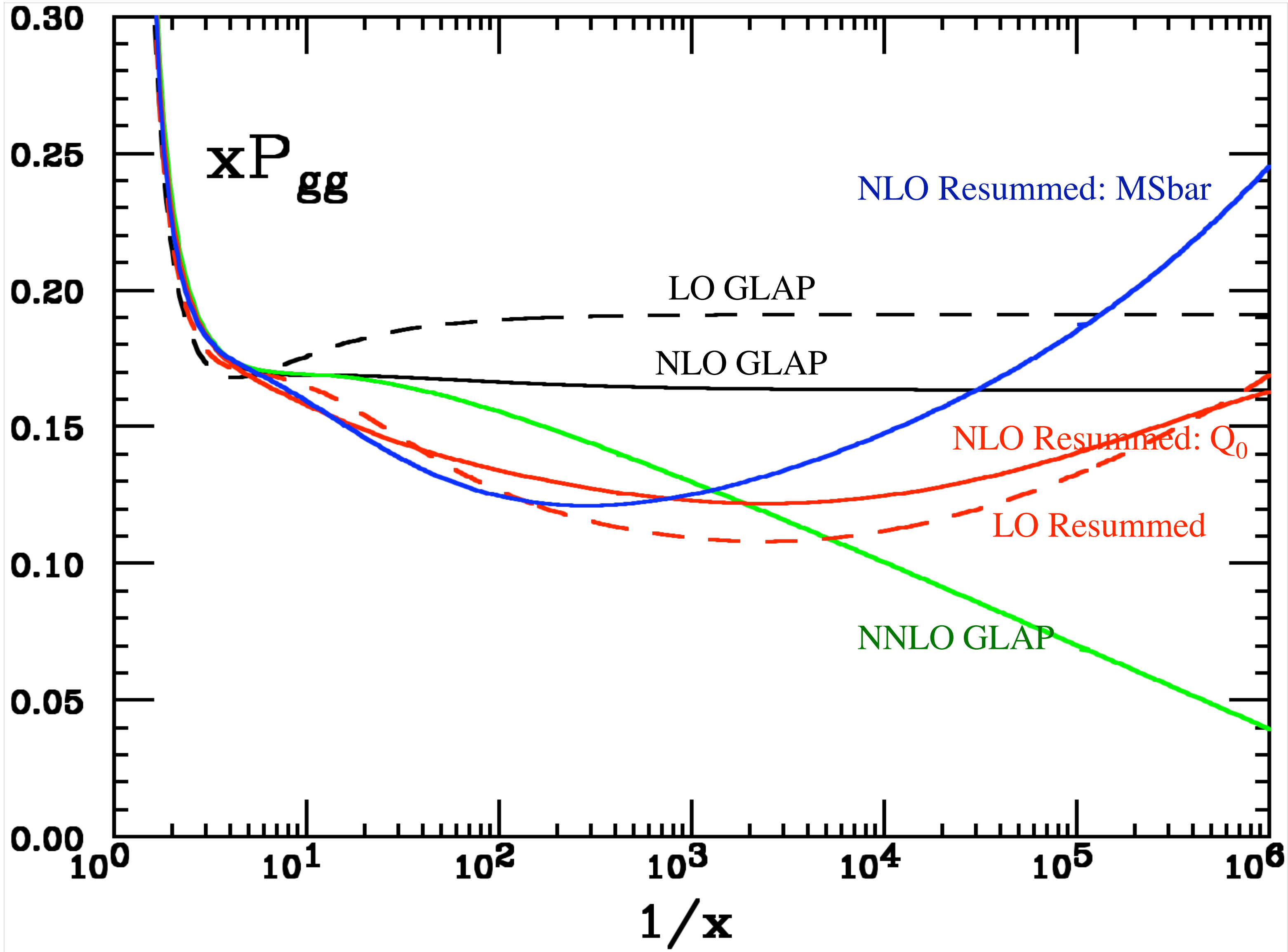
An important progress we have accomplished is the calculation of coefficients and splitting functions (by using running coupling duality) in both $Q_0 MS^{bar}$ and MS^{bar} schemes (complete control of scheme change: could also have DIS or....)

Combining splitting functions and coefficients in the same scheme is needed to obtain the evolution of pdf's and structure functions

The Resummed Splitting Function: P_{gg}

$\alpha_s = 0.2$

$n_f = 4$



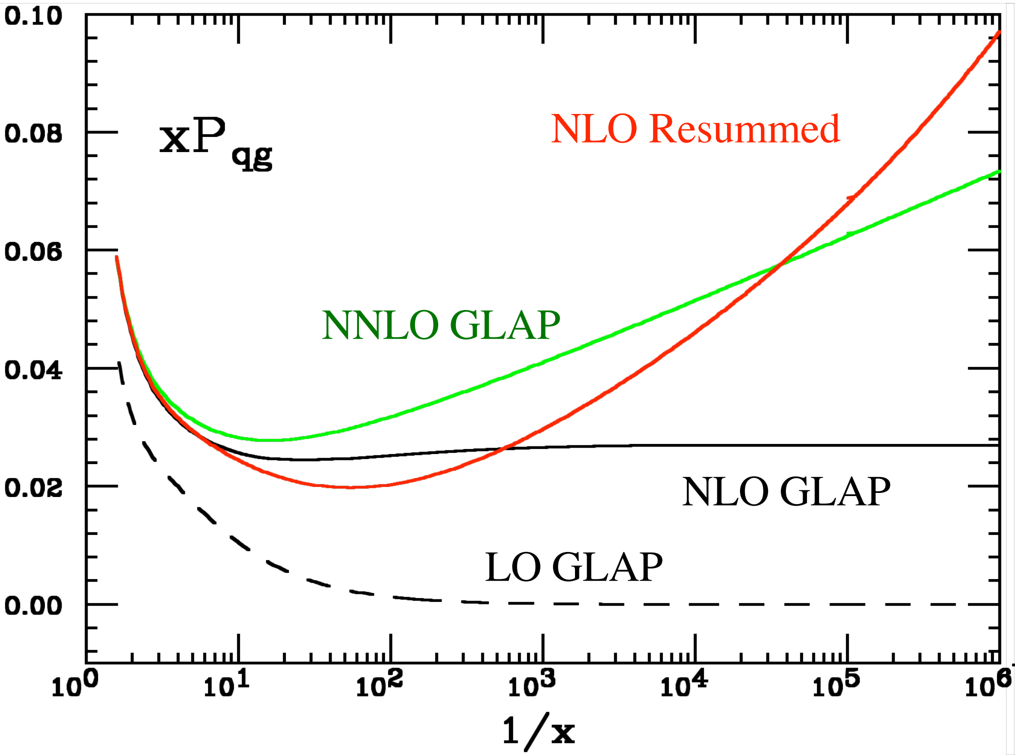
The Resumed Quark Sector

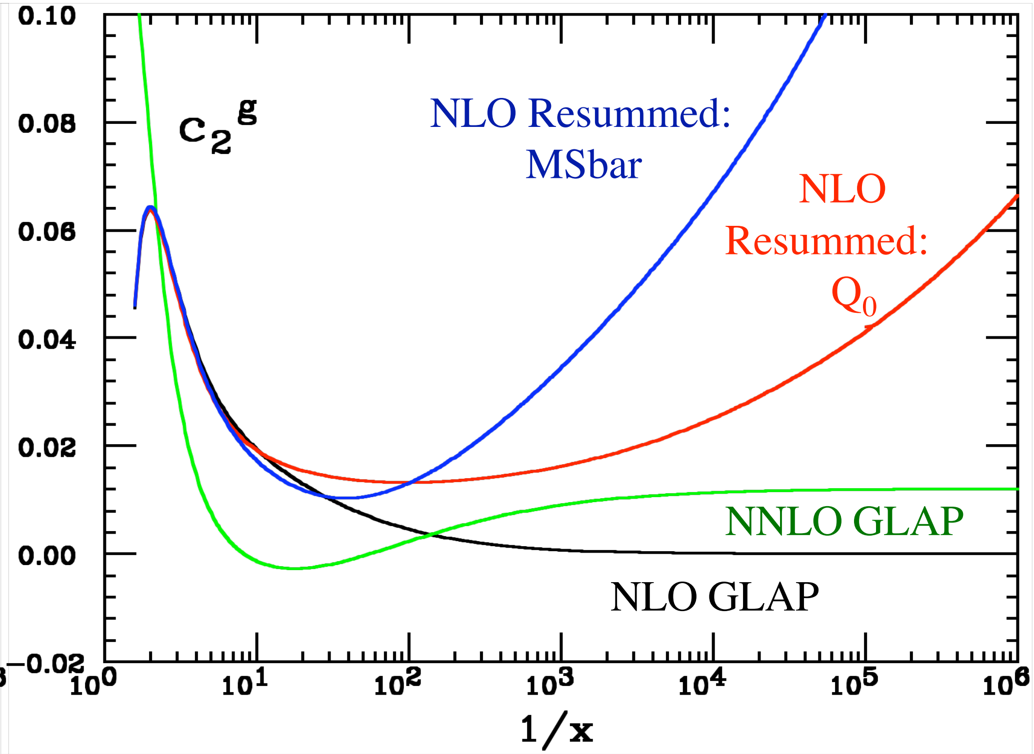
$\alpha_s = 0.2$

$n_f = 4$

Splitting Function P_{qg}

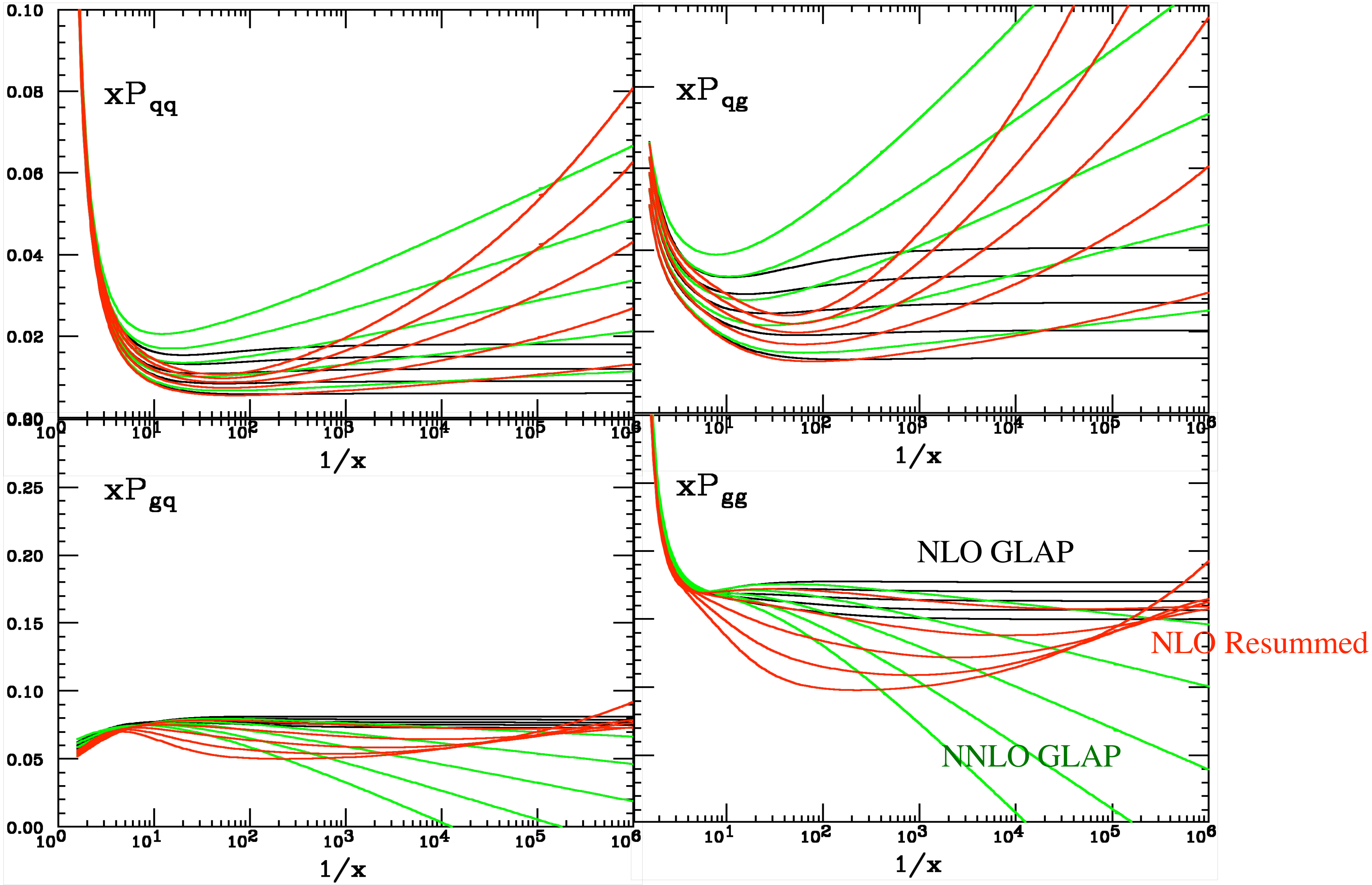
Coefficient Function $c_2{}^g$





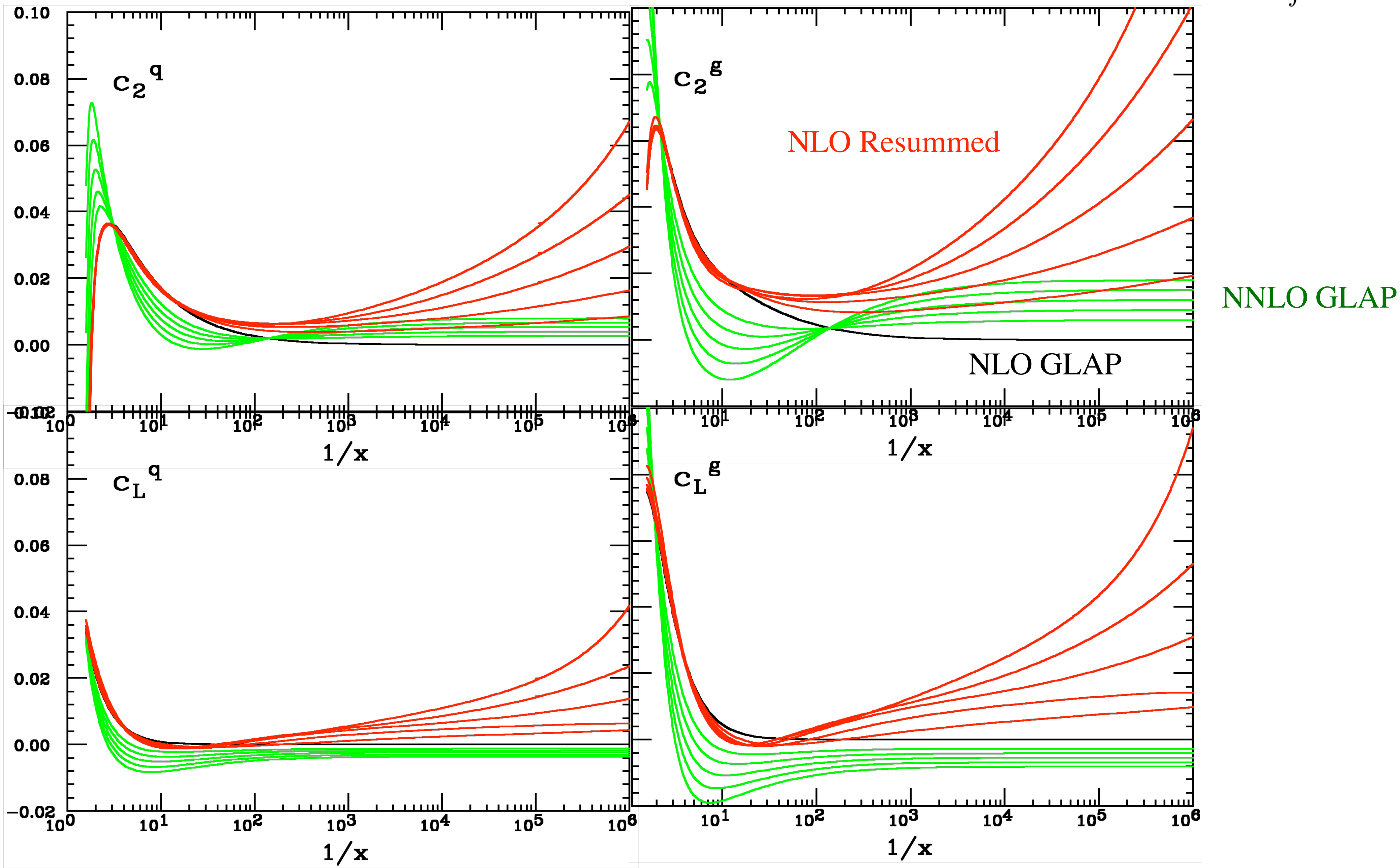
$Q_0\overline{MS}$ Splitting Functions: $\alpha_s = 0.1, 0.15, 0.2, 0.25, 0.3$

$n_f = 4$



$Q_0\overline{MS}$ Coefficient Functions: $\alpha_s = 0.1, 0.15, 0.2, 0.25, 0.3$

$n_f = 4$

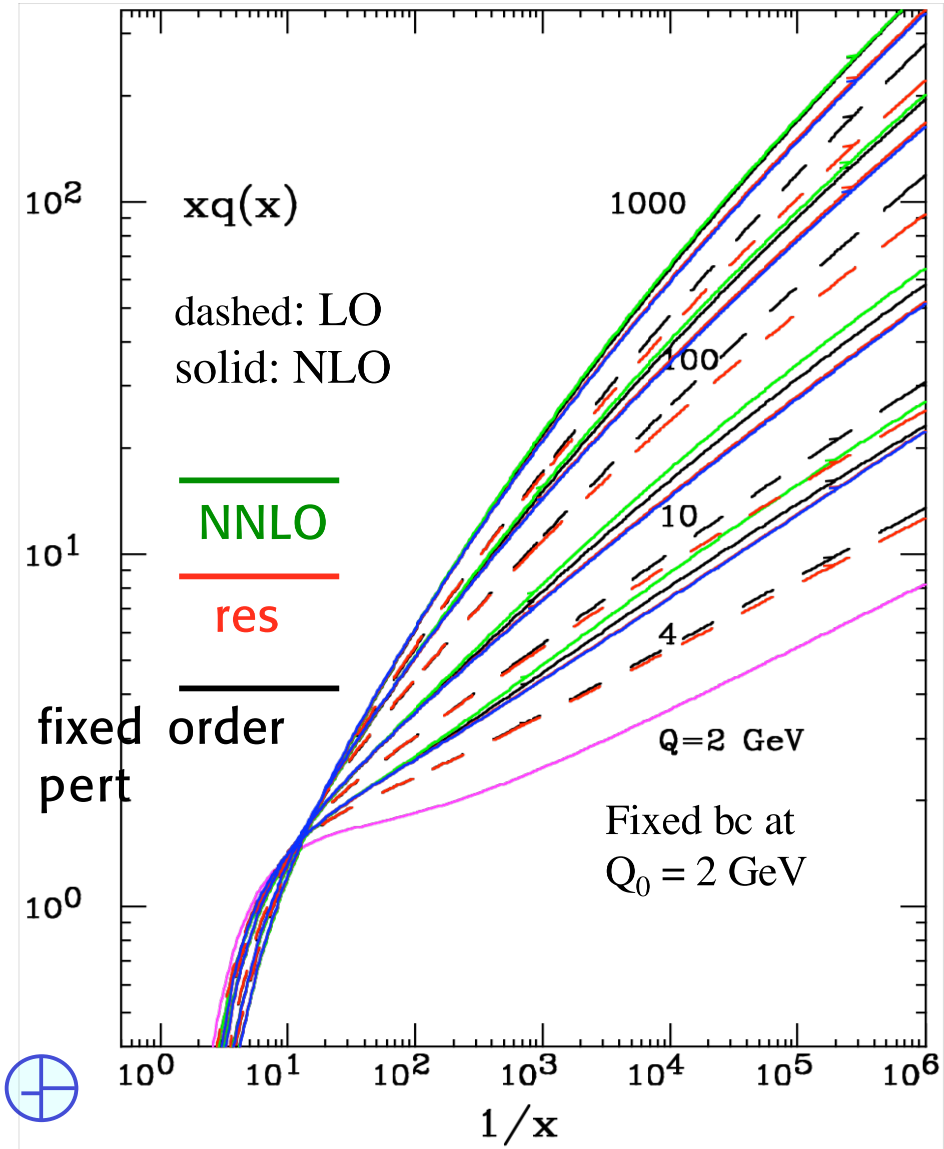


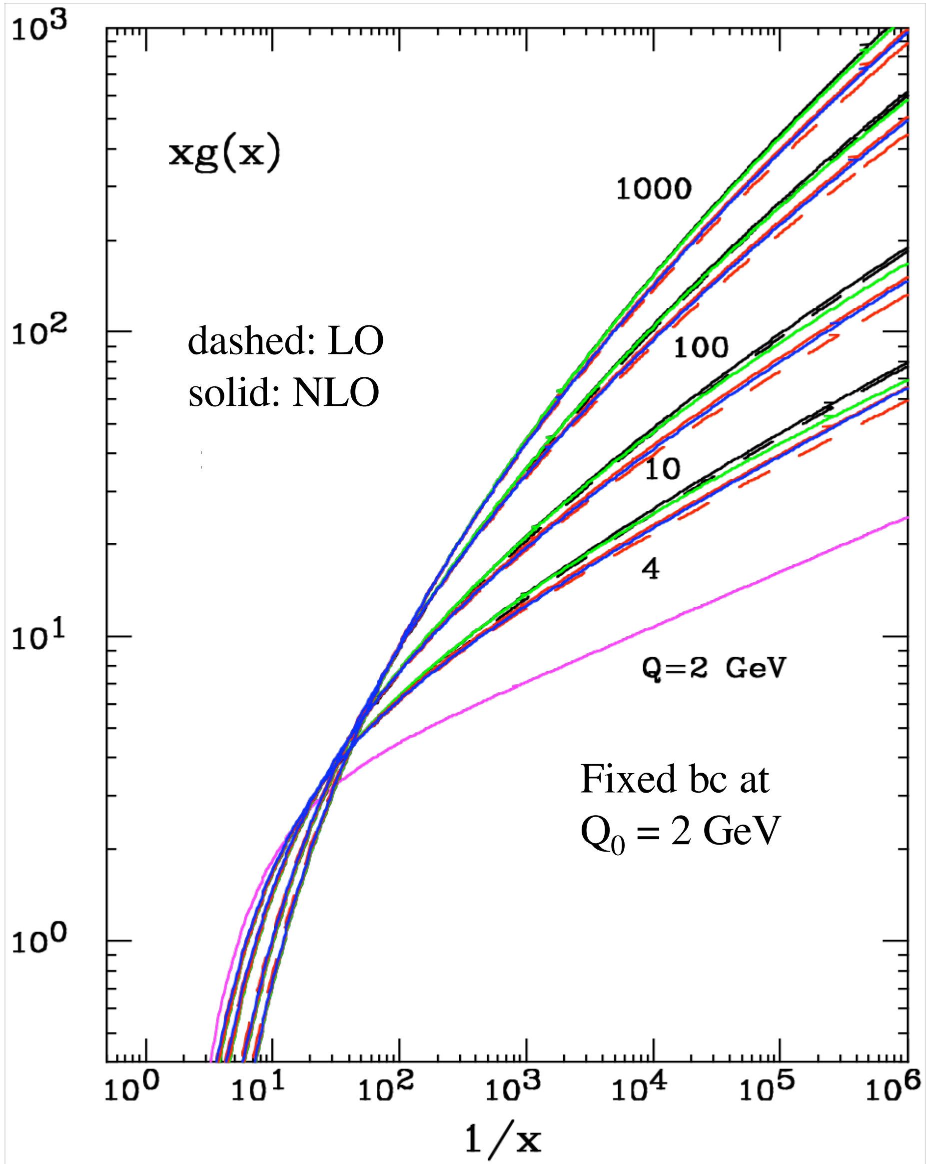
We are finally ready to applications to pdf's and structure functions

At the starting point Q=2 GeV we start with some model for valence, sea and gluon pdf's. Then, going from perturbative to resummed formulae, the pdf's are readjusted such that the initial structure functions (the physical objects!) are the same and then compare their evolution with or without resummation

Evolved Quarks and Gluons

Note: the resummed gluon at not too small x is less enhanced

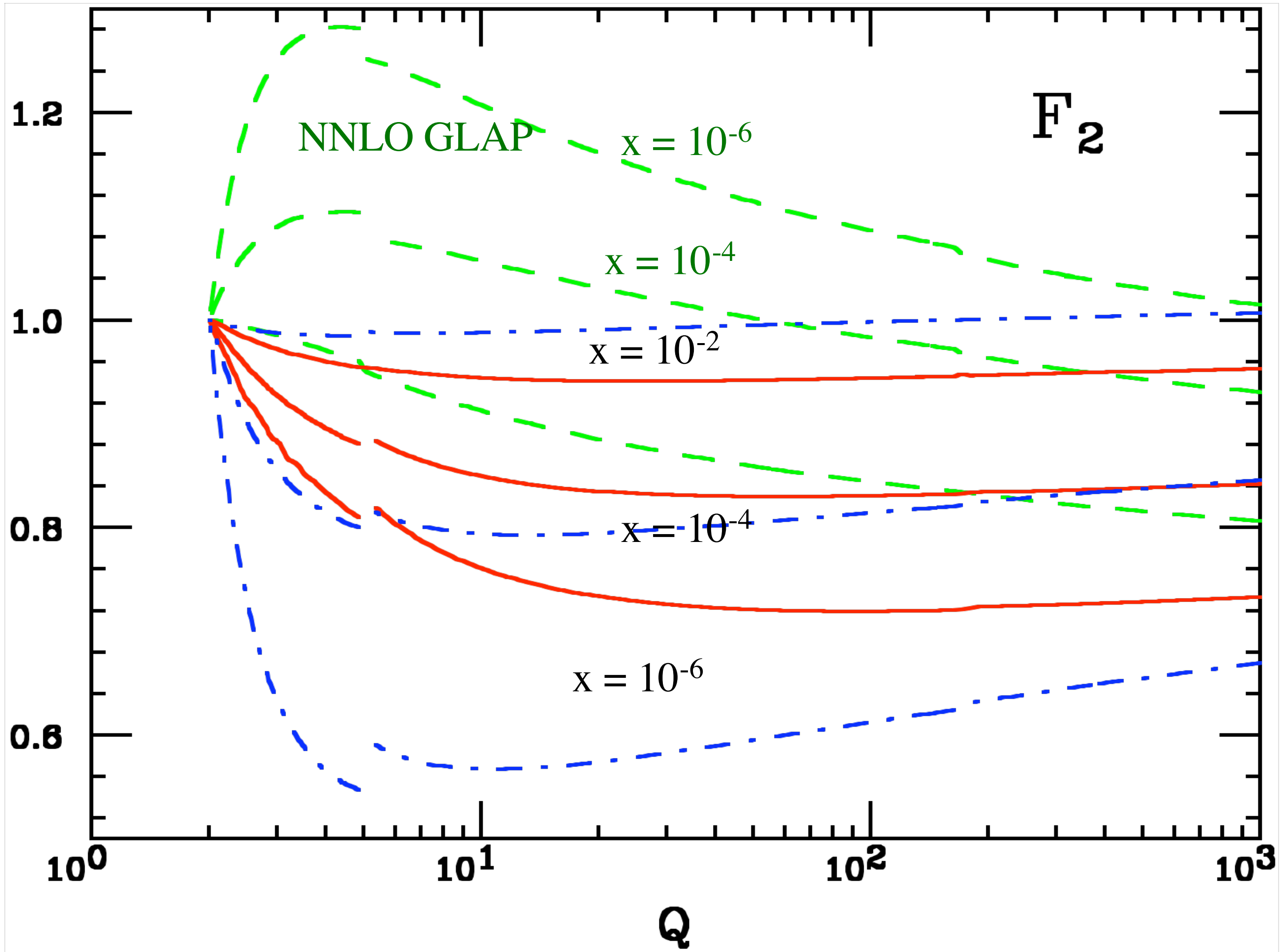




Structure Function F_2

Initial pdfs at Q_0 = 2GeV adjusted so that $F_2^{Res} = F_2^{NLO}$ etc.

$$K \equiv \frac{F_2^{Res}}{F_2^{NLO}}$$

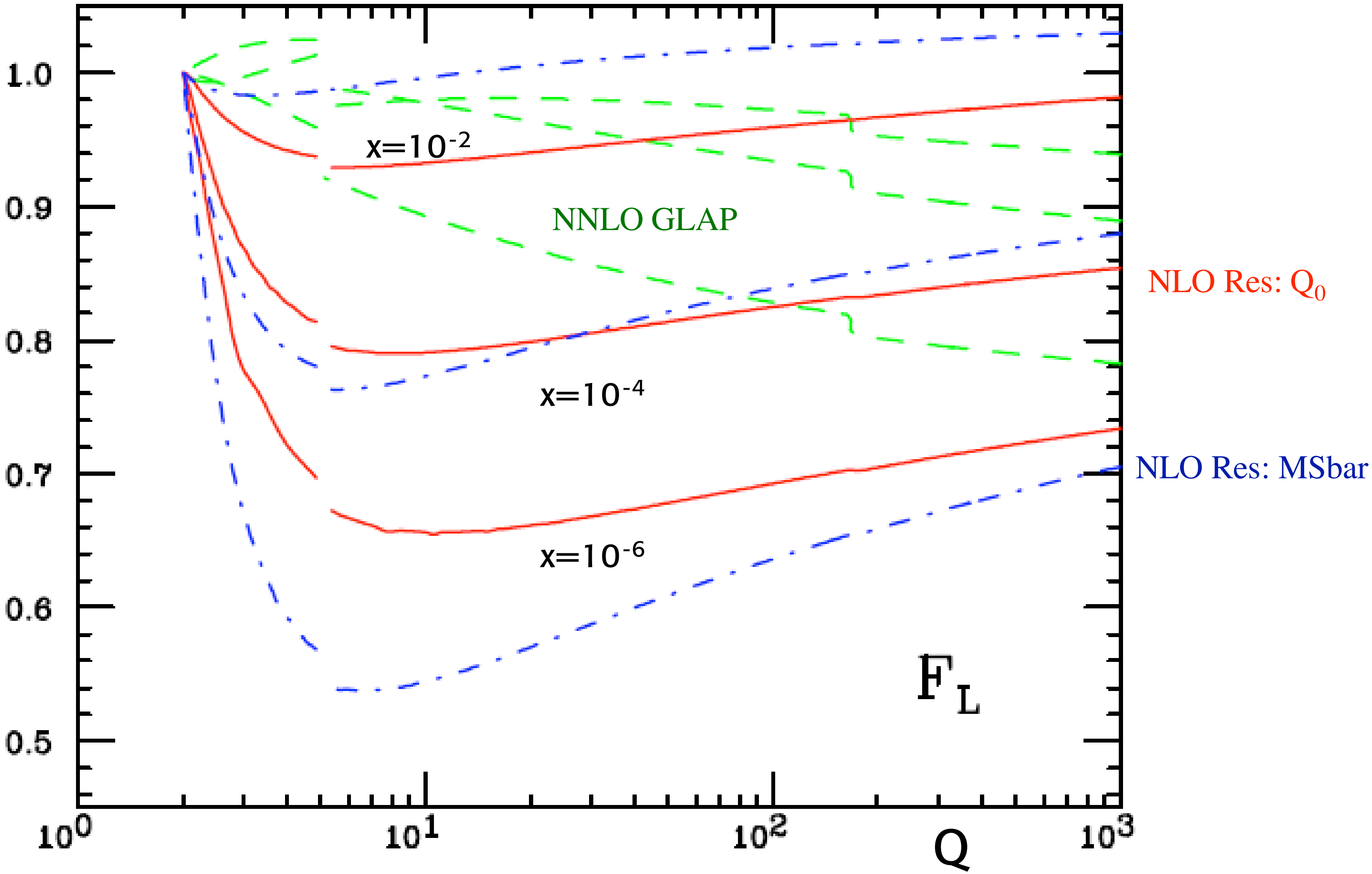


NLO Res: Q_0

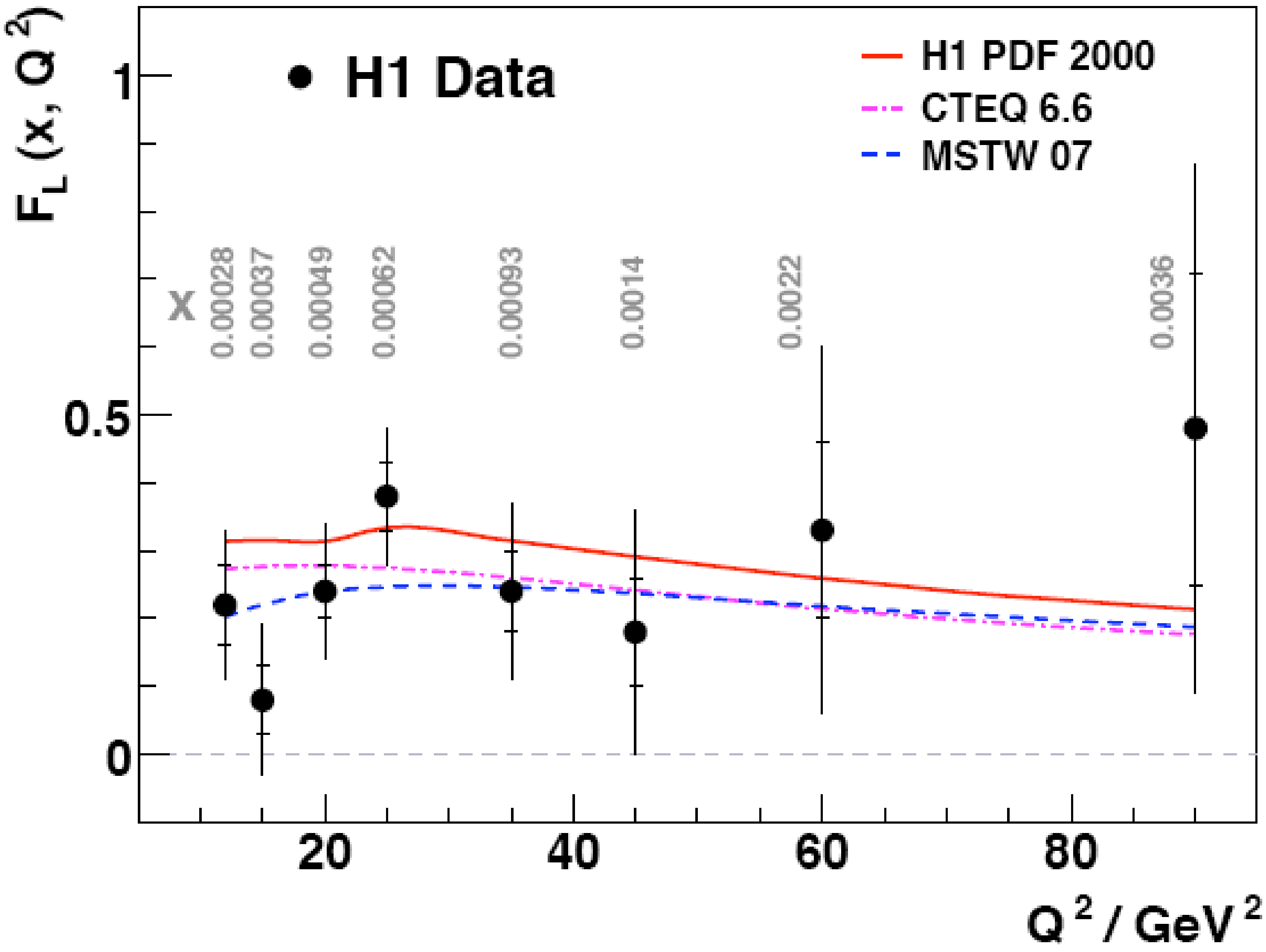
NLO Res: MSbar

Effect of resummation opposite to NNLO

The longitudinal structure function F_L



Well compatible with the new H1 data on FL
(0805.2809[hep-ex])

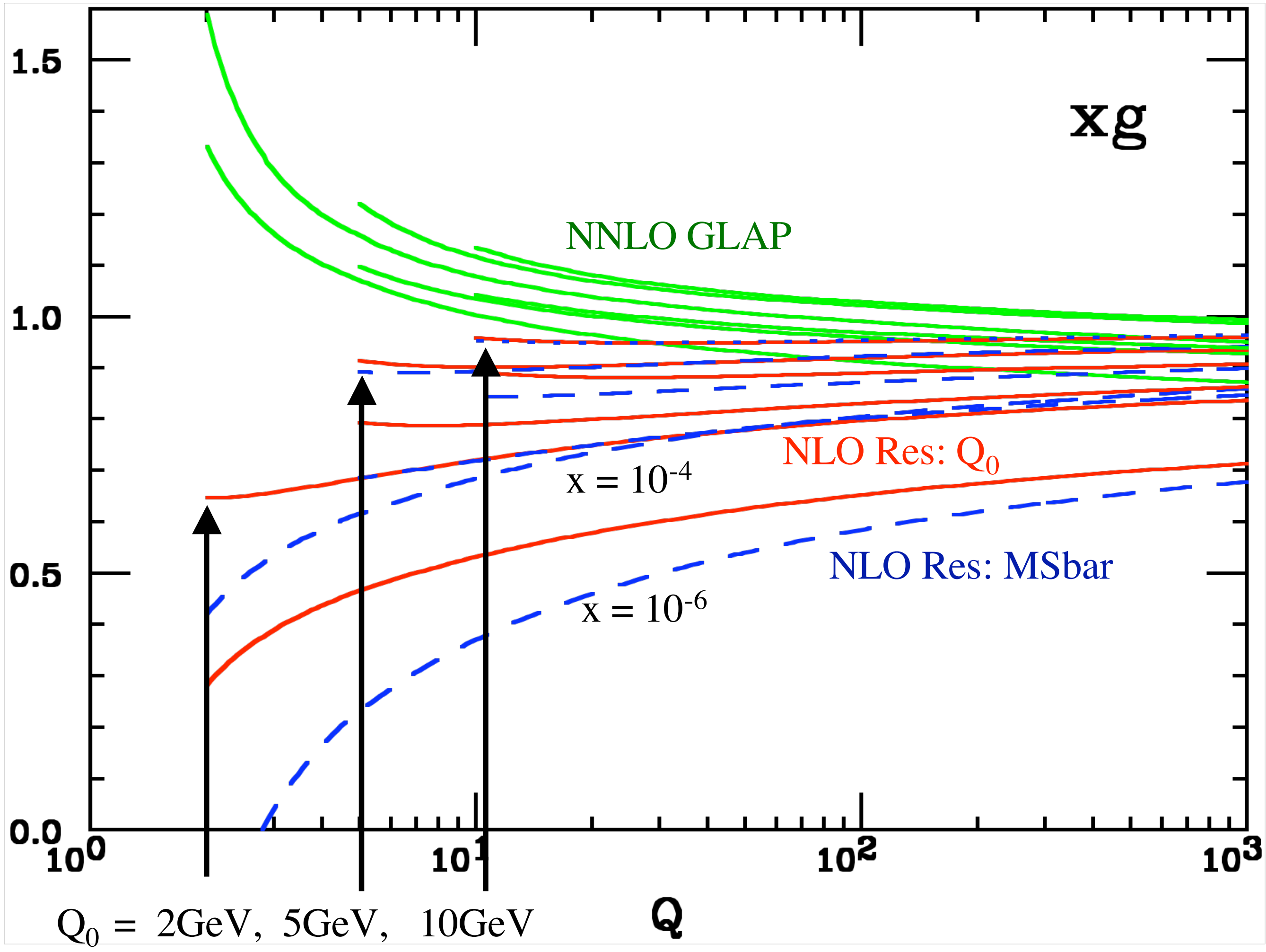


K-factors for gluons: from HERA to LHC

Initial pdfs at Q_0 = 2, 5 and 10GeV adjusted so that $F_2^{Res} = F_2^{NLO}$ etc.

$$K \equiv \frac{g^{Res}}{g^{NLO}}$$

Resummation:
fewer gluons
at LHC



Getting ready to fit the data

We have just produced very dense numerical grids of $P_{ij}^{improved} = P_{ij}^{NLO} + K_{ij}$, and similarly for coefficients c_{ij}^{2}, c_{ij}^{L}.

The eight grids contain 60 by 60 values of the resummation corrections K in uniform steps in logx (for x between 1 and 10^{-6}) and α_s (between 0.08 and 0.36) with n_f fixed, in $Q_0 MS^{bar}$ scheme, sufficient to interpolate for any value of x and α_s in these ranges.

CTEQ will soon include $P_{ij}^{improved}$ in their fitting machine

P. Nadolsky

and also in publicly available evolution codes

J. Rojo

Summary and Conclusion

- The matching of perturbative QCD evolution at large x and of BFKL at small x is now understood.
- Duality, momentum conserv., symm. under gluon exchange of the BFKL kernel and running coupling effects are essential
- The resulting asymptotic small x behaviour is much softened with respect to the naive BFKL, in agreement with the data.
- We have constructed splitting functions and coefficients that reduce to the pert. results at large x and incorporate BFKL with running coupling effects at small x.
- We have results expressed in the commonly used MS^{bar} scheme, but can give them in any scheme.
- All formalism is ready for systematic phenomenology (e.g. at the LHC)