Basic Mathematics

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CERN Accelerator School
Basic Accelerator Science & Technology at CERN
3 – 7 February 2014 – Chavannes de Bogis

Contents

- Vectors & Matrices
- Differential Equations
- Some Units we use

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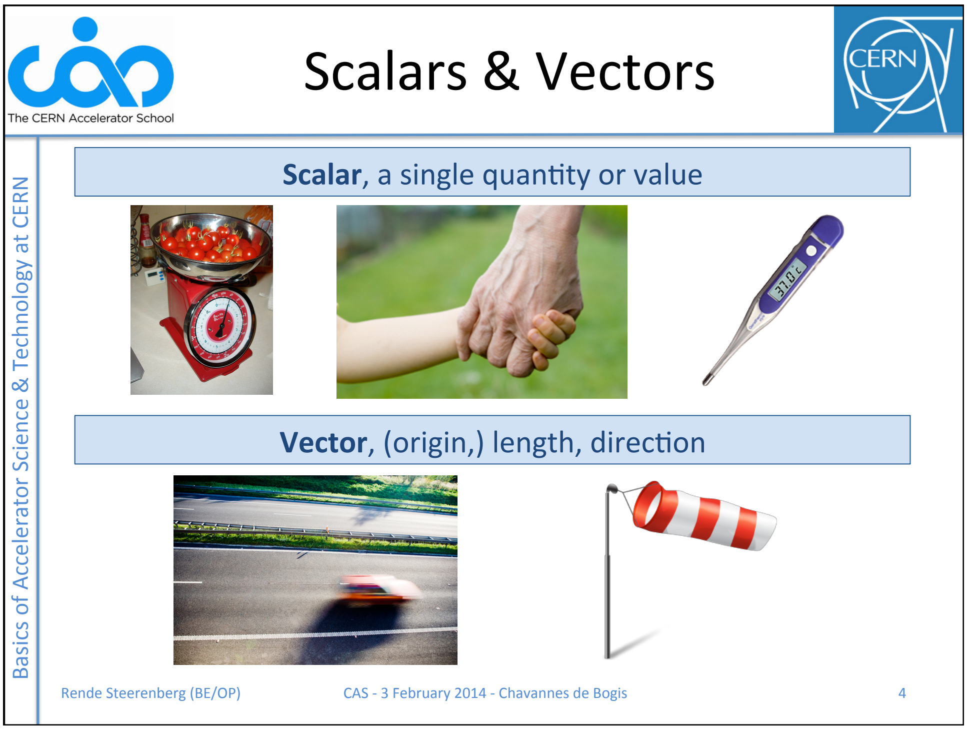
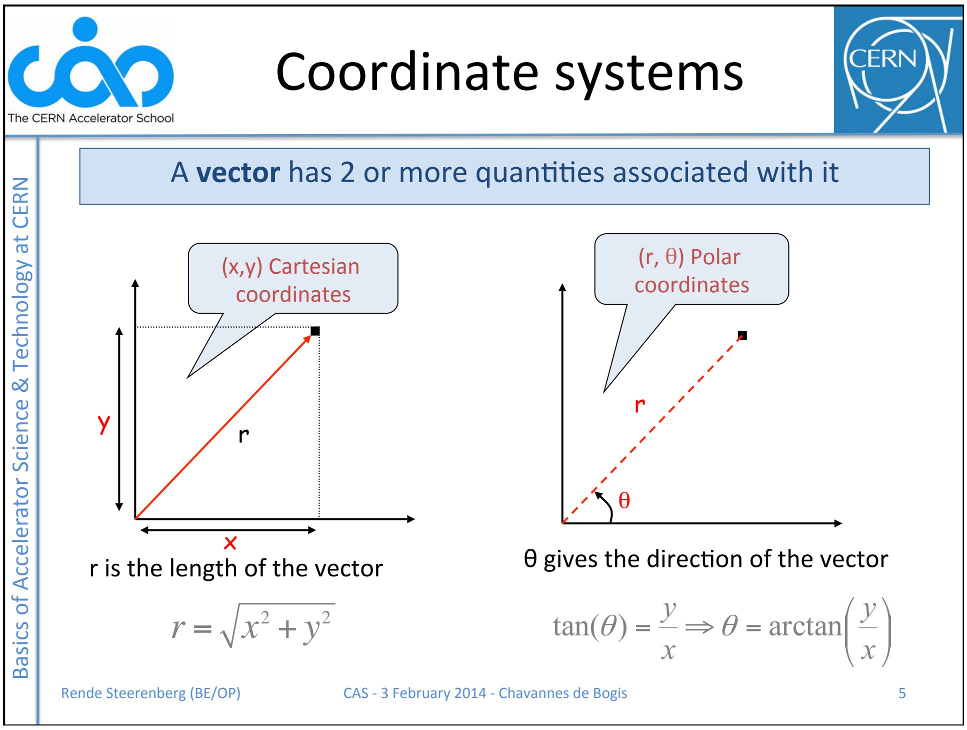
- **Vectors & Matrices**
- Differential Equations
- Some Units we use



Scalars & Vectors

Scalar, a single quantity or value

Vector, (origin,) length, direction



Coordinate systems

A **vector** has 2 or more quantities associated with it

(x,y) Cartesian coordinates

(r, θ) Polar coordinates

r is the length of the vector

$$r = \sqrt{x^2 + y^2}$$

θ gives the direction of the vector

$$\tan(\theta) = \frac{y}{x} \Rightarrow \theta = \arctan\left(\frac{y}{x}\right)$$

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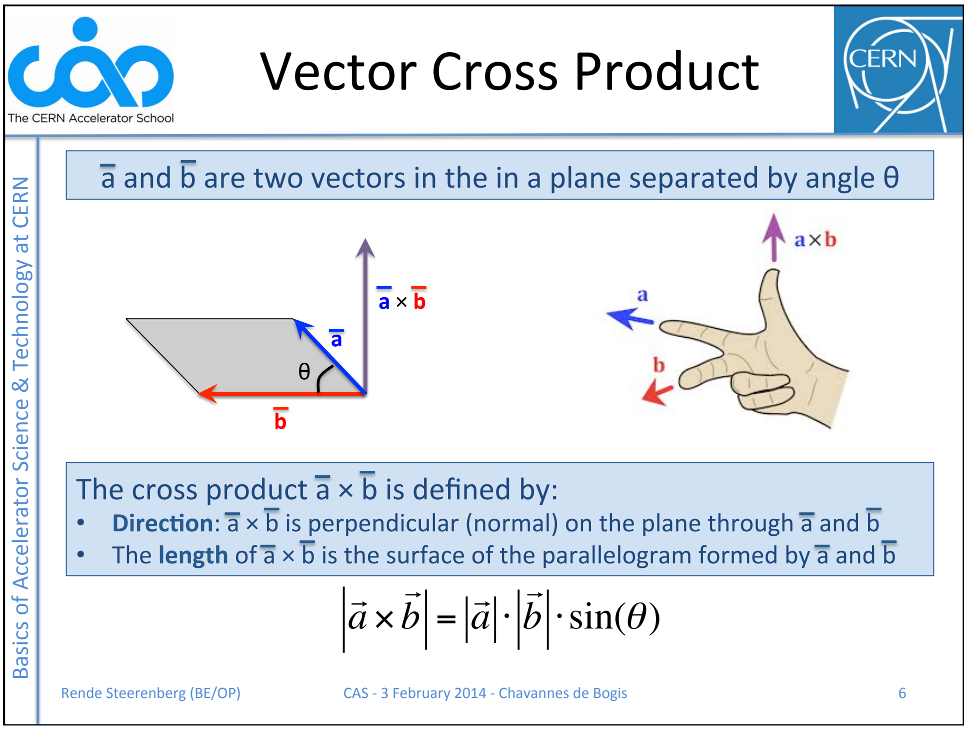

Vector Cross Product

$\overline{a}$ and $\overline{b}$ are two vectors in the in a plane separated by angle θ

The cross product $\overline{a} \times \overline{b}$ is defined by:

- **Direction**: $\overline{a} \times \overline{b}$ is perpendicular (normal) on the plane through $\overline{a}$ and $\overline{b}$
- The **length** of $\overline{a} \times \overline{b}$ is the surface of the parallelogram formed by $\overline{a}$ and $\overline{b}$

$$\left|\vec{a} \times \vec{b}\right| = \left|\vec{a}\right| \cdot \left|\vec{b}\right| \cdot \sin(\theta)$$

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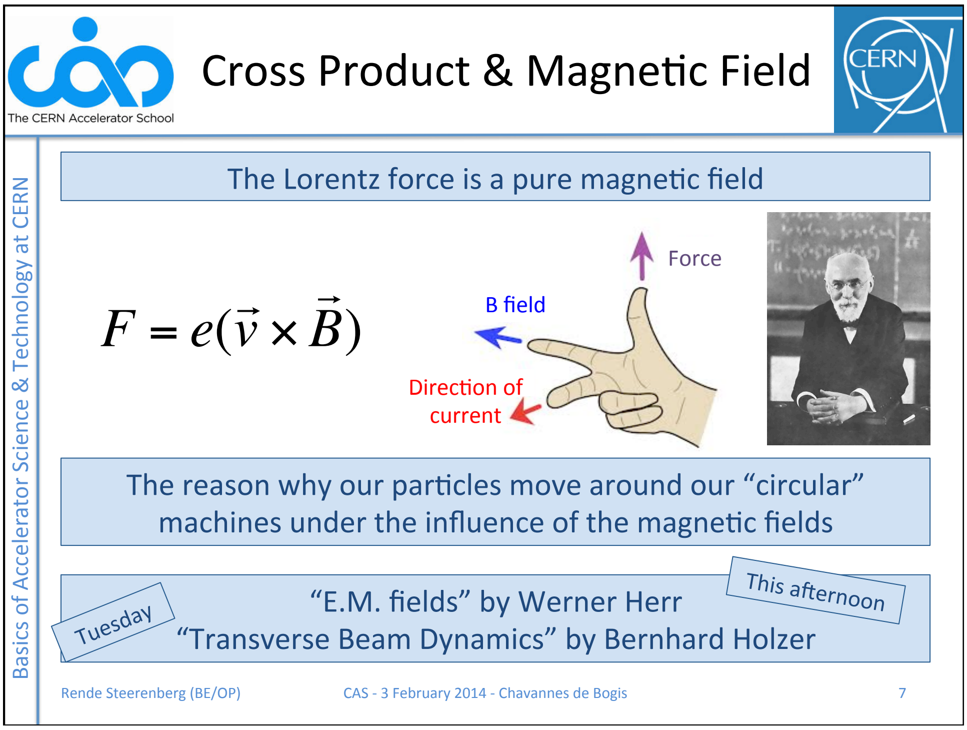


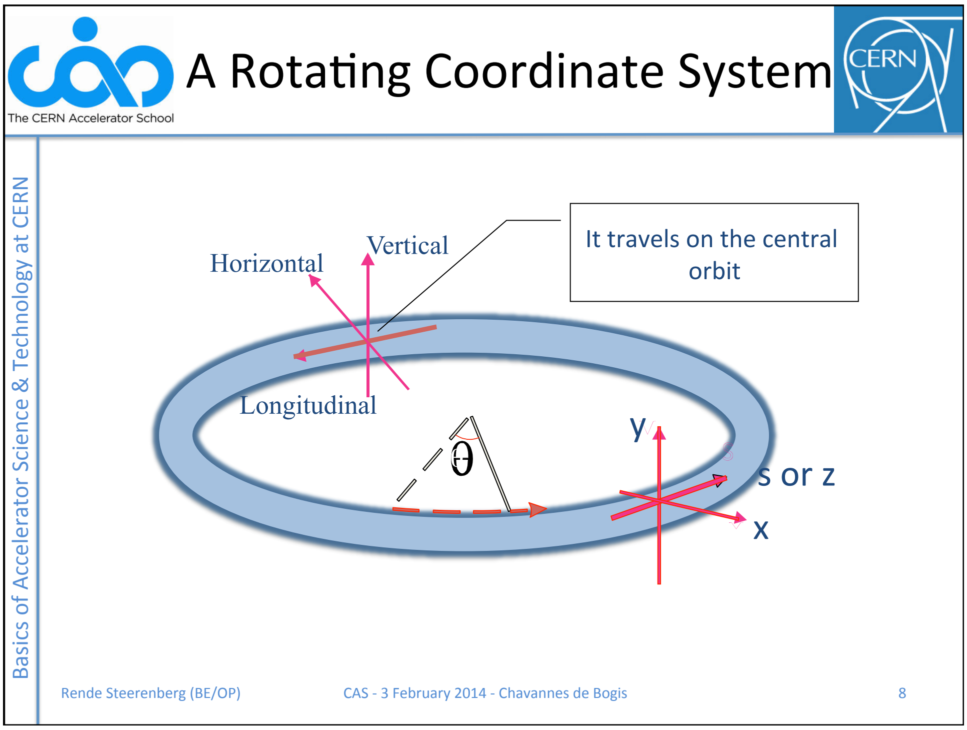
Cross Product & Magnetic Field

The Lorentz force is a pure magnetic field

$$F = e(\vec{v} \times \vec{B})$$

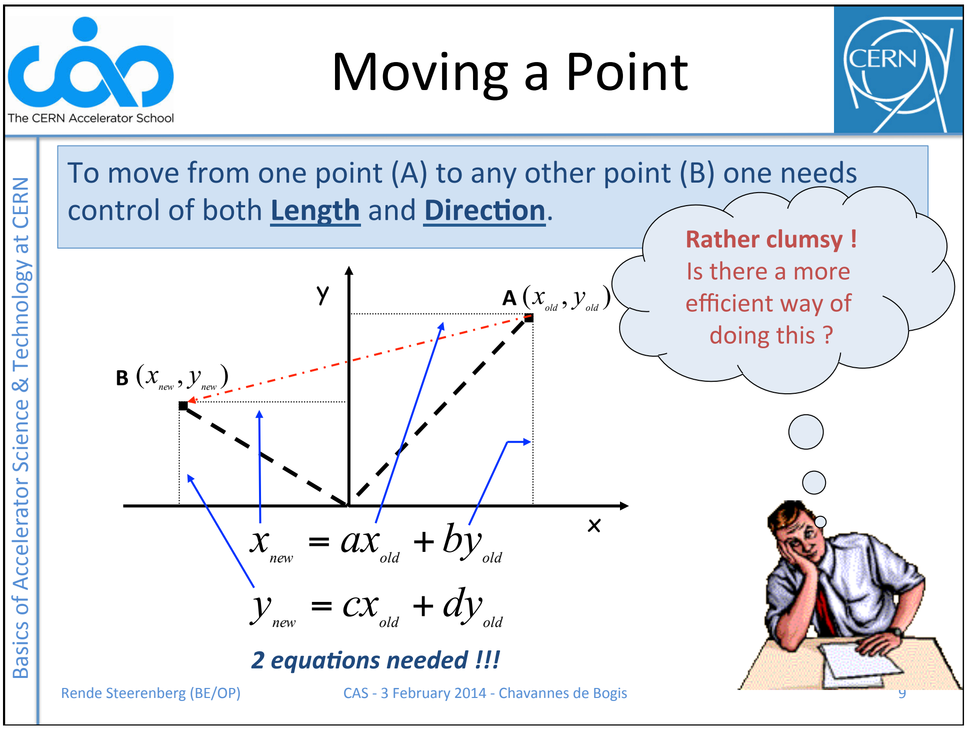
The reason why our particles move around our “circular” machines under the influence of the magnetic fields

“E.M. fields” by Werner Herr
“Transverse Beam Dynamics” by Bernhard Holzer



A Rotating Coordinate System

It travels on the central orbit



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Moving a Point

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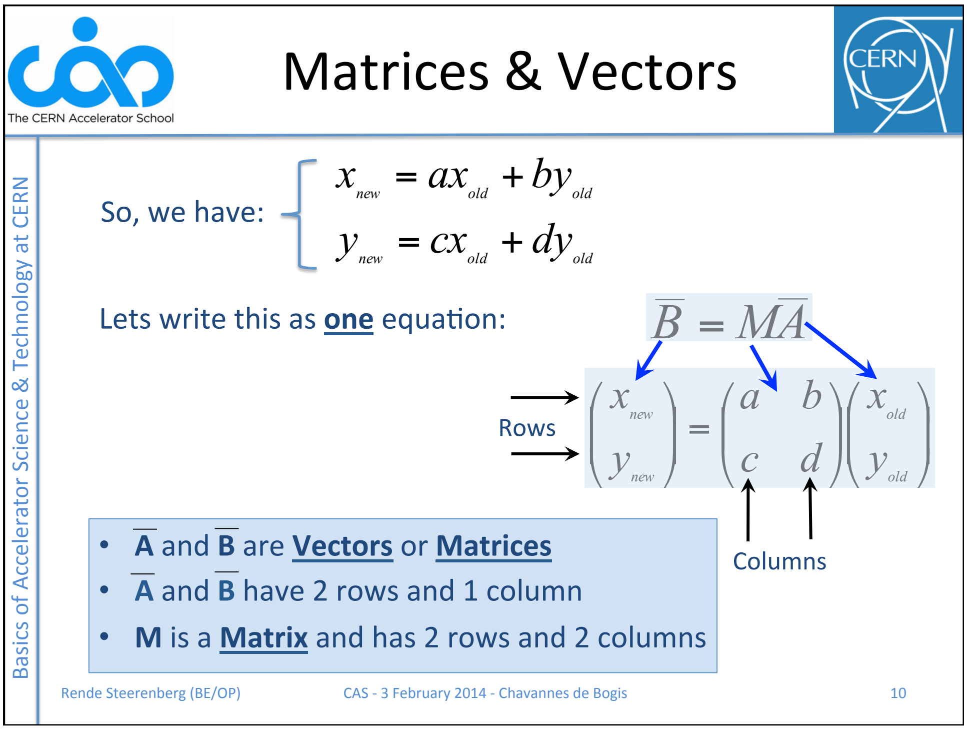
To move from one point (A) to any other point (B) one needs control of both **Length** and **Direction**.

$$x_{new} = ax_{old} + by_{old}$$

$$y_{new} = cx_{old} + dy_{old}$$

2 equations needed !!!

Rather clumsy !
Is there a more
efficient way of
doing this ?



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Matrices & Vectors

Basics of Accelerator Science & Technology at CERN

So, we have:

$$\begin{cases} x_{new} = ax_{old} + by_{old} \\ y_{new} = cx_{old} + dy_{old} \end{cases}$$

Lets write this as **one** equation:

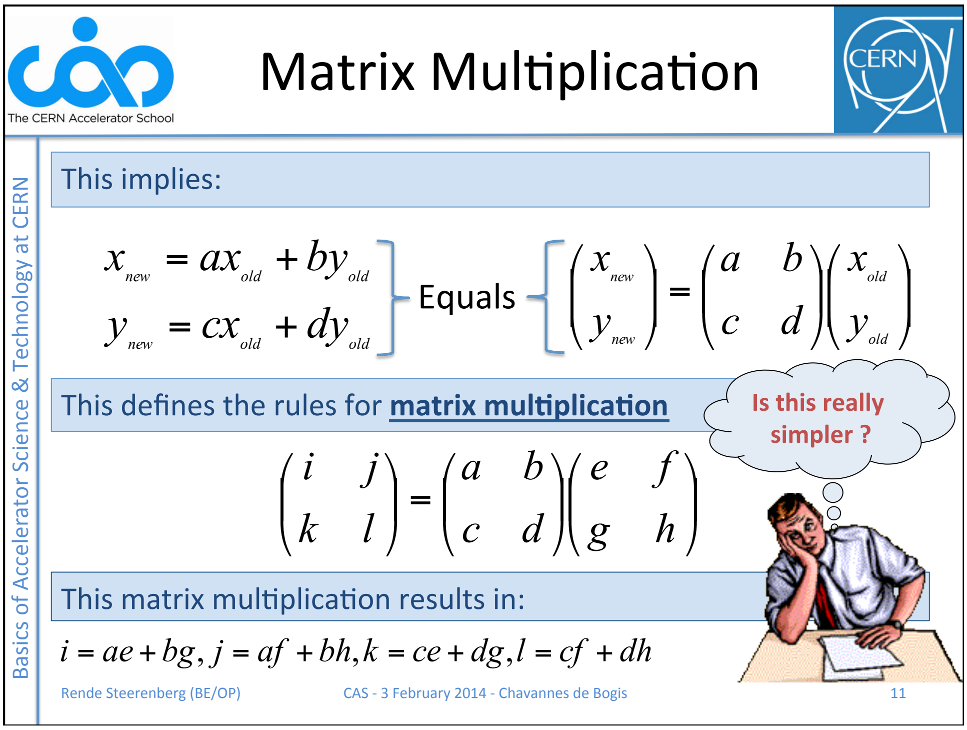
$$\bar{B} = M\bar{A}$$

$$\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix}$$

Rows

Columns

- $\bar{A}$ and $\bar{B}$ are **Vectors** or **Matrices**
- $\bar{A}$ and $\bar{B}$ have 2 rows and 1 column
- **M** is a **Matrix** and has 2 rows and 2 columns



Matrix Multiplication

This implies:

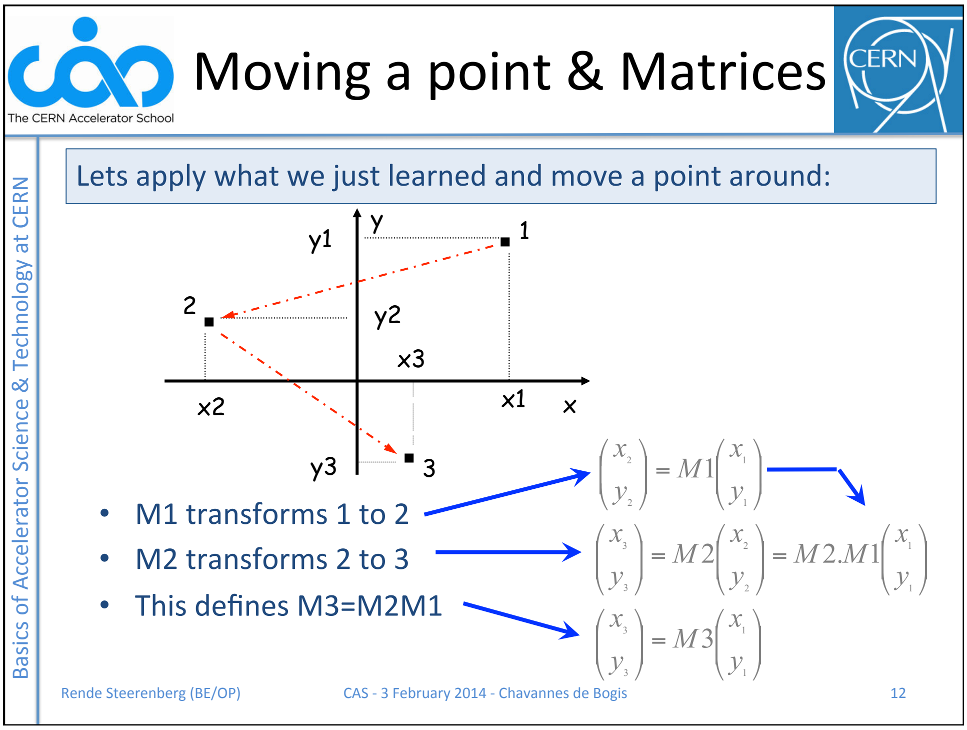
$$\left.\begin{aligned} x_{new} &= ax_{old} + by_{old} \\ y_{new} &= cx_{old} + dy_{old} \end{aligned}\right\} \text{Equals} \left\{ \begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix} \right.$$

This defines the rules for **matrix multiplication**

$$\begin{pmatrix} i & j \\ k & l \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

This matrix multiplication results in:

$$i = ae + bg, j = af + bh, k = ce + dg, l = cf + dh$$



Moving a point & Matrices

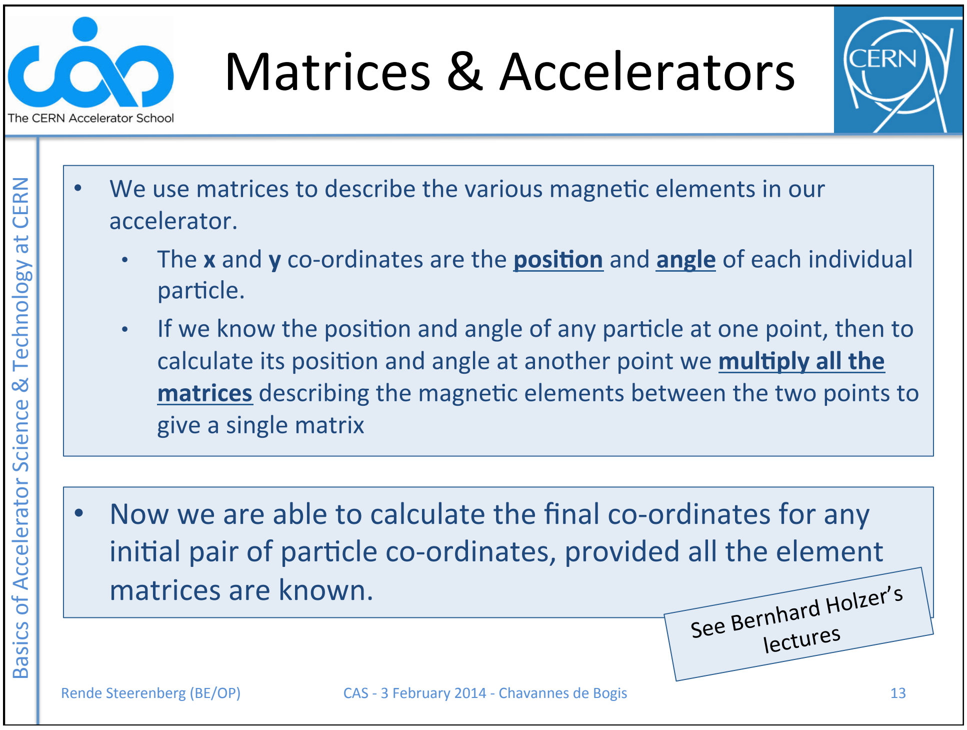
Lets apply what we just learned and move a point around:

- M1 transforms 1 to 2
- M2 transforms 2 to 3
- This defines M3=M2M1

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = M1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = M2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = M2.M1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = M3 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

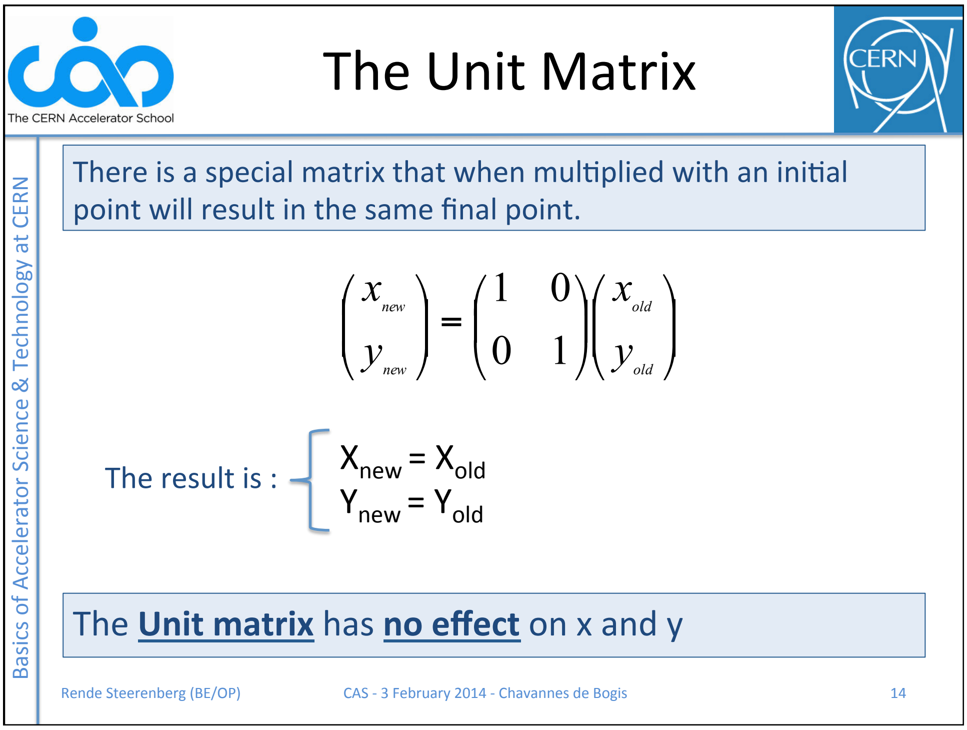


Matrices & Accelerators

- We use matrices to describe the various magnetic elements in our accelerator.
  - The **x** and **y** co-ordinates are the **position** and **angle** of each individual particle.
  - If we know the position and angle of any particle at one point, then to calculate its position and angle at another point we **multiply all the matrices** describing the magnetic elements between the two points to give a single matrix

- Now we are able to calculate the final co-ordinates for any initial pair of particle co-ordinates, provided all the element matrices are known.

See Bernhard Holzer's lectures



The Unit Matrix

There is a special matrix that when multiplied with an initial point will result in the same final point.

$$\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix}$$

The result is : $\begin{cases} X_{new} = X_{old} \\ Y_{new} = Y_{old} \end{cases}$

The **Unit matrix** has **no effect** on x and y

Going backwards

What about **going back** from a **final** point to the corresponding **initial** point ?

$$\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix} \quad \text{or} \quad \overline{B} = M\overline{A}$$

For the reverse we need another matrix M^{-1}

$$\overline{A} = M^{-1}\overline{B} \quad \text{such that} \quad \overline{B} = MM^{-1}\overline{B}$$

The combination of M and M^{-1} does have no effect

$$MM^{-1} = Unit\,Matrix$$

M^{-1} is the "**inverse**" or "**reciprocal**" matrix of **M.**

Calculating the Inverse Matrix

If we have a 2 x 2 matrix:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then the **inverse matrix** is calculated by:

$$M^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The term **(ad – bc)** is called the **determinate**, which is just a **number** (scalar).

A Practical Example

- Changing the current in two sets of quadrupole magnets (F & D) changes the horizontal and vertical tunes (Q_h & Q_v).
- This can be expressed by the following matrix relationship:

$$\begin{pmatrix} \Delta Q_h \\ \Delta Q_v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \Delta I_F \\ \Delta I_D \end{pmatrix} \quad \text{or} \quad \overline{\Delta Q} = M\,\overline{\Delta I}$$

- Change I_F then I_D and measure the changes in Q_h and Q_v
- Calculate the matrix M
- Calculate the inverse matrix M^{-1}
- Use now M^{-1} to calculate the current changes (ΔI_F and ΔI_D) needed for any required change in tune (ΔQ_h and ΔQ_v).

$$\overline{\Delta I} = M^{-1}\,\overline{\Delta Q}$$

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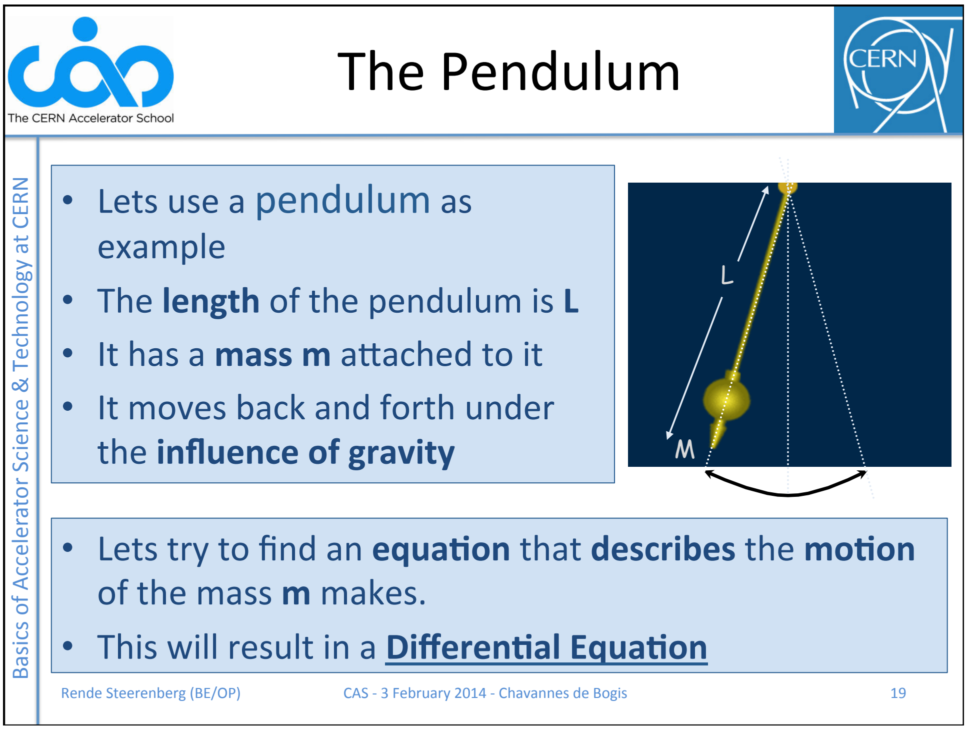
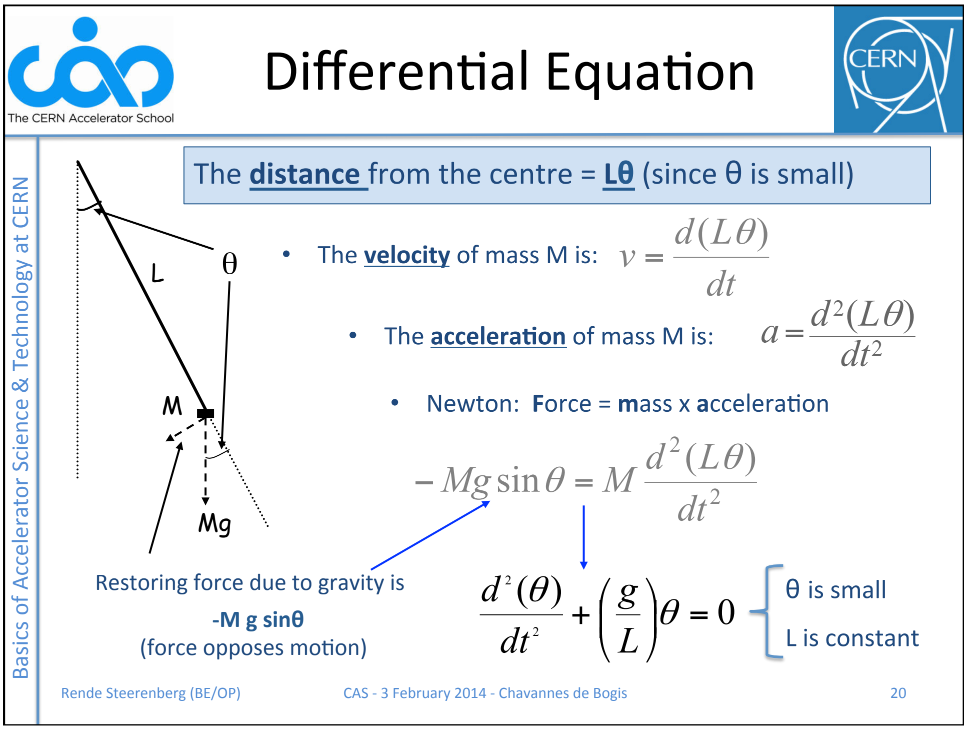
- Vectors & Matrices
- **Differential Equations**
- Some Units we use



The Pendulum

- Lets use a pendulum as example
- The **length** of the pendulum is **L**
- It has a **mass m** attached to it
- It moves back and forth under the **influence of gravity**

- Lets try to find an **equation** that **describes** the **motion** of the mass **m** makes.
- This will result in a **Differential Equation**



Differential Equation

The **distance** from the centre = **Lθ** (since θ is small)

- The **velocity** of mass M is: $v = \frac{d(L\theta)}{dt}$
- The **acceleration** of mass M is: $a = \frac{d^2(L\theta)}{dt^2}$
- Newton: **F**orce = **m**ass x **a**cceleration

$$-Mg\sin\theta = M\frac{d^2(L\theta)}{dt^2}$$

Restoring force due to gravity is

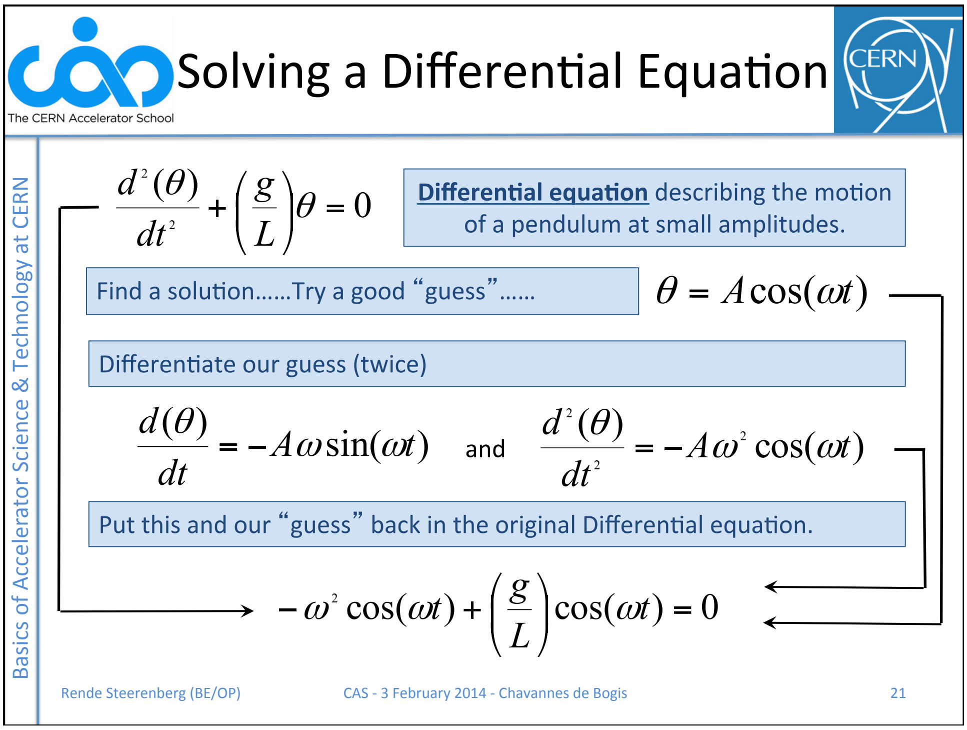
-M g sinθ

(force opposes motion)

$$\frac{d^2(\theta)}{dt^2} + \left(\frac{g}{L}\right)\theta = 0$$

θ is small

L is constant



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Solving a Differential Equation

$$\frac{d^2(\theta)}{dt^2} + \left(\frac{g}{L}\right)\theta = 0$$

Differential equation describing the motion of a pendulum at small amplitudes.

Find a solution……Try a good "guess"……

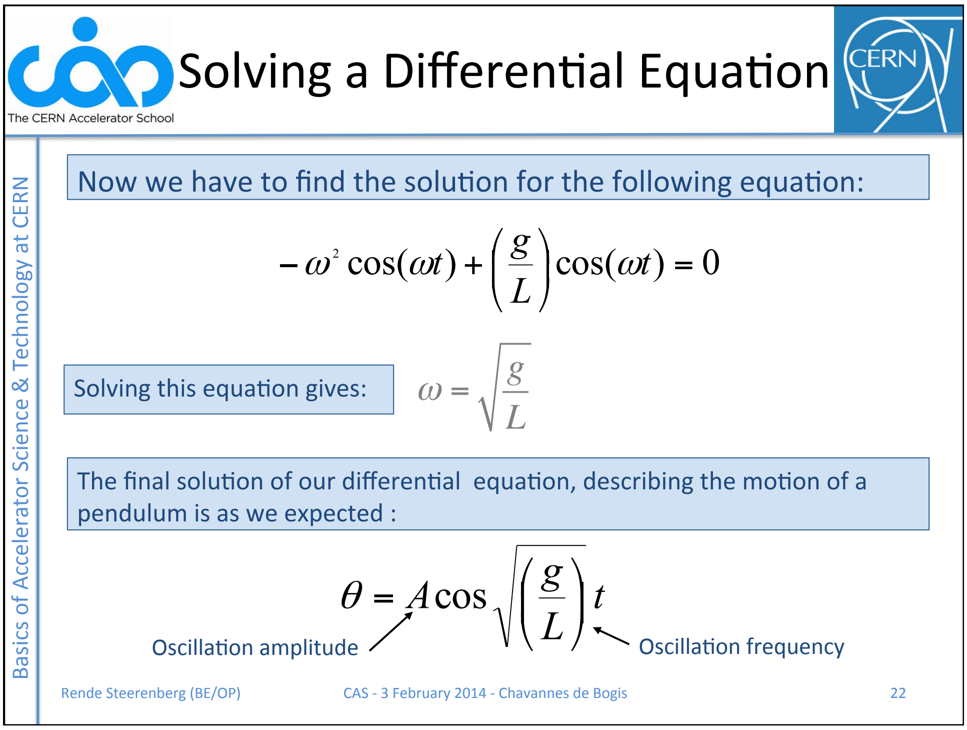
$$\theta = A\cos(\omega t)$$

Differentiate our guess (twice)

$$\frac{d(\theta)}{dt} = -A\omega\sin(\omega t) \quad \text{and} \quad \frac{d^2(\theta)}{dt^2} = -A\omega^2\cos(\omega t)$$

Put this and our "guess" back in the original Differential equation.

$$-\omega^2\cos(\omega t) + \left(\frac{g}{L}\right)\cos(\omega t) = 0$$



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Solving a Differential Equation

Now we have to find the solution for the following equation:

$$-\omega^2\cos(\omega t) + \left(\frac{g}{L}\right)\cos(\omega t) = 0$$

Solving this equation gives: $\omega = \sqrt{\frac{g}{L}}$

The final solution of our differential equation, describing the motion of a pendulum is as we expected :

$$\theta = A\cos\sqrt{\left(\frac{g}{L}\right)}t$$

Oscillation amplitude

Oscillation frequency

Position & Velocity

The differential equation that describes the transverse motion of the particles as they move around our accelerator.

$$\frac{d^2(x)}{dt^2}+\left(K\right)x=0$$

The solution of this second order describes **oscillatory motion**

For any system, where the restoring force is proportional to the displacement, the solution for the displacement will be of the form:

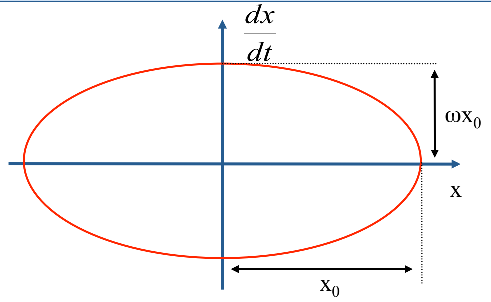
$$x = x_0 \cos(\omega t) \qquad \frac{dx}{dt} = -x_0\omega\sin(\omega t)$$

Phase Space Plot

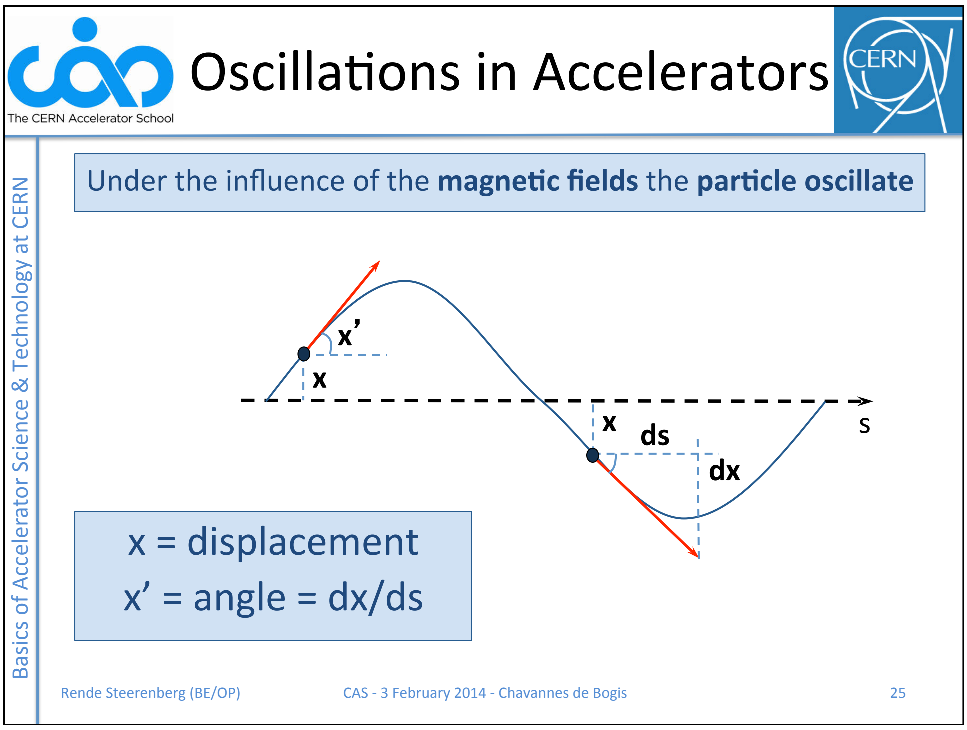
Plot the **velocity** as a function of **displacement**:

$$x = x_0 \cos(\omega t)$$

$$\frac{dx}{dt} = -x_0\omega\sin(\omega t)$$



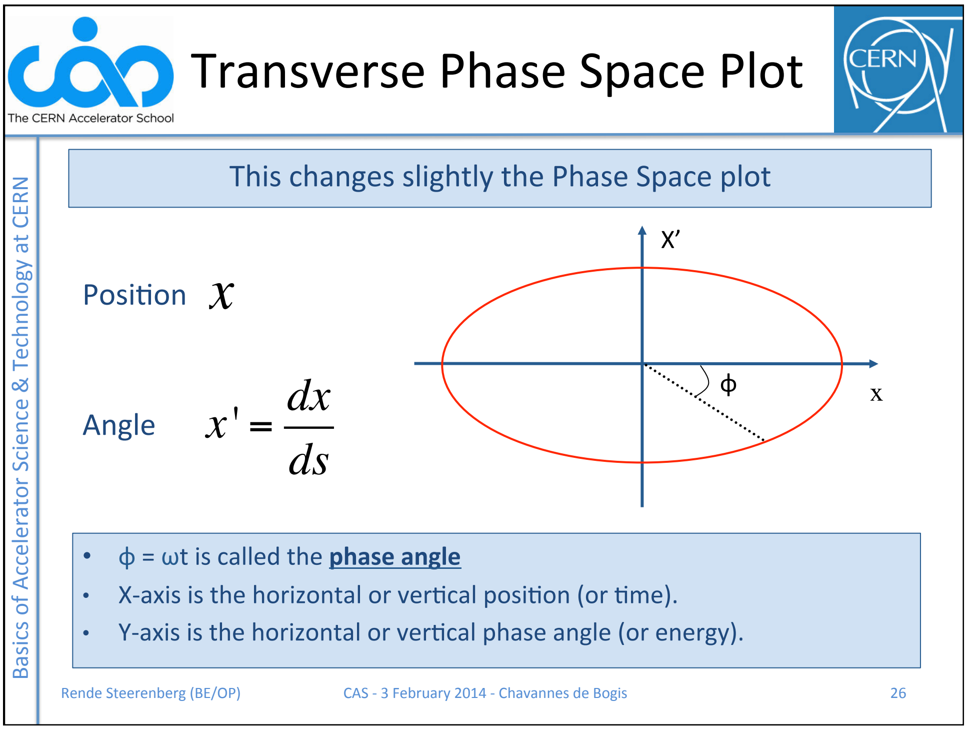
- It is an ellipse.
- As ωt advances by 2 π it repeats itself.
- This continues for (ω t + k 2π), with k=0,±1, ±2,..,..etc



Oscillations in Accelerators

Under the influence of the **magnetic fields** the **particle oscillate**

x = displacement
x' = angle = dx/ds



Transverse Phase Space Plot

This changes slightly the Phase Space plot

Position x

Angle $x' = \frac{dx}{ds}$

- ϕ = ωt is called the **phase angle**
- X-axis is the horizontal or vertical position (or time).
- Y-axis is the horizontal or vertical phase angle (or energy).

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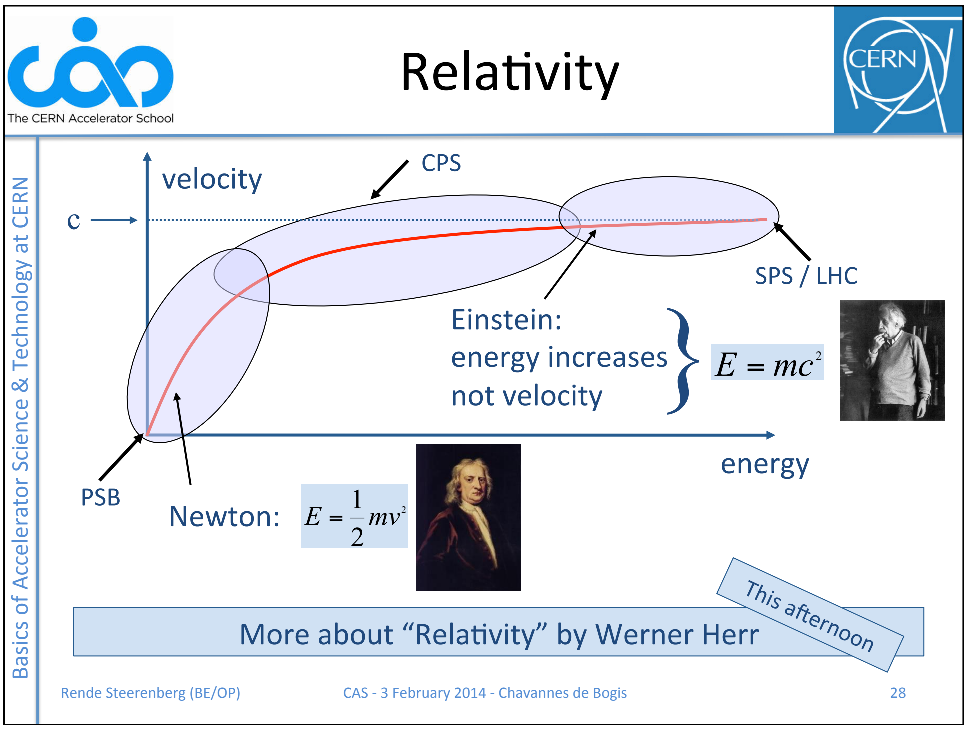
Basics of Accelerator Science & Technology at CERN

- Vectors & Matrices
- Differential Equations
- **Some Units we use**



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Relativity

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Einstein:
energy increases
not velocity

$E = mc^2$

Newton: $E = \frac{1}{2}mv^2$

More about "Relativity" by Werner Herr

This afternoon

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Units: Joules versus eV

- The unit most commonly used for **Energy** is **Joules [J]**
- In accelerator and particle physics we talk about **eV**…!?
- The **energy** acquired by an **electron** in a potential of **1 Volt** is defined as being **1 eV**
- **1 eV** is **1 elementary charge** ‘pushed’ by **1 Volt**.

1 eV = 1.6 x 10^{-19} Joules

- The unit eV is too small to be used currently, we use:

1 keV = 10^3 eV; 1 MeV = 10^6 eV; 1 GeV=10^9 eV; 1 TeV=10^{12} eV,………

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Energy versus Momentum

Einstein’s formula:

$E = mc^2$, which for a mass at rest is: $E_0 = m_0c^2$

The ratio between the total energy and the rest energy is

$$\gamma = \frac{E}{E_0}$$

The ratio between the real velocity and the velocity of light is

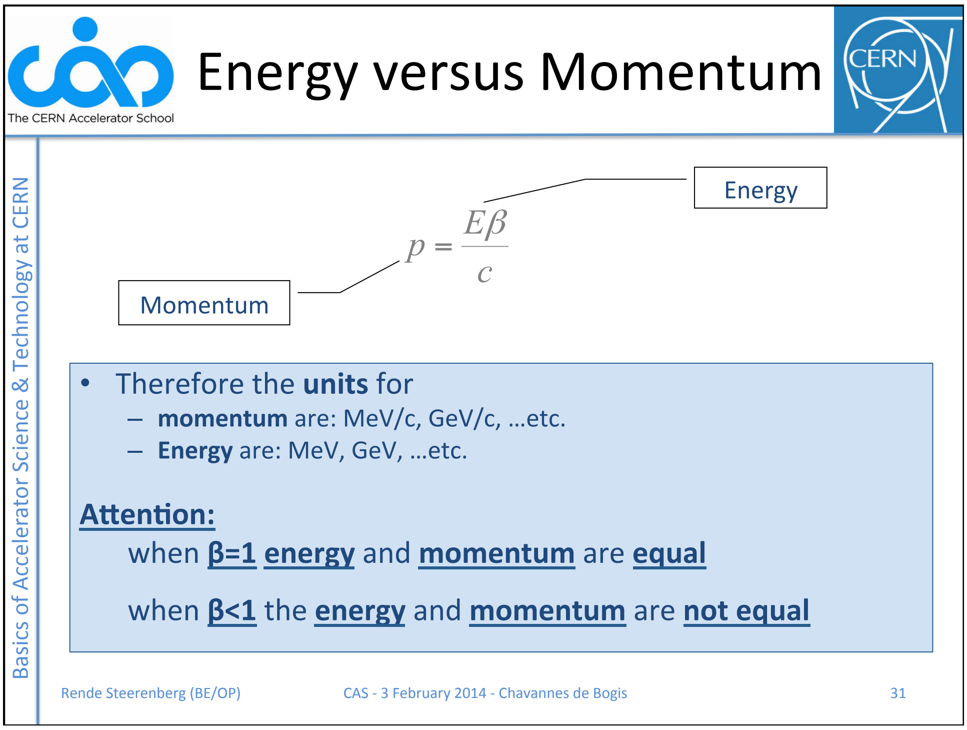
$$\beta = \frac{v}{c}$$

Then the mass of a moving particle is: $m = \gamma m_0$

We can write: $\beta = \dfrac{mvc}{mc^2}$

Momentum is: $p = mv$

$$\left.\begin{matrix} \beta = \dfrac{mvc}{mc^2} \\ p = mv \end{matrix}\right\} \quad \beta = \frac{pc}{E} \quad or \quad p = \frac{E\beta}{c}$$



Energy versus Momentum

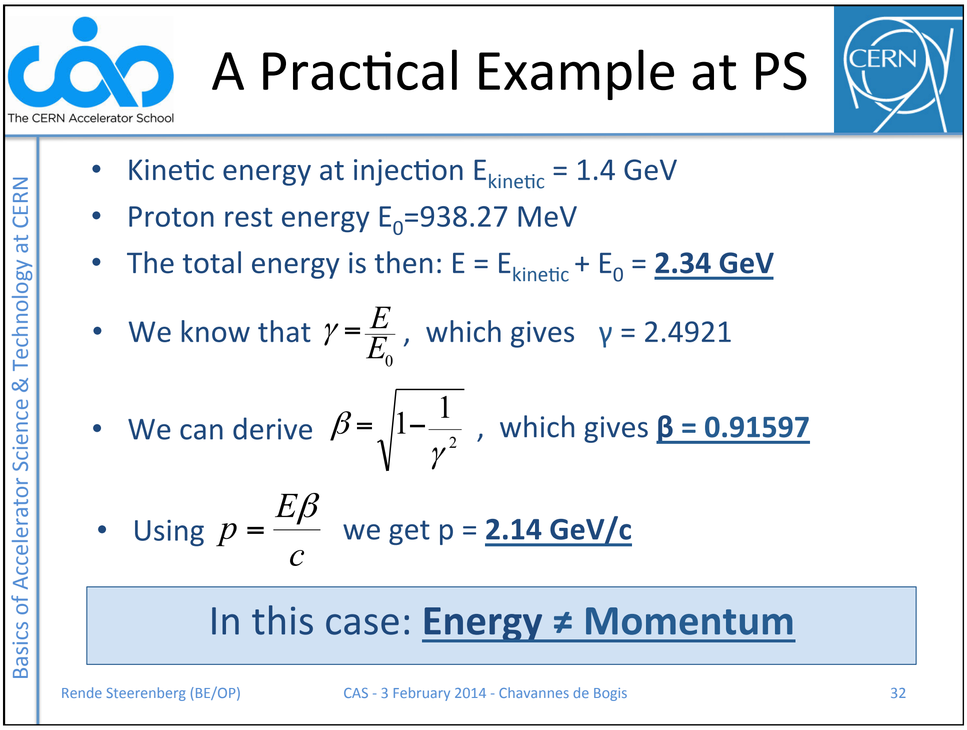
$$p = \frac{E\beta}{c}$$

- Therefore the **units** for
  - **momentum** are: MeV/c, GeV/c, …etc.
  - **Energy** are: MeV, GeV, …etc.

Attention:

when **β=1** **energy** and **momentum** are **equal**

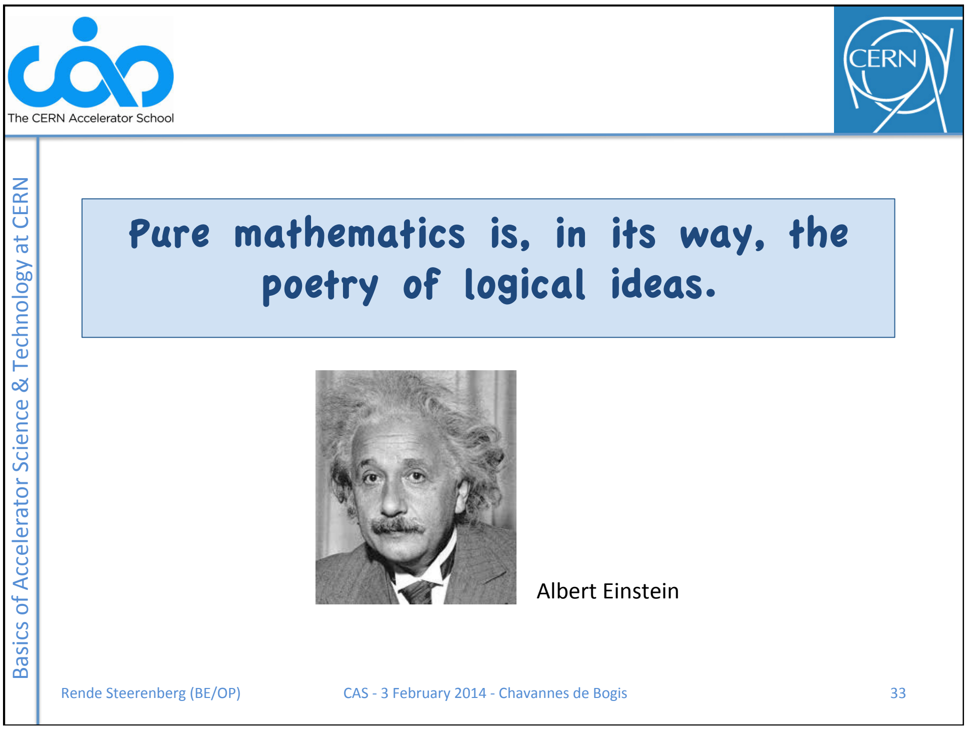
when **β<1** the **energy** and **momentum** are **not equal**



A Practical Example at PS

- Kinetic energy at injection $E_{kinetic}$ = 1.4 GeV
- Proton rest energy E_0=938.27 MeV
- The total energy is then: $E = E_{kinetic} + E_0$ = **2.34 GeV**
- We know that $\gamma = \frac{E}{E_0}$, which gives γ = 2.4921
- We can derive $\beta = \sqrt{1 - \frac{1}{\gamma^2}}$, which gives **β = 0.91597**
- Using $p = \frac{E\beta}{c}$ we get p = **2.14 GeV/c**

In this case: **Energy ≠ Momentum**



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> Pure mathematics is, in its way, the poetry of logical ideas.

Albert Einstein

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