Short Introduction to (Classical) Electromagnetic Theory

(.. and applications to accelerators)

Werner Herr, CERN

(http://cern.ch/Werner.Herr/CAS2014_Chavannes/em.pdf)

Why electrodynamics ?

- Accelerator physics relies on electromagnetic concepts:
  - Beam dynamics
  - Magnets, cavities
  - Beam instrumentation
  - Powering
  - ...

OUTLINE

- Some mathematics (intuitive, mostly illustrations), see also lecture R. Steerenberg
- Basic electromagnetic phenomena
- Maxwell's equations
- Lorentz force
- Motion of particles in electromagnetic fields
- Electromagnetic waves in vacuum
- Electromagnetic waves in conducting media
  - Waves in RF cavities
  - Waves in wave guides

Reading Material

- J.D. Jackson, *Classical Electrodynamics* (Wiley, 1998 ..)
- L. Landau, E. Lifschitz, *KlassischeFeldtheorie*, Vol2. (Harri Deutsch, 1997)
- W. Greiner, *Classical Electrodynamics*, (Springer, February, 22nd, 2009)
- J. Slater, N. Frank, *Electromagnetism*, (McGraw-Hill, 1947, and Dover Books, 1970)
- R.P. Feynman, *Feynman lectures on Physics*, Vol2.

First some mathematics (vectors, potential, calculus ....)

Reminder: mathematics used here

- **Addition to previous lecture (R.S.)**
- **Not all details are strictly needed to understand, but required for calculations**
- **I shall introduce:**
  - **Scalar and vector fields**
  - **Calculation on fields (vector calculus)**
  - **Illustrations and examples ...**

  Remark: many illustrations only in 2 dimensions

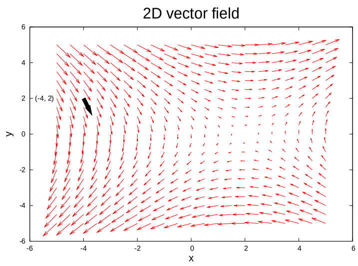
A bit on scalar fields (potentials)

- **At each point in space has assigned a quantity with a value (real or complex)**
- **Described by a scalar $\phi(x, y, z)$ (a number)**

  Example: $\phi(x, y, z) = 0.1x^2 - 0.2 \cdot x \cdot y + z^2$

→ **We get for** $(x = 4, y = 2, z = 1)$**:** $\phi(-4, 2, 1) = 4.2$

A bit on vector fields ...



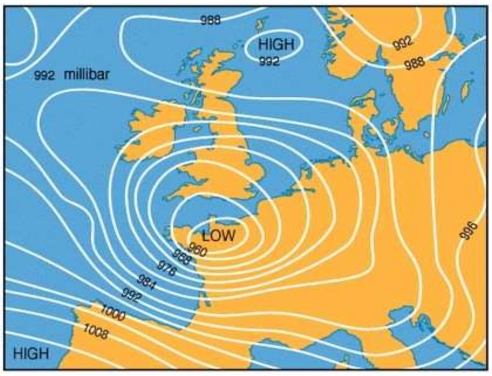
- At each point in space (or plane): a quantity with a **length** and **direction**, (typically 2, 3, 4, 6 components)
- A vector with 3 components: $\vec{F}(x,y,z) = (F_x, F_y, F_z)$
- Example (in 2D): $\vec{F}(x,y) = (0.1y, 0.1x - 0.2)$

→ We get: $\vec{F}(-4,2) = (0.2, -0.6)$

Examples:

- Scalar fields:
  - Atmospheric pressure
  - Temperature in a room
  - Density of molecules in a gas
- Vector fields:
  - Speed and direction of wind ..
  - Heat flow
  - Velocity and direction of moving molecules in a gas

Example: scalar field/potential ...



Lines of pressure (isobars)

Function of longitude, latitude and altitude (x, y, z)

Example: vector field ...

Example for an extreme vector field ..

What we shall talk about

Maxwell's equations relate Electric and Magnetic fields from charge and current distributions (SI units).

$\vec{E}$ = **electric field [V/m]**

$\vec{H}$ = **magnetic field [A/m]**

$\vec{D}$ = **electric displacement [C/m^2]**

$\vec{B}$ = **magnetic flux density [T]**

q = **electric charge [C]**

ρ = **electric charge density [C/m^3]**

$\vec{j}$ = **current density [A/m^2]**

μ_0 = **permeability of vacuum, 4** $\pi \cdot 10^{-7}$ **[H/m or N/A^2]**

ϵ_0 = **permittivity of vacuum, 8.854** $\cdot 10^{-12}$ **[F/m]**

c = **speed of light, 2.99792458** $\cdot 10^{8}$ **[m/s]**

Electromagnetic fields

In electrodynamics we talk about vector fields:

Electric phenomena: $\vec{E}$ **and** $\vec{D}$

Magnetic phenomena: $\vec{H}$ **and** $\vec{B}$

→ **Electrodynamics: need vectors with 3 components**

→ **Need to know how to calculate with vectors**

- **Scalar and vector products**
- **Vector calculus products**

Scalar products

Define a scalar product for (usual) vectors like: $\vec{a} \cdot \vec{b}$,

$$\vec{a} = (x_a, y_a, z_a) \qquad \vec{b} = (x_b, y_b, z_b)$$

$$\vec{a} \cdot \vec{b} = (x_a, y_a, z_a) \cdot (x_b, y_b, z_b) = (x_a \cdot x_b + y_a \cdot y_b + z_a \cdot z_b)$$

This product of two vectors is a <u>scalar</u> (number) not a vector.

(on that account: Scalar Product)

Example:
$(-2, 2, 1) \cdot (2, 4, 3) = -2 \cdot 2 + 2 \cdot 4 + 1 \cdot 3 = 7$

Vector products (sometimes cross product)

Define a vector product for (usual) vectors like: $\vec{a} \times \vec{b}$,

$$\vec{a} = (x_a, y_a, z_a) \qquad \vec{b} = (x_b, y_b, z_b)$$

$$\vec{a} \times \vec{b} = (x_a, y_a, z_a) \times (x_b, y_b, z_b)$$
$$= (\underbrace{y_a \cdot z_b - z_a \cdot y_b}_{x_{ab}}, \underbrace{z_a \cdot x_b - x_a \cdot z_b}_{y_{ab}}, \underbrace{x_a \cdot y_b - y_a \cdot x_b}_{z_{ab}})$$

This product of two vectors is a <u>vector</u>, not a scalar (number), (on that account: Vector Product)

Example 1:
$(-2, 2, 1) \times (2, 4, 3) = (2, 8, -12)$
Example 2 (two components only in the $x - y$ plane):
$(-2, 2, 0) \times (2, 4, 0) = (0, 0, -12)$ **(see R. Steerenberg)**

Vector calculus ...

We can define a special vector ∇ (sometimes written as $\vec{\nabla}$):

$$\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$$

It is called the "gradient" and invokes "partial derivatives". It can operate on a scalar function $\phi(x, y, z)$:

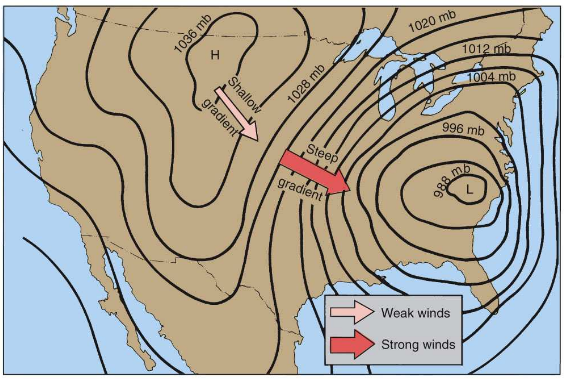
$$\nabla\phi = (\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}) \quad = \quad \vec{G} = (G_x, G_y, G_z)$$

and we get a vector $\vec{G}$. It is a kind of "slope" (steepness ..) in the 3 directions.

Example: $\phi(x, y, z) \; = \; 0.1x^2 - 0.2 \cdot x \cdot y + z^2$

$$\nabla\phi \; = \; \vec{G}(x, y, z) \; = \; (G_x, G_y, G_z) \; = \; (0.2x - 0.2y, \; -0.2x, \; 2z)$$

Gradient (slope) of a scalar field



Lines of pressure (isobars)

Gradient is large (steep) where lines are close (fast change of pressure)

Vector calculus ...

The gradient ∇ can be used as scalar or vector product with a vector $\vec{F}$, sometimes written as $\vec{\nabla}$
Used as:

$$\nabla \cdot \vec{F} \quad \text{or} \quad \nabla \times \vec{F}$$

Same definition for products as before, ∇ treated like a "normal" vector, but results depends on how they are applied:
$\nabla \cdot \Phi$ **is a vector**
$\nabla \cdot \vec{F}$ **is a scalar**
$\nabla \times \vec{F}$ **is a vector**

Operations on vector fields ...

Two operations of ∇ have special names:

Divergence (scalar product of gradient with a vector):

$$\mathsf{div}(\vec{F}) \quad = \quad \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Physical significance: "amount of density", (see later)

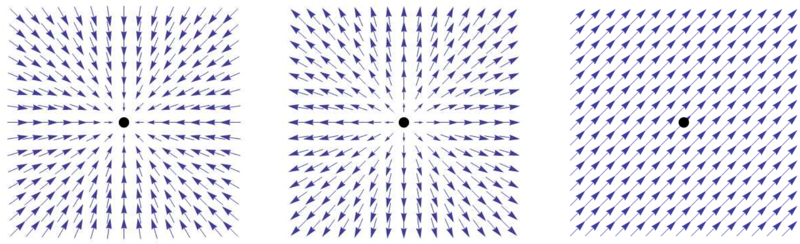
Curl (vector product of gradient with a vector):

$$\mathsf{curl}(\vec{F}) \quad = \quad \nabla \times \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

Physical significance: "amount of rotation", (see later)

Meaning of Divergence of fields ...

Field lines seen from some origin:



$\nabla\vec{F} < 0$ (sink) | $\nabla\vec{F} > 0$ (source) | $\nabla\vec{F} = 0$ (fluid)

The divergence (scalar, a single number) characterizes what comes from (or goes to) the origin

Meaning of Curl of fields ...

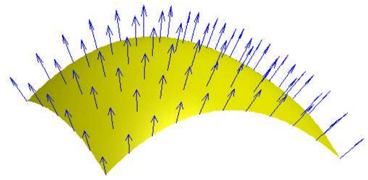
Here we have fields in $x-y$ plane::

$$\vec{F}_1 = (-0.2y, +0.2x, 0) \qquad \vec{F}_2 = (+0.5y, -0.5x, 0)$$

$$\nabla\times\vec{F}_1 = \mathsf{curl}\vec{F}_1 = (0,0,+0.4) \qquad \nabla\times\vec{F}_2 = \mathsf{curl}\vec{F}_2 = (0,0,-1.0)$$

Vectors in z-direction, perpendicular to $x-y$ plane
Values characterize "strength" and "direction" of rotation

Integration of (vector-) fields



Surface integrals: integrate field vectors passing (perpendicular) through a surface S (or area A), we obtain the Flux:

$$\rightarrow \int\int_A \vec{F} \cdot d\vec{A}$$

Density of field lines through the surface

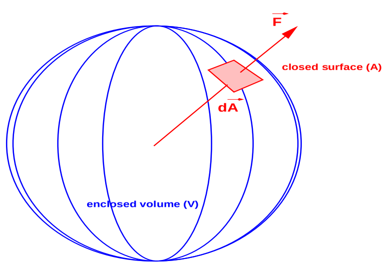
(e.g. amount of heat passing through a surface)

Easier Integration of (vector-) fields

Gauss' Theorem:

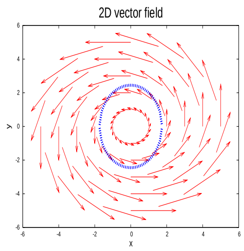
Integral through a closed surface (flux) is integral of divergence in the enclosed volume

$$\int\int_A \vec{F} \cdot d\vec{A} = \int\int\int_V \nabla \cdot \vec{F} \cdot dV$$



Relates surface integral to divergence

Integration of (vector-) fields



Line integrals: integrate field vectors along a line C:

$$\rightarrow \oint_C \vec{F} \cdot d\vec{r}$$

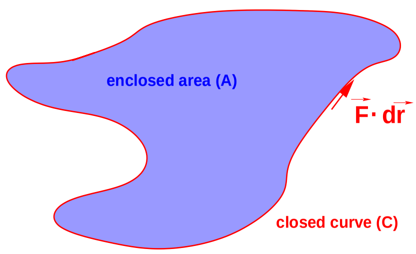
"sum up" vectors (length) in direction of line C
Integral often called Circulation.

(e.g. work performed along a path ...)

Easier Integration of (vector-) fields

Stokes' Theorem:
Integral along a closed line is integral of curl in the enclosed area

$$\oint_C \vec{F} \cdot d\vec{r} \;=\; \int\int_A \nabla \times \vec{F} \cdot d\vec{A}$$



Relates line integral to curl

To remember: ...

Not really rigorous, but:

- ➡ ***DIV*** **measures what is coming out (or going in), integral is called the FLUX**
- ➡ ***CURL*** **measures what is circulating, integral is called the CIRCULATION**

In general: a closed surface or closed line "measures" what is happening inside ...

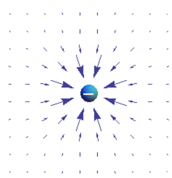
- BACK to ELECTRODYNAMICS -

How do we use all that stuff ?

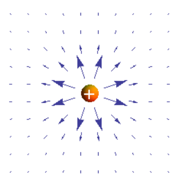
Some generalities

- Electric fields $\vec{E}$ are generated by charges
- Magnetic fields $\vec{B}$ are generated by moving charges
- Quantified by strength and density of field vectors

Electric fields from charges



(negative charges)



(positive charges)

Assume fields from a positive or negative charge q

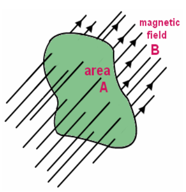
Electric field $\vec{E}$ is written as (Coulomb law):

$$\vec{E} = \frac{\pm q}{4\pi\epsilon_0} \cdot \frac{\vec{r}}{|r|^3}$$

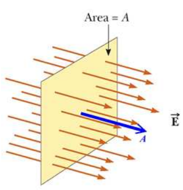
with:

$$\vec{r} = (x, y, z), \qquad |r| = \sqrt{x^2 + y^2 + z^2}$$

Electric and Magnetic flux



$$\int\int_A \vec{B} \cdot d\vec{A}$$



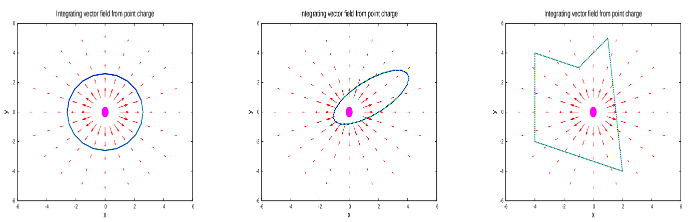
$$\int\int_A \vec{E} \cdot d\vec{A}$$

Integrate (count) field vectors through an area (or surface)

"Measures" the strength of the fields

Gives flux of electric and magnetic fields

Integrating fields from charges (2D !) ..



- **To compute the flux, add field lines through the surface:** $\int\int_A \vec{E} \cdot d\vec{A}$
- **Put any <u>closed</u> surface around charges (sphere, box, ...). If all charges are enclosed: independent of shape !**

→ **If <u>positive</u>: total net charge enclosed <u>positive</u>**

→ **If <u>negative</u>: total net charge enclosed <u>negative</u>**

Applying Divergence and charges ..

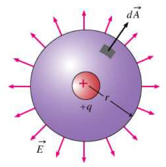
We can do the (non-trivial) computation of the divergence:

$$\text{div}\vec{E} \;=\; \nabla\vec{E} \;=\; \frac{dE_x}{dx} + \frac{dE_y}{dy} + \frac{dE_z}{dz} \;=\; \frac{\rho}{\epsilon_0}$$

(negative charges) $\nabla \cdot \vec{E} < 0$ **(positive charges)** $\nabla \cdot \vec{E} > 0$

Divergence related to charge density ρ **generating the field** $\vec{E}$

More formal: Gauss's Theorem
(Maxwell's first equation ...)



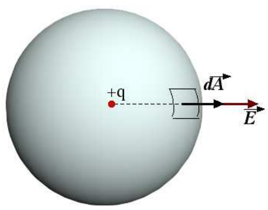
$\frac{1}{\epsilon_0} \int \int_A \vec{E} \cdot d\vec{A} = \frac{1}{\epsilon_0} \int \int \int_V \nabla \vec{E} \cdot dV = \frac{q}{\epsilon_0}$

$\nabla \vec{E} = \frac{\rho}{\epsilon_0}$

Flux of electric field $\vec{E}$ through a closed surface proportional to net electric charge q enclosed in the region (Gauss's Theorem). Written with charge density ρ we get Maxwell's first equation:

$$\text{div}\vec{E} = \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

Example: field from a charge q



A charge q generates a field $\vec{E}$ according to:

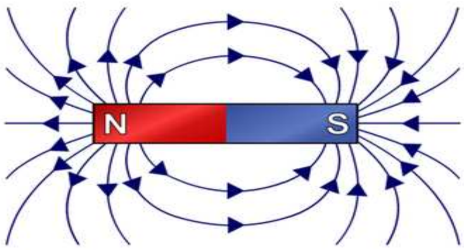
$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\vec{r}}{r^3}$$

Enclose it by a sphere: $\vec{E} = const.$ on a sphere (area is $4\pi \cdot r^2$):

$$\int \int_{sphere} \vec{E} \cdot d\vec{A} = \frac{q}{4\pi\epsilon_0} \int \int_{sphere} \frac{dA}{r^2} = \frac{q}{\epsilon_0}$$

Surface integral through sphere A is charge inside the sphere

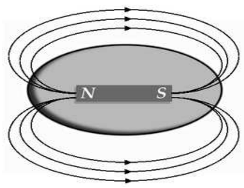
Divergence of magnetic fields



Definitions

- **Magnetic field lines from North to South**
- **Q: which is the direction of the earth magnetic field lines ?**

Maxwell's second equation ...



$$\int\int_A \vec{B} \;\; d\vec{A} = \int\int\int_V \nabla\vec{B} \;\; dV \;\; = \;\; 0$$
$$\nabla\vec{B} = 0$$

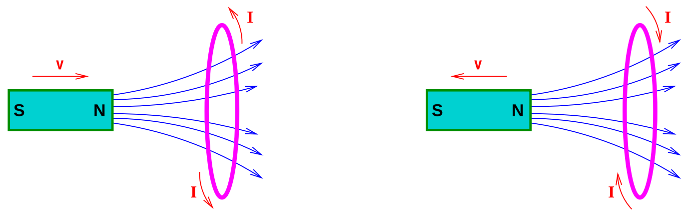
Closed field lines of magnetic flux density ($\vec{B}$): What goes out ANY closed surface also goes in, Maxwell's second equation:

$$\nabla\vec{B} = \mu_0\nabla\vec{H} = 0$$

→ **Physical significance: no Magnetic Monopoles**

Maxwell's third equation ...

Faradays law:



- **Changing magnetic flux through area of a coil introduces electric current I**
- **Can be changed by moving magnet or coil**

Maxwell's third equation ...

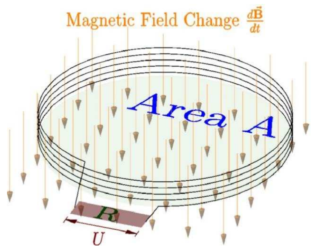
A changing flux Ω through an area A produces circulating electric field $\vec{E}$, i.e. a current I (Faraday)

$$-\frac{\partial \Omega}{\partial t} = \frac{\partial}{\partial t}\underbrace{\int_A \vec{B} d\vec{A}}_{flux\ \Omega} = \oint_C \vec{E} \cdot d\vec{r}$$

➤ **Flux can be changed by:**

- **Change of magnetic field $\vec{B}$ with time t (e.g. transformers)**
- **Change of area A with time t (e.g. dynamos)**

Formally: Maxwell's third equation ...



$$-\int_A \frac{\partial \vec{B}}{\partial t} d\vec{A} = \underbrace{\int_A \nabla \times \vec{E} \; d\vec{A} = \oint_C \vec{E} \cdot d\vec{r}}_{Stoke's formula}$$

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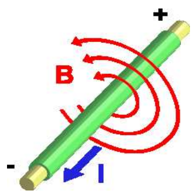
Changing magnetic field through an area induces electric field in coil around the area (Faraday)

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Remember: strong *curl* = strong circulating field

Maxwell's fourth equation (part 1) ...

From Ampere's law, for example current density $\vec{j}$:



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<u>Static</u> electric current induces circulating magnetic field

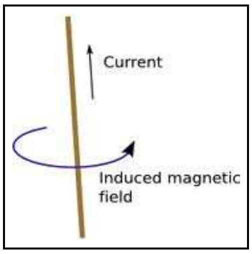
$$\nabla \times \vec{B} = \mu_0 \vec{j}$$

or in integral form the currect density becomes the current I:

$$\int\int_A \nabla \times \vec{B} \;\; d\vec{A} \;\; = \;\; \int\int_A \mu_0 \vec{j} \;\; d\vec{A} \;\; = \;\; \mu_0 \vec{I}$$

Maxwell's fourth equation - application

For a static electric current I in a single wire we get Biot-Savart law (we have used Stoke's theorem and area of a circle $A = r^2 \cdot \pi$):

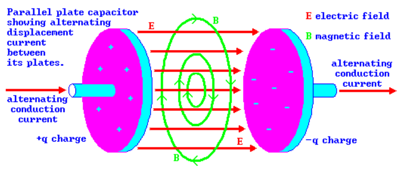


$$\vec{B} = \frac{\mu_0}{4\pi} \oint \vec{I} \cdot \frac{\vec{r} \cdot d\vec{r}}{r^3}$$
$$\vec{B} = \frac{\mu_0}{2\pi} \frac{\vec{I}}{r}$$

For magnetic field calculations in electromagnets

Maxwell's fourth equation (part 2)...

From displacement current, for example charging capacitor $\vec{j}_d$:



- Defining a Displacement Current $\vec{I}_d$:

  Not a current from moving charges

  But a current from time varying electric fields

Maxwell's fourth equation (part 2) ...

Displacement current I_d produces magnetic field, just like "actual currents" do ...

→ **Time varying electric field induce magnetic field (using the current density $\vec{j}_d$**

$$\nabla \times \vec{B} = \mu_0 \vec{j}_d = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

Remember: strong *curl* = strong circulating field

Maxwell's complete fourth equation ...

Magnetic fields $\vec{B}$ can be generated by two ways:

$\nabla \times \vec{B} = \mu_0 \vec{j}$ **(electrical current)**

$\nabla \times \vec{B} = \mu_0 \vec{j}_d = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$ **(changing electric field)**

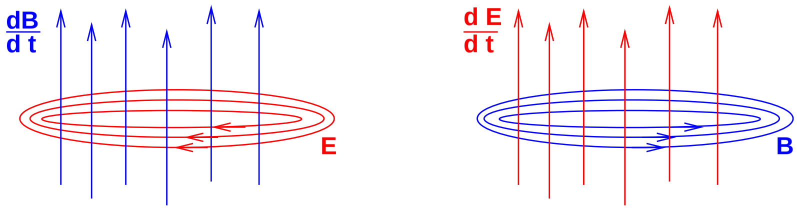
or putting them together:

$$\nabla \times \vec{B} = \mu_0 (\vec{j} + \vec{j}_d) = \mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

or in integral form (using Stoke's formula):

$$\underbrace{\oint_C \vec{B} \cdot d\vec{r} = \int_A \nabla \times \vec{B} \cdot d\vec{A}}_{Stoke's formula} = \int_A \left(\mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{A}$$

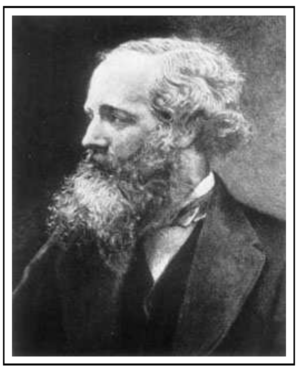
Summary: Static and Time Varying Fields



- Time varying magnetic fields produce circulating electric field: $\mathrm{curl}(\vec{E}) = \nabla \times \vec{E} = -\frac{d\vec{B}}{\partial t}$
- Time varying electric fields produce circulating magnetic field: $\mathrm{curl}(\vec{B}) = \nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{d\vec{E}}{\partial t}$

because of the $\times$ they are perpendicular: $\vec{E} \perp \vec{B}$

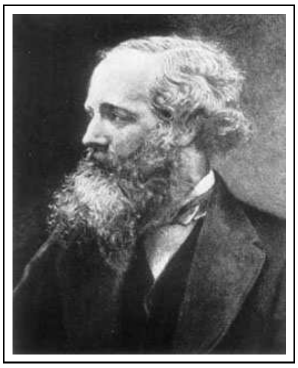
Summary: Maxwell's Equations



$$\int_A \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$
$$\int_A \vec{B} \cdot d\vec{A} = 0$$
$$\oint_C \vec{E} \cdot d\vec{r} = -\int_A \left(\frac{d\vec{B}}{dt}\right) \cdot d\vec{A}$$
$$\oint_C \vec{B} \cdot d\vec{r} = \int_A \left(\mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}\right) \cdot d\vec{A}$$

Written in Integral form

Summary: Maxwell's Equations



$$\nabla \vec{E} = \frac{\rho}{\epsilon_0}$$
$$\nabla \vec{B} = 0$$
$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$
$$\nabla \times \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}$$

Written in Differential form

Summary: Maxwell's Equations

1. **Electric fields $\vec{E}$ are generated by charges and proportional to total charge**
2. **Magnetic monopoles do <u>not</u> exist**
3. **Changing magnetic flux generates circulating electric fields/currents**

4.1 Changing electric flux generates circulating magnetic fields

4.2 Static electric current generates circulating magnetic fields

Written in Physical terms

Interlude and Warning !!

Maxwell's equation can be written in other forms.
Often used: cgs (Gaussian) units instead of SI units, example:

Starting from (SI):

$$\nabla \cdot \vec{E} \;=\; \frac{\rho}{\epsilon_0}$$

we would use:

$$\vec{E}_{cgs} \;=\; \tfrac{1}{c} \cdot \vec{E}_{SI} \qquad \text{and} \qquad \epsilon_0 \;=\; \tfrac{1}{4\pi \cdot c}$$

and arrive at (cgs):

$$\nabla \cdot \vec{E} \;=\; 4\pi \cdot \rho$$

Beware: there are more different units giving: $\nabla \cdot \vec{E} \;=\; \rho$

Electromagnetic fields in material

In vacuum:

$$\vec{D} = \epsilon_0 \cdot \vec{E}, \qquad \vec{B} = \mu_0 \cdot \vec{H}$$

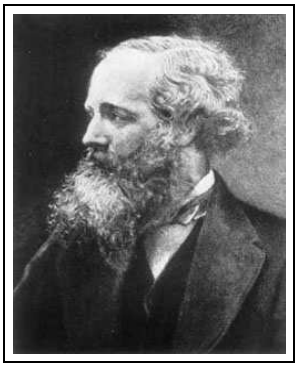
In a material:

$$\vec{D} = \epsilon_r \cdot \epsilon_0 \cdot \vec{E}, \qquad \vec{B} = \mu_r \cdot \mu_0 \cdot \vec{H}$$

ϵ_r **is relative permittivity** $\approx \; [1 - 10^5]$
μ_r **is relative permeability** $\approx \; [0(!) - 10^6]$

Origin: polarization and Magnetization

Once more: Maxwell's Equations



$$\nabla \vec{D} = \rho$$
$$\nabla \vec{B} = 0$$
$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$
$$\nabla \times \vec{H} = \vec{j} + \frac{d\vec{D}}{dt}$$

Re-factored in terms of the free current density $\vec{j}$ and free charge density ρ ($\mu_0 = 1, \epsilon_0 = 1$):

Applications of Maxwell's Equations

- **Lorentz force, motion in EM fields**
  - **- Motion in electric fields**
  - **- Motion in magnetic fields**
- **EM waves (in vacuum and in material)**
- **Boundary conditions**
- **EM waves in cavities and wave guides**

Lorentz force on charged particles

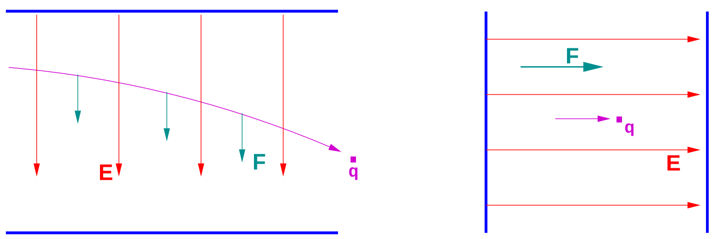
Moving ($\vec{v}$) charged (q) particles in electric ($\vec{E}$) and magnetic ($\vec{B}$) fields experience a force $\vec{f}$ like (Lorentz force):

$$\vec{f} = q \cdot (\vec{E} + \vec{v} \times \vec{B})$$

for the equation of motion we get (using Newton's law and relativistic γ);

$$\frac{d}{dt}(m_0 \gamma \vec{v}) = \vec{f} = q \cdot (\vec{E} + \vec{v} \times \vec{B})$$

Motion in electric fields



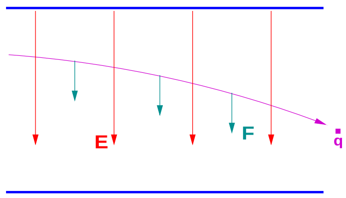
$\vec{v} \perp \vec{E}$ $\qquad$ $\vec{v} \parallel \vec{E}$

Assume no magnetic field:

$$\frac{d}{dt}(m_0 \gamma \vec{v}) = \vec{f} = q \cdot \vec{E}$$

Force always in direction of field $\vec{E}$, also for particles at rest.

Motion in electric fields



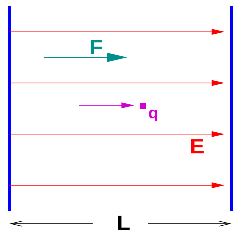
$$\frac{d}{dt}(m_0\gamma\vec{v}) \;=\; \vec{f} \;=\; q\cdot\vec{E}$$

The solution is:

$$\vec{v} \;=\; \frac{q\cdot\vec{E}}{\gamma\cdot m_0}\cdot t \quad\Rightarrow\quad \vec{x} \;=\; \frac{q\cdot\vec{E}}{\gamma\cdot m_0}\cdot t^2 \qquad \textbf{(parabola)}$$

Constant E-field deflects beams: TV, electrostatic separators (SPS,LEP)

Motion in electric fields



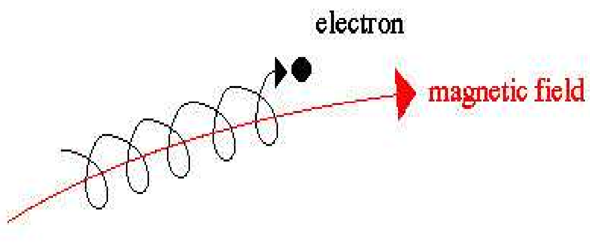
$$\frac{d}{dt}(m_0\gamma\vec{v}) \;=\; \vec{f} \;=\; q\cdot\vec{E}$$

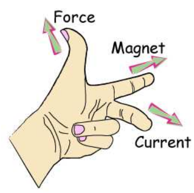
For constant field $\vec{E} = (E,0,0)$ in x-direction the energy gain is:

$$m_0c^2(\gamma-1) \;=\; qE\cdot L$$

It is a line integral of the force along the path !
Constant E-field gives uniform acceleration over length L

Motion in magnetic fields



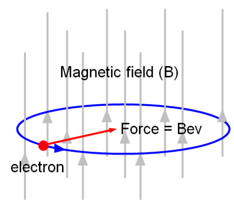


Assume first no electric field:

$$\frac{d}{dt}(m_0\gamma\vec{v}) = \vec{f} = q \cdot \vec{v} \times \vec{B}$$

Force is perpendicular to both, $\vec{v}$ and $\vec{B}$
No forces on particles at rest !
Particles will spiral around the magnetic field lines ...

Motion in magnetic fields



Assuming that $v_\perp$ is perpendicular to $\vec{B}$
We get a circular motion with radius ρ:

$$\rho = \frac{m_0\gamma v_\perp}{q \cdot B}$$

defines the Magnetic Rigidity: $B \cdot \rho = \frac{m_0\gamma v}{q} = \frac{p}{q}$

Magnetic fields deflect particles, but no acceleration (synchrotron, ..)

Motion in magnetic fields

Practical units:

$$B[T] \cdot \rho[m] \quad = \quad \frac{p[ev]}{c[m/s]}$$

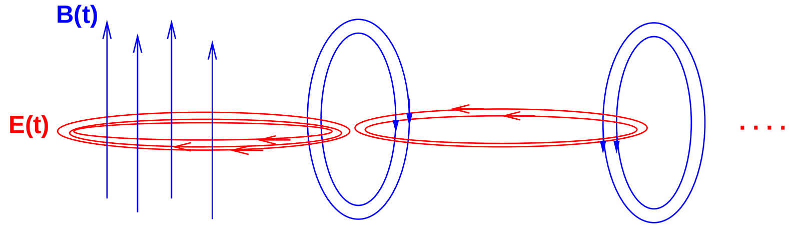
Example LHC:

B = 8.33 T, p = 7000 GeV/c → ρ **= 2804 m**

Use of static fields (some examples, incomplete)

- **Magnetic fields**
  - **Bending magnets**
  - **Focusing magnets (quadrupoles)**
  - **Correction magnets (sextupoles, octupoles, orbit correctors, ..)**
- **Electric fields**
  - **Electrostatic separators (beam separation in particle-antiparticle colliders)**
  - **Very low energy machines**
- **What about non-static, time-varying fields ?**

Time Varying Fields



Time varying magnetic fields produce circulating electric fields

Time varying electric fields produce circulating magnetic fields

→ Can produce self-sustaining, propagating fields (i.e. waves)

Electromagnetic waves in vacuum

Vacuum: only fields, no charges ($\rho = 0$), no current ($j = 0$) ...

From: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\begin{aligned} \Longrightarrow \quad \nabla \times (\nabla \times \vec{E}) &= -\nabla \times (\tfrac{\partial \vec{B}}{\partial t}) \\ \Longrightarrow \quad -(\nabla^2 \vec{E}) &= -\tfrac{\partial}{\partial t}(\nabla \times \vec{B}) \\ \Longrightarrow \quad -(\nabla^2 \vec{E}) &= -\mu_0 \epsilon_0 \tfrac{\partial^2 \vec{E}}{\partial t^2} \end{aligned}$$

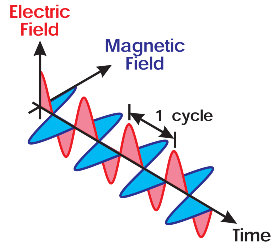
$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \cdot \epsilon_0 \cdot \frac{\partial^2 \vec{E}}{\partial t^2}$$

Similar expression for the magnetic field:

$$\nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = \mu_0 \cdot \epsilon_0 \cdot \frac{\partial^2 \vec{B}}{\partial t^2}$$

Equation for a plane wave with velocity: $c = \frac{1}{\sqrt{\mu_0 \cdot \epsilon_0}}$

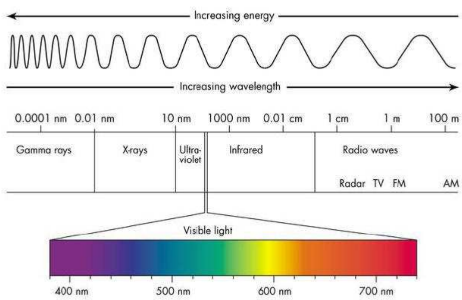
Electromagnetic waves



$$\vec{E} = \vec{E}_0 e^{i(\omega t - \vec{k}\cdot\vec{x})}$$
$$\vec{B} = \vec{B}_0 e^{i(\omega t - \vec{k}\cdot\vec{x})}$$
$$|\vec{k}| = \frac{2\pi}{\lambda} = \frac{\omega}{c} \quad \textbf{(propagation vector)}$$
$$\lambda = \textbf{(wave length, 1 cycle)}$$
$$\omega = (\textbf{frequency} \cdot 2\pi)$$

Magnetic and electric fields are transverse to direction of propagation: $\vec{E} \perp \vec{B} \perp \vec{k}$

Spectrum of Electromagnetic waves



Example: yellow light → $\approx 5 \cdot 10^{14}$ **Hz (i.e.** $\approx$ **2 eV !)**
gamma rays → $\leq 3 \cdot 10^{21}$ **Hz (i.e.** $\leq$ **12 MeV !)**
LEP (SR) → $\leq 2 \cdot 10^{20}$ **Hz (i.e.** $\approx$ **0.8 MeV !)**

Waves hitting material

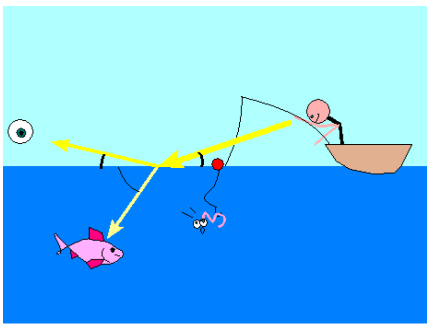
Need to look at the behaviour of electromagnetic fields at boundaries between different materials (air-glass, air-water, vacuum-metal, ...).

Important for highly conductive materials, e.g.:

- RF systems
- Wave guides
- Impedance calculations

Can be derived from Maxwell's equations, here only the results !

Observation: between air and water

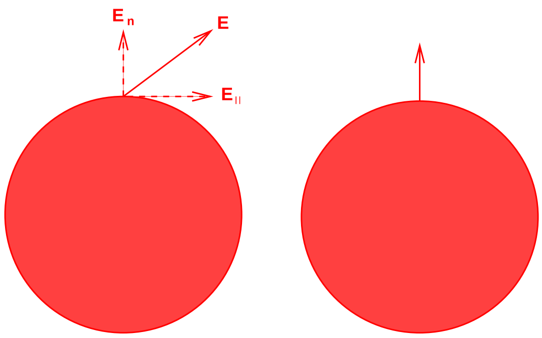


- Some of the light is reflected
- Some of the light is transmitted and refracted

→ Reason are boundary conditions for fields

Boundary conditions: air and conductor

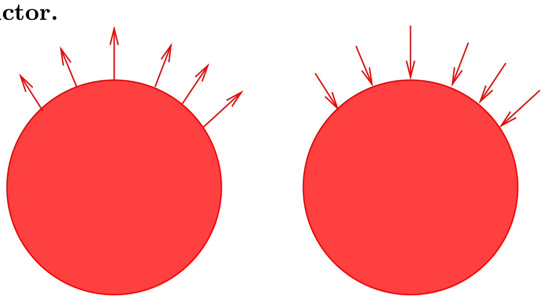
A simple case as demonstration ($\vec{E}$-fields on a conducting sphere):



- **Field parallel to surface $E_{\|}$ cannot exist (it would move charges and we get a surface current)**
- **Only field normal to surface E_n is possible**

Boundary conditions for fields

All electric field lines must be normal (perpendicular) to surface of a conductor.



- **All conditions for $\vec{E}, \vec{D}, \vec{H}, \vec{B}$ can be derived from Maxwell's equations (see bibliography, e.g. R.P.Feynman or J.D.Jackson)**

Boundary conditions for fields*

Electromagnetic fields at boundaries between different materials with different permittivity and permeability ($\epsilon^a, \epsilon^b, \mu^a, \mu^b$).

The requirements for the components are (summary of the results, not derived here !):

- $(E^a_\parallel \;=\; E^b_\parallel), (E^a_n \;\neq\; E^b_n)$
- $(D^a_\parallel \;\neq\; D^b_\parallel), (D^a_n \;=\; D^b_n)$
- $(H^a_\parallel \;=\; H^b_\parallel), (H^a_n \;\neq\; H^b_n)$
- $(B^a_\parallel \;\neq\; B^b_\parallel), (B^a_n \;=\; B^b_n)$

Conditions are used to compute reflection, refraction and refraction index n.

Extreme case: ideal conductor

For an ideal conductor (i.e. no resistance) the tangential electric field must vanish, otherwise a surface current becomes infinite. Similar conditions for magnetic fields. We must have:

$$\vec{E}_\parallel \;=\; 0, \qquad \vec{B}_n \;=\; 0$$

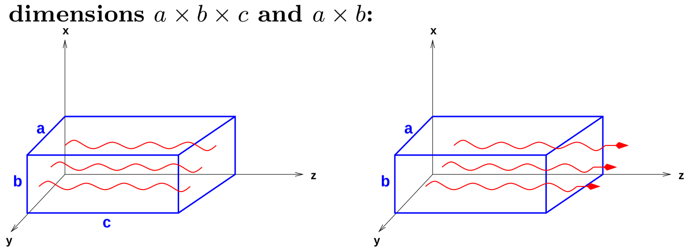
This implies:

- All energy of an electromagnetic wave is reflected from the surface.
- Fields at any point in the conductor are zero.
- Only some fieldpatterns are allowed in waveguides and RF cavities

A very nice lecture in R.P.Feynman, Vol. II

Examples: cavities and wave guides

Rectangular, conducting cavities and wave guides (schematic) with dimensions $a \times b \times c$ and $a \times b$:



- **RF cavity, fields can persist and be stored (reflection !)**
- **Plane waves can propagate along wave guides, here in z-direction**

Fields in RF cavities

Assume a rectangular RF cavity ($a,\ b,\ c$), ideal conductor.

Without derivations, the components of the fields are:

$$E_x = E_{x0} \cdot cos(k_x x) \cdot sin(k_y y) \cdot sin(k_z z) \cdot e^{-i\omega t}$$
$$E_y = E_{y0} \cdot sin(k_x x) \cdot cos(k_y y) \cdot sin(k_z z) \cdot e^{-i\omega t}$$
$$E_z = E_{z0} \cdot sin(k_x x) \cdot sin(k_y y) \cdot cos(k_z z) \cdot e^{-i\omega t}$$

$$B_x = \frac{i}{\omega}(E_{y0}k_z - E_{z0}k_y) \cdot sin(k_x x) \cdot cos(k_y y) \cdot cos(k_z z) \cdot e^{-i\omega t}$$
$$B_y = \frac{i}{\omega}(E_{z0}k_x - E_{x0}k_z) \cdot cos(k_x x) \cdot sin(k_y y) \cdot cos(k_z z) \cdot e^{-i\omega t}$$
$$B_z = \frac{i}{\omega}(E_{x0}k_y - E_{y0}k_x) \cdot cos(k_x x) \cdot cos(k_y y) \cdot sin(k_z z) \cdot e^{-i\omega t}$$

Consequences for RF cavities

Field must be zero at conductor boundary, only possible under the condition:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

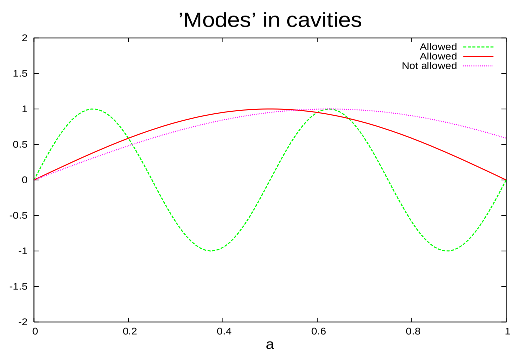
and for k_x, k_y, k_z we can write:

$$k_x = \frac{m_x \pi}{a}, \qquad k_y = \frac{m_y \pi}{b}, \qquad k_z = \frac{m_z \pi}{c},$$

The integer numbers m_x, m_y, m_z are called mode numbers, important for shape of cavity !

It means that a half wave length $\lambda/2$ must always fit exactly the size of the cavity.

Allowed modes



- **Only modes which 'fit' into the cavity are allowed**
- $\frac{\lambda}{2} = \frac{a}{4}$, $\frac{\lambda}{2} = \frac{a}{1}$, $\frac{\lambda}{2} = \frac{a}{0.8}$
- **No electric field at boundaries**

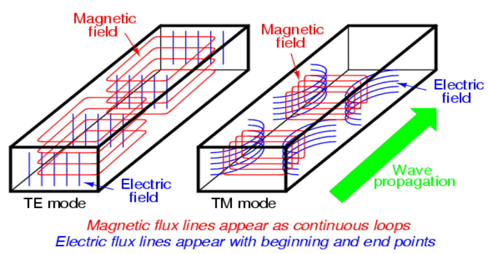
Fields in wave guides

Similar considerations lead to (propagating) solutions in (rectangular) wave guides:

$$E_x = E_{x0} \cdot cos(k_x x) \cdot sin(k_y y) \cdot e^{-i(k_z z - \omega t)}$$
$$E_y = E_{y0} \cdot sin(k_x x) \cdot cos(k_y y) \cdot e^{-i(k_z z - \omega t)}$$
$$E_z = i \cdot E_{z0} \cdot sin(k_x x) \cdot sin(k_y y) \cdot e^{-i(k_z z - \omega t)}$$

$$B_x = \frac{1}{\omega}(E_{y0}k_z - E_{z0}k_y) \cdot sin(k_x x) \cdot cos(k_y y) \cdot e^{-i(k_z z - \omega t)}$$
$$B_y = \frac{1}{\omega}(E_{z0}k_x - E_{x0}k_z) \cdot cos(k_x x) \cdot sin(k_y y) \cdot e^{-i(k_z z - \omega t)}$$
$$B_z = \frac{1}{i \cdot \omega}(E_{x0}k_y - E_{y0}k_x) \cdot cos(k_x x) \cdot cos(k_y y) \cdot e^{-i(k_z z - \omega t)}$$

The fields in wave guides



- **Electric and magnetic fields through a wave guide**
- **Shapes are consequences of boundary conditions !**
- **Can be Transverse Electric (TE, no E-field in z-direction) or Transverse Magnetic (TM, no B-field in z-direction)**

Consequences for wave guides

Similar considerations as for cavities, no field at boundary.
We must satisfy again the condition:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

This leads to modes like:

$$k_x = \frac{m_x \pi}{a}, \qquad k_y = \frac{m_y \pi}{b},$$

The numbers m_x, m_y are called mode numbers for planar waves in wave guides !

Consequences for wave guides

Re-writing the condition as:

$$k_z^2 = \frac{\omega^2}{c^2} - k_x^2 - k_y^2$$

Propagation without losses requires k_z to be real, i.e.:

$$\frac{\omega^2}{c^2} > k_x^2 + k_y^2 = (\frac{m_x \pi}{a})^2 + (\frac{m_y \pi}{b})^2$$

which defines a cut-off frequency ω_c.

- **Above cut-off frequency: propagation without loss**
- **Below cut-off frequency: attenuated wave (means it does not "really fit" and k is complex).**

Done ...

- Review of basics and Maxwell's equations
- Lorentz force
- Motion of particles in electromagnetic fields
- Electromagnetic waves in vacuum
- Electromagnetic waves in conducting media
  - Waves in RF cavities
  - Waves in wave guides

- BACKUP SLIDES -

Some popular confusion ..

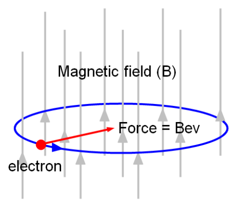
V.F.A.Q: why this strange mixture of $\vec{E}, \vec{D}, \vec{B}, \vec{H}$??

Materials respond to an applied electric E field and an applied magnetic B field by producing their own internal charge and current distributions, contributing to E and B. Therefore H and D fields are used to re-factor Maxwell's equations in terms of the free current density $\vec{j}$ and free charge density ρ:

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$
$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

$\vec{M}$ and $\vec{P}$ are $Magnetization$ and $Polarisation$ in material

Is that the full truth ?



If we have a circulating E-field along the circle of radius R ?
➡ should get acceleration !
Remember Maxwell's third equation:

$$\oint_C \vec{E} \cdot d\vec{r} = -\frac{d}{dt} \int_A \vec{B} \cdot d\vec{A}$$

$$\Rightarrow 2\pi R E_\theta = -\frac{d\Phi}{dt}$$

Motion in magnetic fields

- This is the principle of a Betatron
  - Time varying magnetic field creates circular electric field !
  - Time varying magnetic field deflects the charge !

For a constant radius we need:

$$-\frac{m \cdot v^2}{R} = e \cdot v \cdot B \quad \rightarrow \quad B = -\frac{p}{e \cdot R}$$

$$\frac{\partial}{\partial t} B(r,t) = -\frac{1}{e \cdot R}\frac{dp}{dt}$$

$$\rightarrow \quad B(r,t) = \frac{1}{2}\frac{1}{\pi R^2}\int\int B dS$$

B-field on orbit must be half the average over the circle

→ **Betatron condition**

Other case: finite conductivity

Assume conductor with finite conductivity ($\sigma_c = \rho_c^{-1}$) , waves will penetrate into surface. Order of the skin depth is:

$$\delta_s = \sqrt{\frac{2\rho_c}{\mu\omega}}$$

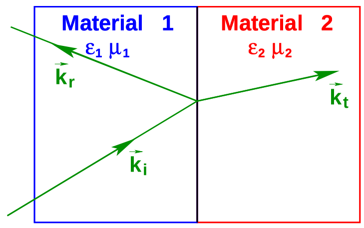
i.e. depend on resistivity, permeability and frequency of the waves (ω).

We can get the surface impedance as:

$$Z = \sqrt{\frac{\mu}{\epsilon}} = \frac{\mu\omega}{k}$$

the latter follows from our definition of k and speed of light. Since the wave vector k is complex, the impedance is also complex. We get a phase shift between electric and magnetic field.

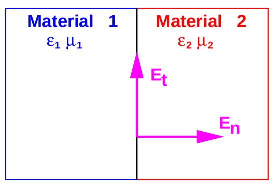
Boundary conditions for fields

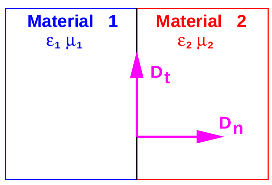


What happens when an incident wave $(\vec{K}_i)$ encounters a boundary between two different media ?

- Part of the wave will be reflected $(\vec{K}_r)$, part is transmitted $(\vec{K}_t)$
- What happens to the electric and magnetic fields ?

Boundary conditions for fields

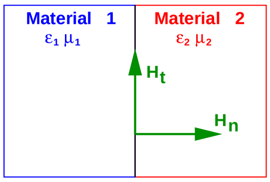


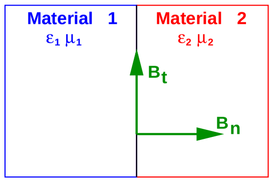


Assuming no surface charges:

- tangential $\vec{E}$-field constant across boundary $(E_{1t} = E_{2t})$
- normal $\vec{D}$-field constant across boundary $(D_{1n} = D_{2n})$

Boundary conditions for fields





Assuming no surface currents:

- **tangential $\vec{H}$-field constant across boundary** $(H_{1t} = H_{2t})$
- **normal $\vec{B}$-field constant across boundary** $(B_{1n} = B_{2n})$