

Special relativity

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(http://cern.ch/Werner.Herr/CAS2014_Chavannes/rel.pdf)



Why Special Relativity ?

- Most beams at CERN are relativistic
- Strong implications for beam dynamics:
 - Transverse dynamics (e.g. momentum compaction, radiation, ...)
 - Longitudinal dynamics (e.g. transition, ...)
 - Collective effects (e.g. space charge, beam-beam, ...)
 - Luminosity in colliders
 - Particle lifetime and decay (e.g. μ , π , Z_0 , Higgs, ...)

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Small history

- 1678 (Römer, Huygens): Speed of light c is finite ($c \approx 3 \cdot 10^8$ m/s)
- 1630-1687 (Galilei, Newton): **Principles of Relativity**
- 1863 (Maxwell): Electromagnetic theory, light are waves moving through static ether with speed c
- 1887 (Michelson, Morley): Speed c independent of direction, \rightarrow no ether
- 1892 (Lorentz, FitzGerald, Poincaré): **Lorentz transformations, Lorentz contraction**
- 1897 (Larmor): **Time dilation**
- 1905 (Einstein): **Principles of Special Relativity**
- 1907 (Einstein, Minkowski): Concepts of Spacetime

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OUTLINE

- Principle of Relativity (Newton, Galilei)
 - Motivation and Idea
 - Formalism, examples
- Principle of Special Relativity (Einstein)
 - Why ?
 - Formalism and consequences
 - Four-vectors and applications (accelerators)

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Mathematical derivations and proofs are mostly avoided ...

Reading Material

- A. Einstein, Zur Elektrodynamik bewegter Körper, Ann. Phys. 17, (1905).
- R.P. Feynman, Feynman lectures on Physics, Vol. 1 + 2, (Basic Books, 2011).
- R.P. Feynman, Six not-so-easy pieces, (Basic Books, 2011).
- J. Freund, Special Relativity, (World Scientific, 2008).
- J. Hafele and R. Keating, Science 177, (1972) 166.

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Setting the scene ..

Where we describe physics and physics laws:

- In space coordinates: $\vec{x} = (x, y, z)$
- In time: t

A "Frame":

- Where we observe physical phenomena and properties as function of their position \vec{x} and moment in time t .
- In different frames \vec{x} and t are usually different.

An "Event":

- Something happening at \vec{x} at time t is an "event", given by four numbers $(x, y, z), t$

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Setting the scene ..

Assuming two frames F and F' :

- Event described in F using: $\vec{x} = (x, y, z)$ and t
- Event described in F' using: $\vec{x}' = (x', y', z')$ and t'

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Principles of Relativity (Newton)

We always observe and describe physics in a certain Frame

Laws of physics should be **invariant**

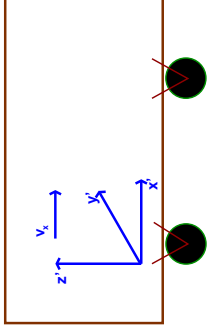
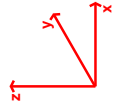
i.e. the same when we change the frame, for example:

- Frame displaced in Space
- Frame displaced in Time
- Frame moving at constant speed, an Inertial System
- More formal: "Physical laws have the same form in all inertial systems, they are invariant"

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Principles of Relativity (Galilei, Newton)

Assume a frame at rest (F) and another frame (F') moving in x -direction with constant velocity $\vec{v} = (v_x, 0, 0)$



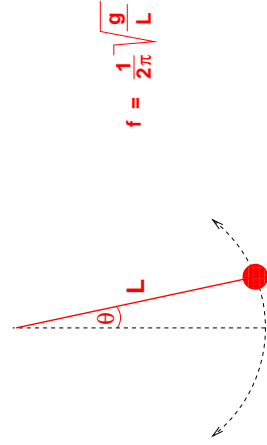
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Example: we would like to have

$$Force = m \cdot a \quad \text{and} \quad Force' = m \cdot a'$$

(Mass m is the same in all frames)

Example: Pendulum

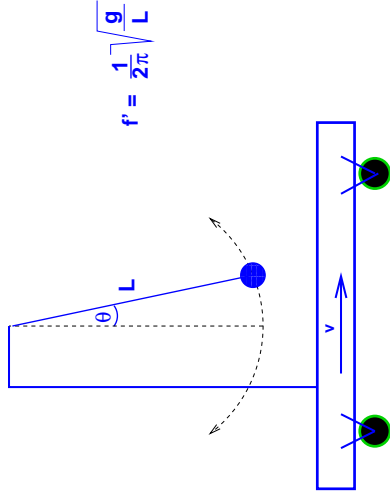


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Motion described by coordinate Θ

Frequency of a pendulum is f

Example: Pendulum



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Motion described by coordinate Θ

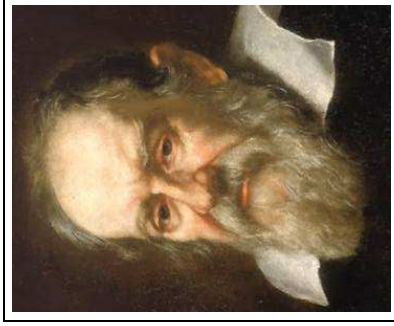
Frequency of a pendulum the same in all inertial systems: $f = f'$

How do we describe invariant laws ?

- We have described an experiment in rest frame F
- How can we describe it seen from a moving frame F' ?
- Need to transform coordinates (x, y, z) and time t to describe (translate) results of measurements and observations to the moving system (x', y', z') and t' .
- For Newton's principle of relativity need Galilei transformation for:
 (x, y, z) and t \rightarrow (x', y', z') and t' .
- Then laws should look the same, have the same form

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Galilei transformation



$$\begin{aligned}x' &= x - v_x t \\y' &= y \\z' &= z \\t' &= t\end{aligned}$$

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Consequences of Galilei transformation

Velocity and transformation in x -coordinates.
Only for simplicity, can always rotate the coordinate system.

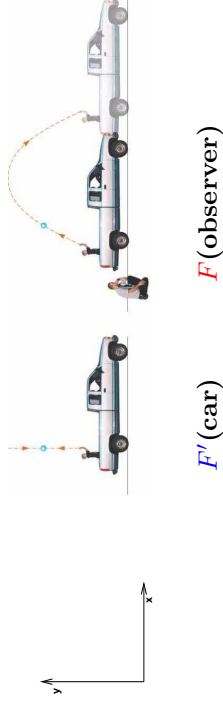
For Galilei transformation:

Space and time are independent quantities

Space coordinates are changed, time is not changed !

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Illustration of Galilei transformation:



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- In car frame F' moving with speed v_x :
Ball starts with vertical velocity $v'_y = v'_0$
Ball goes up and down
- In rest frame for observer F : ball describes a curve (parabola ?)

Illustration of Galilei transformation:

Equation of motion in moving frame $x'(t')$ and $y'(t')$:

$$x'(t') = 0, \quad v'_y(t') = v'_0 - g \cdot t'$$
$$y'(t') = \int v'_y(t') dt' = v'_0 \cdot t' - \frac{1}{2} g t'^2$$

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Illustration of Galilei transformation:

From moving frame:

$$y'(t') = v'_0 \cdot t' - \frac{1}{2}gt'^2$$

To get equation of motion in rest frame $x(t)$ and $y(t)$:

Galilei transform: $y(t) \equiv y'(t')$, $t \equiv t'$, $x(t) = x' + v_x \cdot t = v_x \cdot t$

To get y as function of x we can re-write:

$$\rightarrow t' = t = \frac{x}{v_x}$$

and get for the trajectory in the rest frame:

$$y(x) = \frac{v_0}{v_x} \cdot x - \frac{1}{2}g\frac{x^2}{v_x^2}$$

This is a parabola, observed from the rest frame.

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Consequences of Galilei transformation

Velocities can be added

- From Galilei transformation, take derivative:

$$x' = x - v_x t$$

$$\dot{x}' = \dot{x} - v_x \quad \rightarrow \quad v' = v - v_x$$

- A car moving with speed v' in a frame moving with speed v_x we have in rest frame $v = v' + v_x$

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Problems with Galilei transformation

- Maxwell's equations are different when Galilei transformations are applied (because they predict the speed of light)
- Could exceed it when velocities are added:
 $0.8 \cdot c + 0.5 \cdot c = 1.3 \cdot c$?
 $c = 299792458,000 \text{ m/s}$
- From experiments: Speed of light is upper limit and the same in all frames and all directions
- Enter Einstein: principles of special relativity

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Principle(s) of Special Relativity (Einstein)

All physical laws (e.g. Maxwell's) in inertial frames must have equivalent forms, in particular:

Speed of light c must be the same in all frames

- Cannot distinguish between inertial frames by measuring speed of light:
 - Cannot determine absolute speed of an inertial frame
 - No absolute space, no absolute time
- Need **Transformations** (not Galilean) which make the physics laws (Maxwell !) look the same !

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Coordinates must be transformed differently

- Transformation must keep speed of light constant

Constant speed of light requires:

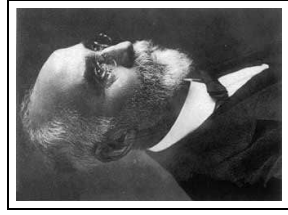
$$x^2 + y^2 + z^2 - c^2t^2 = 0 \quad \rightarrow \quad x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$$

(front of a light wave)

- To fulfill this condition: Time must be changed by transformation as well as space coordinates
- Transform $(x, y, z), t \rightarrow (x', y', z'), t'$
- Defines the Lorentz transformation

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Lorentz transformation



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$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma \cdot (x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{v \cdot x}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma \cdot \left(t - \frac{v \cdot x}{c^2}\right)$$

Transformation for constant velocity v along x-axis

Time is now also transformed

Definitions: relativistic factors

$$\beta_r = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{(1 - \frac{v^2}{c^2})}} = \frac{1}{\sqrt{(1 - \beta_r^2)}}$$

β_r relativistic speed: $\beta_r = [0, 1]$

γ relativistic factor: $\gamma = [1, \infty]$

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(unfortunately, you will also see other β and γ ... !)

Consequences of Einstein's interpretation

- Space and time and NOT independent quantities
- Time has no absolute meaning
- Relativistic phenomena:
 - ▶ Velocities cannot exceed speed of light
 - ▶ (Non-) Simultaneity of events in independent frames
 - ▶ Lorentz contraction
 - ▶ Time dilation
- Formalism with four-vectors introduced (see later)

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Addition of velocities

Galilei: $v = v_1 + v_2$

With Lorentz transform we have:

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

for $v_1, v_2, v_3, \dots = 0.5c$ we get:

$$0.5c + 0.5c = 0.8c$$

$$0.5c + 0.5c + 0.5c = 0.93c$$

$$0.5c + 0.5c + 0.5c + 0.5c = 0.976c$$

$$0.5c + 0.5c + 0.5c + 0.5c + 0.5c = 0.992c$$

→ Speed of light can never be exceeded by adding velocities !

Special case: $0.5c + 1.0c = 1.0c$

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- Simultaneity -

Simultaneity between moving frames (the least intuitive concept)

- Assume two events in frame F at positions x_1 and x_2 happen simultaneously at times $t_1 = t_2$:

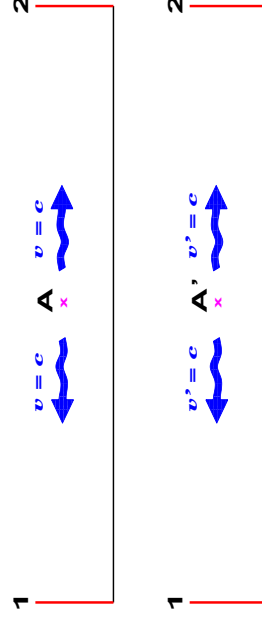
$$t'_1 = \frac{t_1 - \frac{v \cdot x_1}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{and} \quad t'_2 = \frac{t_2 - \frac{v \cdot x_2}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$x_1 \neq x_2$ in F implies that $t'_1 \neq t'_2$ in frame F' !!

- Two events simultaneous at positions x_1 and x_2 in F are not simultaneous in F'

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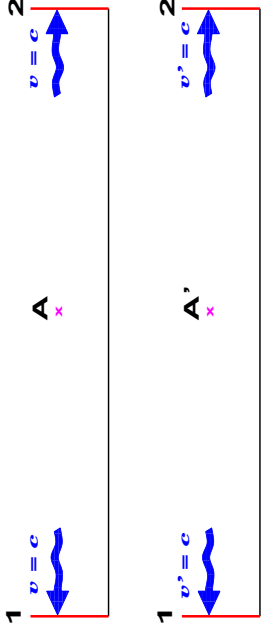
Simultaneity between moving frames



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- System with a light source (x) and detectors (1, 2) and one observer (A) in this frame, another (A') outside
- System at rest → observation the same in A and A'

Simultaneity between moving frames



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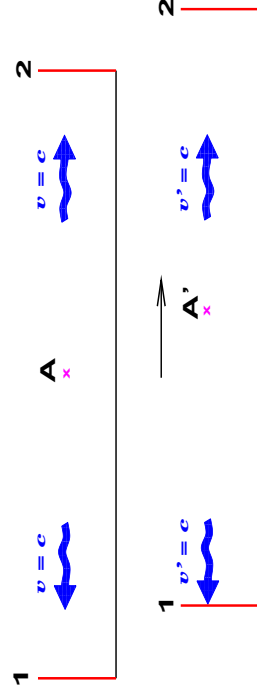
Speed of light is the same in both frames, (no adding of speeds):

For A: both flashes arrive simultaneously in 1,2

For A': both flashes arrive simultaneously in 1,2

What if the frame is moving relative to A' ?

Simultaneity between moving frames



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Speed of light is the same in both frames, (no adding of speeds):

For A: both flashes arrive simultaneously in 1,2

For A': flash arrives first in 1, later in 2

A simultaneous event in F is not simultaneous in F'

Why do we care ??

Why care about simultaneity ?

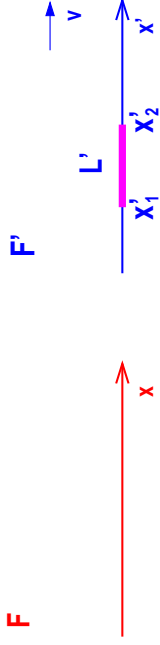
- Simultaneity is **not** frame independent
- This is a key in special relativity
- Most paradoxes are explained by that (although not the twin paradox) !
- Different observers see a different reality, in particular the sequence of events can change !
 - For $t_1 < t_2$ we may find (not always !) a frame where $t_1 > t_2$ (concept of **before** and **after** depends on the observer)

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- Lorentz contraction -

Consequences: length measurement

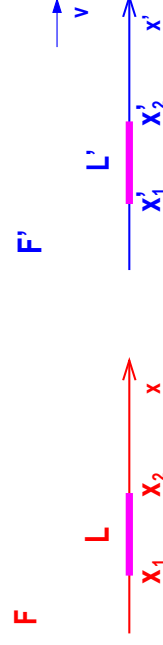


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How to measure the length of an object ?

Have to measure position of both ends simultaneously !
 Length of a rod in F' is $L' = x'_2 - x'_1$, measured simultaneously at a fixed time t' in frame F' , what is the length L seen in F ??

Consequences: length measurement



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We have to measure simultaneously (!) the ends of the rod at a fixed time t in frame F , i.e.: $L = x_2 - x_1$ \rightarrow

$$x'_1 = \gamma \cdot (x_1 - vt) \quad \text{and} \quad x'_2 = \gamma \cdot (x_2 - vt)$$

$$L' = x'_2 - x'_1 = \gamma \cdot (x_2 - x_1) = \gamma \cdot L$$

$$\rightarrow L = L' / \gamma$$

Lorentz contraction

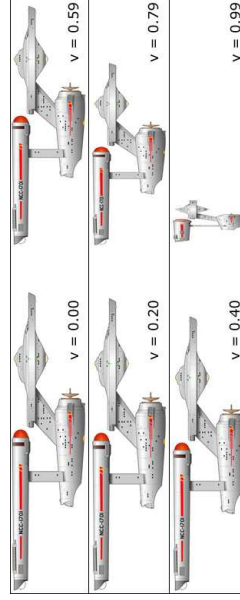
- In moving frame an object has always the same length (it is invariant, our principle !)
- From stationary frame moving objects appear contracted by a factor γ (Lorentz contraction)
- Why do we care ?
- Turn the argument around: assume length of a proton bunch appears always at 0.1 m in laboratory frame (e.g. in the RF bucket), what is the length in its own (moving) frame ?

➤ At 5 GeV ($\gamma \approx 5.3$) $\rightarrow L' = 0.53$ m

➤ At 450 GeV ($\gamma \approx 480$) $\rightarrow L' = 48.0$ m

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Lorentz contraction - schematic



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Spaceship seen from earth

➤ Appears shorter at higher velocities

Lorentz contraction - schematic



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Earth seen from spaceship

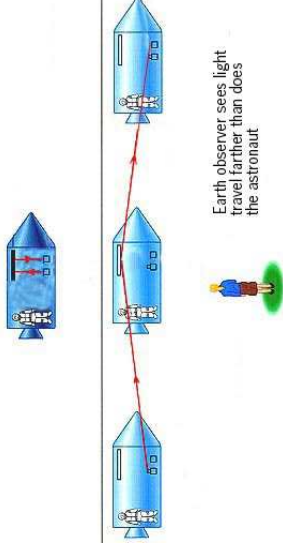
- ↑ Both observers see the other object contracted
- ↑ No inertial frame is privileged

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- Time dilation -

Time dilation - the dilemma

Reflection of light between 2 mirrors seen in rocket and earth



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Does longer travel for the same time mean that c is different ?

Time dilation - derivation

A clock measures time differences:

$$\Delta t = t_2 - t_1 \text{ in frame F}$$
$$\Delta t' = t'_2 - t'_1 \text{ in frame F'}$$

For Lorentz transformation of time in moving frame to rest frame we have:

$$t'_1 = \gamma \left(t_1 - \frac{v \cdot x}{c^2} \right) \quad \text{and} \quad t'_2 = \gamma \left(t_2 - \frac{v \cdot x}{c^2} \right)$$

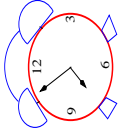
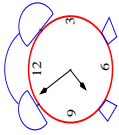
$$\Delta t' = t'_2 - t'_1 = \gamma \cdot (t_2 - t_1) = \gamma \cdot \Delta t$$

$$\Delta t' = \gamma \Delta t$$

Seen from the rest frame: time in moving frame goes slower ..

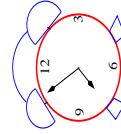
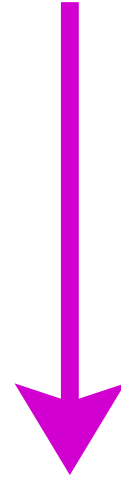
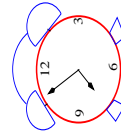
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Moving clocks go slower



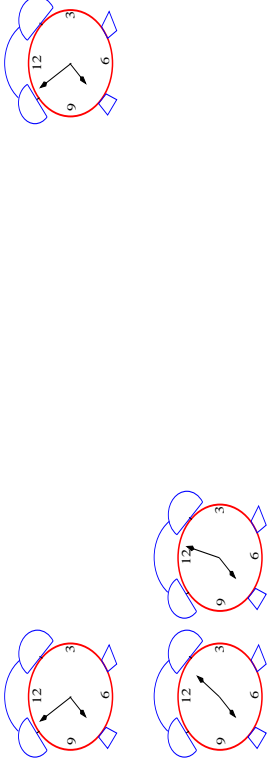
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Moving clocks go slower



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Ten minutes later ...

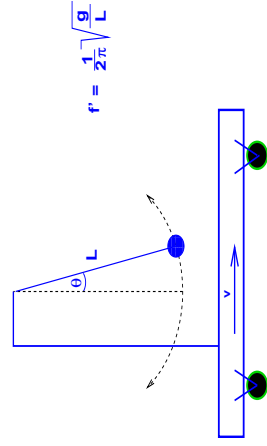


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Travel by airplane:

On a flight from Montreal to Geneva, the time is slower by 25 - 30 ns (considering only special relativity) !

Remember the pendulum ?



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From the outside observer:

Galilei: frequency f the same in all inertial systems

Einstein: frequency f' smaller by factor γ (seen from rest system)

Time dilation

In moving frame time appears to run slower

Why do we care ?

- μ have lifetime of $2 \mu\text{s}$ ($\equiv 600 \text{ m}$)
- For $\gamma \geq 150$, they survive 100 km to reach earth from upper atmosphere
- They can survive more than $2 \mu\text{s}$ in a μ -collider
- Generation of neutrinos from the SPS beams

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Proper Length and Proper Time

Time and distances are relative :

- τ is a fundamental time: **proper time** τ
- The time measured by an observer in its **own** frame
- From frames moving relative to it, time appears longer

- \mathcal{L} is a fundamental length: **proper length** \mathcal{L}
- The length measured by an observer in its **own** frame
- From other frames it appears shorter

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The importance of "proper time"

$\Delta\tau$ is the time interval measured inside the moving frame

Back to μ -decay

- ▶ μ lifetime is $\approx 2 \mu\text{s}$
- ▶ μ decay in $\approx 2 \mu\text{s}$ in their frame, i.e. using the "proper time"
- ▶ μ decay in $\approx \gamma \cdot 2 \mu\text{s}$ in the laboratory frame, i.e. earth
- ▶ μ appear to exist longer than $2 \mu\text{s}$ in the laboratory frame, i.e. earth

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The meaning of "proper time"

How to make neutrinos (e.g. CNGS) ??

- Let pions decay: $\pi \rightarrow \mu + \nu_\mu$
- π -mesons have lifetime of $2.6 \cdot 10^{-8}$ s (i.e. 7.8 m)
- For 40 GeV π -mesons: $\gamma = 288$
- In laboratory frame: decay length is 2.25 km (required length of decay tunnel)

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First summary

- Physics laws the same in different moving frames ...
- Speed of light c is maximum possible speed and constant
- Constant speed of light requires Lorentz transformation
- Moving objects appear shorter
- Moving clocks seem to go slower
- No absolute space or time: **where** it happens and **when** it happens is not independent
- ➔ Next: how to calculate something and applications ...

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Introducing four-vectors

Since space and time are not independent, must reformulate physics taking both into account:

Separated time and space (Euclidean space):

$$t, \quad \vec{a} = (x, y, z)$$

Replace by vector including the time (Minkowski space):

$$A = (ct, x, y, z)$$

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(time t multiplied by c to get the same units)

This is a position four-vector, you also find a^μ instead of A

Definitions of four-vectors

Not a unique definition in literature, one can find:

$$\begin{aligned} &= (ct, x, y, z) \\ &= (ct, -x, -y, -z) \\ &= (x, y, z, ct) \\ &= (-x, -y, -z, ct) \\ &= (ict, x, y, z) \\ &= (-ict, x, y, z) \\ &= (\dots) \end{aligned}$$

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Always define them when you use them !

With four-vectors, Lorentz transformation can be written in a compact form with matrix multiplication:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

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Nota bene: this matrix is also used for Lorentz transformation of fields, Lorentz force, derivatives, etc. ...

Scalar products revisited

Define a scalar product for (usual) vectors like: $\vec{a} \cdot \vec{b}$,

$$\vec{a} = (x_a, y_a, z_a) \quad \vec{b} = (x_b, y_b, z_b)$$

Standard definition (Euclidean geometry):

$$\vec{a} \cdot \vec{b} = (x_a, y_a, z_a) \cdot (x_b, y_b, z_b) = x_a \cdot x_b + y_a \cdot y_b + z_a \cdot z_b$$

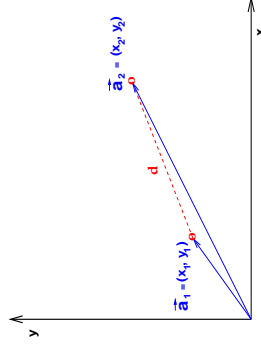
This product of two vectors is a scalar (number) not a vector, and it has a meaning:

$$\vec{a} \cdot \vec{a} = (x_a, y_a, z_a) \cdot (x_a, y_a, z_a) = x_a \cdot x_a + y_a \cdot y_a + z_a \cdot z_a = d^2$$

d is the length of the vector \vec{a} !

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More: distance between events in space



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Distance between two points (here in 2D): d

$$d^2 = (\vec{a}_2 - \vec{a}_1)^2$$

$$d^2 = (x_2 - x_1, y_2 - y_1)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Distance d is always positive !

Scalar products for four-vectors

Define a scalar product for four-vectors like: $A \odot B$

$$A = (ct_a, x_a, y_a, z_a) \quad B = (ct_b, x_b, y_b, z_b)$$

$$A \odot B = ct_a \cdot ct_b - \vec{a} \cdot \vec{b} = (ct_a \cdot ct_b - x_a \cdot x_b - y_a \cdot y_b - z_a \cdot z_b)$$

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Note the **-** sign !!

Does it have a meaning ?

Distance between events in space-time

We can describe a **distance** in the space-time between two points A_1 and A_2 :

$$\Delta X = A_2 - A_1 = (ct_2 - ct_1, x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

Scalar product of the difference is the distance² = D^2 :

$$D^2 = \Delta A^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

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D^2 can be positive (time-like) or negative (space-like)

Another example: $X = (ct, \vec{x})$, $X' = (ct', \vec{x}')$

$$X \odot X = c^2 t^2 - x^2 - y^2 - z^2$$

and

$$X' \odot X' = c^2 t'^2 - x'^2 - y'^2 - z'^2$$

we have:

$$X \odot X = X' \odot X'$$

because this is our condition for constant speed of light c !

This product is an **invariant**

Invariant Quantities have the **same value** in all inertial frames (like c)

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Why bother about four-vectors ?

We have seen the importance of **invariants**:

- Ensure equivalence of physics laws in different frames
- The solution: write the laws of physics in terms of **four vectors**
- Without proof: any four-vector (scalar) product $F \odot F$ has the same value in all coordinate frames moving at constant velocities with respect to each other:

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$$F \odot F = F' \odot F'$$

Scalar products of four-vectors are invariant !

We have important four-vectors:

$$\begin{aligned} \text{Coordinates: } X &= (ct, x, y, z) = (ct, \vec{x}) \\ \text{Velocities: } V &= \frac{dX}{dt} \cdot \frac{dt}{dt} = \gamma(c, \vec{v}) = \gamma(c, \vec{v}) \\ \text{Momenta: } P &= mV = m\gamma(c, \vec{v}) = \gamma(mc, \vec{p}) \\ \text{Force: } F &= \frac{dP}{dt} = \gamma \frac{d}{dt}(mc, \vec{p}) \end{aligned}$$

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Any scalar product of two four-vectors:

$X \odot X, V \odot V, P \odot P, P \odot X, V \odot F, \dots$ are ALL invariants

A special invariant

From the velocity four-vector V :

$$V = \gamma(c, \vec{v})$$

we get the scalar product:

$$V \odot V = \gamma^2(c^2 - v^2) = c^2 !!$$

→ c is an invariant, has the same value in all inertial frames

$$V \odot V = V' \odot V'$$

→ The invariant of the velocity four-vector V is the speed of light c !

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Transformation of mass

We want the invariance of the formula:

$$\vec{F} = m \cdot \vec{a} = m \cdot \frac{d\vec{v}}{dt}$$

Without proof (see e.g. Feynman) momentum conservation in all directions requires that the mass m must also be transformed:

$$m = m' / \sqrt{1 - \left(\frac{v}{c}\right)^2} = \gamma \cdot m'$$

In a frame with $v = 0$ we call the mass the rest mass m_0 . If the frame moves with a velocity v relative to an observer, she will find the mass increased by the factor γ

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Dynamics with four-vectors

Using the expression for the mass m :

$$m = m_0 \sqrt{1 - \left(\frac{v}{c}\right)^2} = \gamma \cdot m_0$$

and expand it for small velocities:

$$m \cong m_0 + \frac{1}{2} m_0 v^2 \left(\frac{1}{c^2}\right)$$

and multiplied by c^2 :

$$m c^2 \cong m_0 c^2 + \frac{1}{2} m_0 v^2$$

The second term is the kinetic energy

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Relativistic energy

Interpretation:

$$E = mc^2 = m_0c^2 + T$$

- Total energy E is $E = mc^2$
- Sum of kinetic energy plus rest energy
- Energy of particle at rest is $E_0 = m_0c^2$

$$E = m \cdot c^2 = \gamma m_0 \cdot c^2$$

using the definition of relativistic mass again: $m = \gamma m_0$

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Interpretation of relativistic energy

- For any object, $m \cdot c^2$ is the total energy
- Object can be composite, like proton ..
- m is the mass (energy) of the object "in motion"
- m_0 is the mass (energy) of the object "at rest"
- The mass m is not the same in all inertial systems, the rest mass m_0 is ! (To prove it, try $P \odot P'$)
- For discussion: what is the mass m of a photon ?

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Practical units

Standard units are not very convenient, easier to use:

$$[E] = \text{eV} \quad [p] = \text{eV}/c \quad [m] = \text{eV}/c^2$$

Mass of a proton: $m_p = 1.672 \cdot 10^{-27} \text{ Kg}$

Energy(at rest): $m_p c^2 = 938 \text{ MeV} = 0.15 \text{ nJ}$

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■ Other example, 1 gram of material equivalent to:

- 250000 times the full LHC beam ($9 \cdot 10^{13} \text{ J}$)

- 21.5 kilotons of TNT

Relativistic mass

The mass of a fast moving particle is increasing like:

$$m = \gamma m_0 = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

■ Why do we care ?

- Particles cannot go faster than c !
- What happens when we accelerate ?

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Relativistic mass

When we accelerate:

■ For $v \ll c$:

- E , m , p , v increase ...

■ For $v \approx c$:

- E , m , p increase, but v does not !

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Relativistic energy

We remember that:

$$T = m_0(\gamma - 1)c^2$$

therefore:

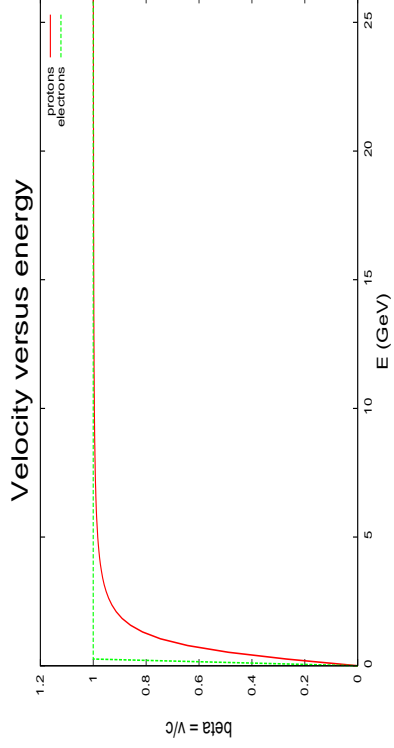
$$\gamma = 1 + \frac{T}{m_0c^2}$$

we get for the speed v , i.e. β :

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}}$$

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Velocity versus energy (protons)



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Why do we care ??

E (GeV)	v (km/s)	γ	β	T (LHC)
450	299791.82	479.74	0.99999787	88.92465 μ s
7000	299792.455	7462.7	0.999999999	88.92446 μ s

- ▶ For identical circumference very small change in revolution time
- ▶ If path for faster particle slightly longer, the faster particle arrives later !

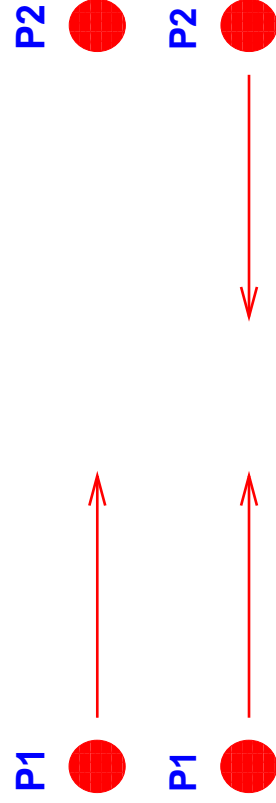
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Four vectors

- Use of four-vectors simplify calculations significantly
- Follow the rules and look for invariants
- In particular kinematic relationships, e.g.
 - Particle decay (find mass of parent particle)
 - Particle collisions →

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Particle collisions



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What is the available collision energy ?

Particle collisions - collider

Assume identical particles and beam energies, colliding head-on



The four momentum vectors are:

$$P1 = (E, \vec{p}) \quad P2 = (E, -\vec{p})$$

The four momentum vector in centre of mass system is:

$$P^* = P1 + P2 = (E + E, \vec{p} - \vec{p}) = (2E, \vec{0})$$

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Particle collisions - collider

The four momentum vector in centre of mass system is:

$$P^* = P1 + P2 = (E + E, \vec{p} - \vec{p}) = (2E, \vec{0})$$

The square of the total available energy s in the centre of mass system is the momentum invariant:

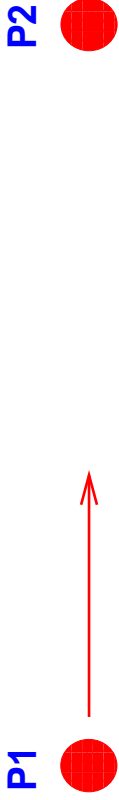
$$s = P^* \odot P^* = 4E^2$$

$$E_{cm} = \sqrt{P^* \odot P^*} = 2E$$

i.e. in a (symmetric) collider the total energy is twice the beam energy

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Particle collisions - fixed target



The four momentum vectors are:

$$P1 = (E, \vec{p}) \quad P2 = (m_0, \vec{0})$$

The four momentum vector in centre of mass system is:

$$P^* = P1 + P2 = (E + m_0, \vec{p})$$

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Particle collisions - fixed target

With the above it follows:

$$P^* \odot P^* = E^2 + 2m_0E + m_0^2 - \vec{p}^2$$

since $E^2 - \vec{p}^2 = m_0^2$ we get:

$$s = 2m_0E + m_0^2 + m_0^2$$

if E much larger than m_0 we find:

$$E_{cm} = \sqrt{s} = \sqrt{2m_0E}$$

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Particle collisions - fixed target

Homework: try for $E1 \neq E2$ and $m1 \neq m2$

Examples:

collision	beam energy	\sqrt{s} (collider)	\sqrt{s} (fixed target)
pp	315 (GeV)	630 (GeV)	24.3 (GeV)
pp	7000 (GeV)	14000 (GeV)	114.6 (GeV)
e+e-	100 (GeV)	200 (GeV)	0.320 (GeV)

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Kinematic invariant

We need to make cross sections (and therefore luminosity) invariant !

This is done by a calibration factor which is (without derivation):

$$K = \sqrt{(v_1^2 - v_2^2)^2 - (v_1 \times v_2)^2} / c^2$$

Here v_1 and v_2 are the velocities of the two (relativistic) beams.

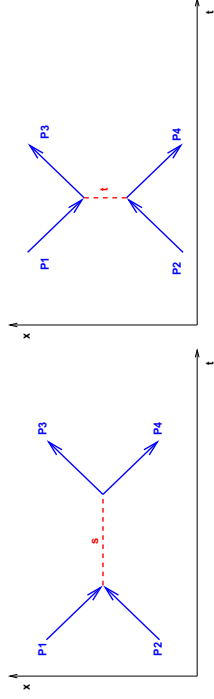
For a (symmetric) collider, e.g. LHC, we have:

$$v_1 = -v_2, \quad v_1 \times v_2 = 0 \quad \text{head-on!}$$

→ $K = 2 \cdot c!$

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For completeness ...



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Squared centre of mass energy:

$$s = (P1 + P2)^2 = (P3 + P4)^2$$

Squared momentum transfer in particle scattering
 (small t - small angle, see again lecture on Luminosity):

$$t = (P1 - P3)^2 = (P2 - P4)^2$$

Kinematic relations

We have already seen a few, e.g.:

- $T = E - E_0 = (\gamma - 1)E_0$
- $E = \gamma \cdot E_0$
- $E_0 = \sqrt{E^2 - c^2p^2}$
- etc. ...

Very useful for everyday calculations →

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Kinematic relations

	cp	T	E	γ
$\beta =$	$\frac{1}{\sqrt{(\frac{E_0}{cp})^2 + 1}}$	$\sqrt{1 - \frac{1}{(1 + \frac{T}{E_0})^2}}$	$\sqrt{1 - (\frac{E_0}{E})^2}$	$\sqrt{1 - \gamma^{-2}}$
cp =	cp	$\sqrt{T(2E_0 + T)}$	$\sqrt{E^2 - E_0^2}$	$E_0 \sqrt{\gamma^2 - 1}$
$E_0 =$	$\frac{cp}{\sqrt{\gamma^2 - 1}}$	$T/(\gamma - 1)$	$\sqrt{E^2 - c^2 p^2}$	E/γ
T =	$cp \sqrt{\frac{\gamma - 1}{\gamma + 1}}$	T	$E - E_0$	$E_0(\gamma - 1)$
$\gamma =$	$cp/E_0\beta$	$1 + T/E_0$	E/E_0	γ

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Kinematic relations

Example: CERN Booster

At injection: T = 50 MeV

→ E = 0.988 GeV, p = 0.311 GeV/c

→ $\gamma = 1.0533$, $\beta = 0.314$

At extraction: T = 1.4 GeV

→ E = 2.338 GeV, p = 2.141 GeV/c

→ $\gamma = 2.4925$, $\beta = 0.916$

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Kinematic relations - logarithmic derivatives

	$\frac{d\beta}{\beta}$	$\frac{dp}{p}$	$\frac{dT}{T}$	$\frac{dE}{E} = \frac{d\gamma}{\gamma}$
$\frac{d\beta}{\beta} =$	$\frac{d\beta}{\beta}$	$\frac{1}{\gamma^2} \frac{dp}{p}$	$\frac{1}{\gamma(\gamma+1)} \frac{dT}{T}$	$\frac{1}{(\beta\gamma)^2} \frac{d\gamma}{\gamma}$
$\frac{dp}{p} =$	$\gamma^2 \frac{d\beta}{\beta}$	$\frac{dp}{p}$	$[\gamma/(\gamma+1)] \frac{dT}{T}$	$\frac{1}{\beta^2} \frac{d\gamma}{\gamma}$
$\frac{dT}{T} =$	$\gamma(\gamma+1) \frac{d\beta}{\beta}$	$(1 + \frac{1}{\gamma}) \frac{dp}{p}$	$\frac{dT}{T}$	$\frac{\gamma}{\gamma-1} \frac{d\gamma}{\gamma}$
$\frac{dE}{E} =$	$(\beta\gamma)^2 \frac{d\beta}{\beta}$	$\beta^2 \frac{dp}{p}$	$(1 - \frac{1}{\gamma}) \frac{dT}{T}$	$\frac{d\gamma}{\gamma}$
$\frac{d\gamma}{\gamma} =$	$(\gamma^2 - 1) \frac{d\beta}{\beta}$	$\frac{dp}{p} - \frac{d\beta}{\beta}$	$(1 - \frac{1}{\gamma}) \frac{dT}{T}$	$\frac{d\gamma}{\gamma}$

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Example LHC (7 TeV): $\frac{\Delta^2}{p} \approx 10^{-4}$ \rightarrow $\frac{\Delta\beta}{\beta} = \frac{\Delta v}{v} \approx 2 \cdot 10^{-12}$

Summary

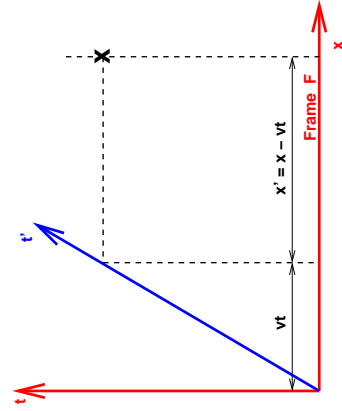
- Special Relativity is very simple, derived from basic principles
- Relativistic effects vital in accelerators:
 - Lorentz contraction and Time dilation
 - Invariants !
 - Relativistic mass effects
 - Modification of electromagnetic field
- Find back in later lectures ...

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- BACKUP SLIDES -

Galilei transformation - schematic



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➤ Rest frame and Galilei transformation ...

Forces and fields

Motion of charged particles in electromagnetic fields \vec{E} , \vec{B} determined by Lorentz force

$$\vec{f} = \frac{d}{dt}(m_0\gamma\vec{v}) = q \cdot (\vec{E} + \vec{v} \times \vec{B})$$

or as four-vector:

$$F = \frac{dP}{d\tau} = \gamma \left(\frac{\vec{v} \cdot \vec{f}}{c}, \vec{f} \right) = \gamma \left(\frac{1}{c} \frac{dE}{dt}, \frac{d\vec{p}}{dt} \right)$$

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Field tensor

Electromagnetic field described by field-tensor $F^{\mu\nu}$:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$

derived from four-vector $A_\mu = (\Phi, \vec{A})$ like:

$$F^{\mu\nu} = \delta^\mu A^\nu - \delta^\nu A^\mu$$

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Lorentz transformation of fields

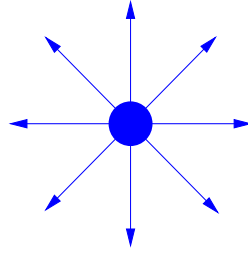
$$\begin{aligned}\vec{E}'_{\perp} &= \gamma(\vec{E}_{\perp} + \vec{v} \times \vec{B}) \\ \vec{B}'_{\perp} &= \gamma\left(\vec{B}_{\perp} - \frac{\vec{v} \times \vec{E}}{c^2}\right) \\ \vec{E}'_{\parallel} &= \vec{E}_{\parallel} \\ \vec{B}'_{\parallel} &= \vec{B}_{\parallel}\end{aligned}$$

➤ Field perpendicular to movement transform

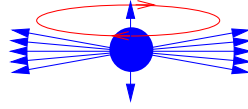
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Lorentz transformation of fields

$\gamma = 1$



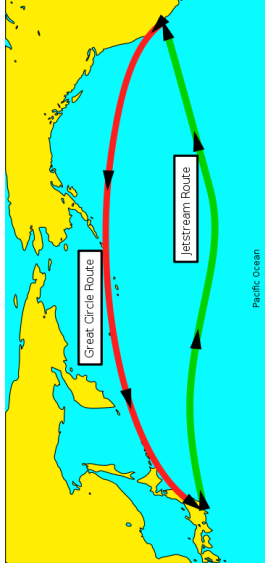
$\gamma \gg 1$



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- In rest frame purely electrostatic forces
- In moving frame \vec{E} transformed and \vec{B} appears

Addition of velocities (Galilei)
(an everyday example ...)



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Jetstream up to 350 - 400 km/hour !
Can save one 1 hour or more on an eastbound flight !