Entanglement

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Introduction

The history of the principle of entanglement is as old as the Quantum Theory itself. In a philosophical experiment, Einstein, Podolsky and Rosen proposed two "Quantum correlated systems, whose complete knowledge of one implies complete knowledge of the other, regardless of their spatial separation. An example of such systems is what is called the spin singlet. A system of two subsystems of spin half, where the total spin of the system is 0. This EPR state can be represented mathematically as:

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right).$$

They argued the existence of such systems violates Einstein's theory of Special Relativity, and hence Quantum Mechanics is an incomplete theory and there must be some hidden variables. In 1964, John Bell was able to put their argument into mathematical terms, in what is called Bell's inequality. The violation of this inequality proved the completeness of Quantum Theory.

Motivation

In our search we were interested in how entanglement could be quantified. We considered two measures of entanglement; the historical Bell's inequality and the well agreed Von-Neumann entropy.

Entanglement Measures of Bipartite Systems

Bell's inequality

Bell's inequality was the evidence of the completeness of Quantum Theory. Classical systems would obey the inequality. However, a pure entangled quantum system would violate it. Taking a bipartite system whose subsystems are of dimension 2, then;

$$E(qs) + E(rs) + E(qt) - E(rt) \le 2.$$

where;
$E(qs)$, $E(rs)$, $E(qt)$, $E(rt)$ are the expectation values of the outcomes combination of the whole system (measuring both subsystems together at the same time).
The stronger the entanglement the bigger the violation. The highest violation of Bell's inequality is given by Tsirelson's inequality that the maximum violation of Bell's inequality is $2\sqrt{2}$.
For mixed states, Reinhard F. Werner showed, in 1989, that some entangled mixed-state systems do not violate Bell's Inequality.

Von-Neumann Entropy

Von-Neumann entropy $S(\rho)$ is the accurate and well accepted measure of entanglement for pure states. A fully entangled pure bipartite system has a Von-Neumann entropy of unity while a separable system has Von-Neumann entropy equal null.
Von-Neumann entropy can be found from the reduced matrices of the bipartite systems.

$$S(\rho) = -tr\left(\rho^A ln\rho^A\right) = -\sum_{ni}(p_n\lambda_{ni}^2 \ln p_n\lambda_{ni}^2) = -tr\left(\rho^B ln\rho^B\right),$$

where $p_n\lambda_{ni}$ is the eigenvalue of the spectral decomposition of the system which is easily findable through the Schmidt decomposition;

$$|\psi\rangle = \sum_i \lambda_i\,|v_i\rangle\,|w_i\rangle\,.$$

A mixed state has a density operator $\rho = \sum_n p_n\,|\psi_n\rangle\,\langle\psi_n|$. We can see that it is not possible to use Von-Neumann entropy to measure the quantum correlation. However,

" *all entanglement measures should coincide on pure bipartite states and be equal to the Von-Neumann entropy* ", Magistra K. Durstberger.

Definitions

- Two (or more) systems are said to be entangled if they were prepared in quantum mechanical device. Such systems are quantum correlated. The knowledge of the state or an observable of one system gives a complete knowledge of the state or observable of the other.

  > *"Another way of expressing the peculiar situation is: the best possible knowledge of a whole does not necessarily include the best possible knowledge of all its parts, even though they may be entirely separate and therefore virtually capable of being "best possibly known, i.e., of possessing, each of them, a representative of its own."*
  > Erwin Schroedinger

- A complete system of subsystems that can suffer entanglement is called "composite system". It can be made of two subsystems (bipartite) or more (multi-partite). This system can be said to have a pure state (one definite state), or mixed state (number of states assigned to probabilities. These probabilities sum to the unity).
- A composite system is represented in the tensor product Hilbert space. Then an entangled state has a density operator that can not be written as a convex linear decomposition

  $$\rho = \sum_i p_i\rho_A^i \otimes \rho_B^i.$$

  where, ρ^A is the reduced density operator of the part A , ρ^B is the reduced density operator of the part B and p_i is the assigned probability.
- An Entanglement measure is a means of determining how strong parts of a composite system are quantum correlated.

References

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Acknowledgement

The authors would like to thank prof. Bernd J. Schroers for his supervision and kind guidance. She would also like to thank African Institute for Mathematical Science (AIMS)- Ghana where she did this research as a part of the master program in mathematical sciences, 2013.