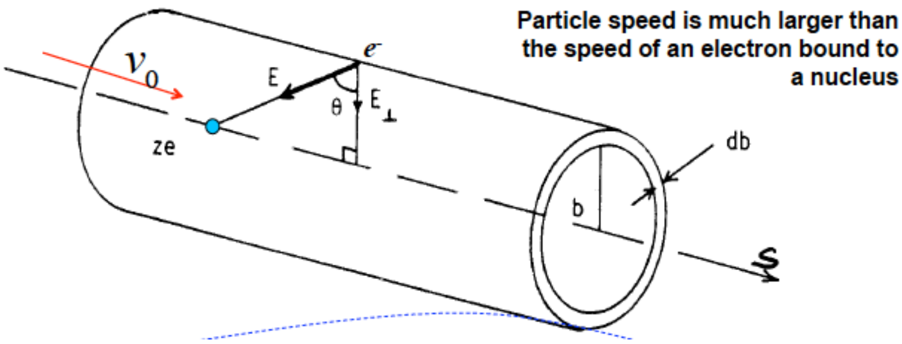
Exercises for the lecture in particle detectors:

1) Classical derivation of the Bethe Bloch formula using the hints given in the lecture:

$$-\frac{dE}{ds} = -\int_0^\infty \frac{dE}{db} db = \frac{4\pi z^2 e^4 k^2}{m_e v_0^2} n_e \ln\frac{b_{\max}}{b_{\min}}; \quad b_{\min} = \frac{z \cdot e^2 k^2}{\gamma m_e v_0^2}; \quad b_{max} = \gamma v_0 \overline{T}$$



We consider a heavy(than an electron) charged particle of mass *m* and charge *ze* and velocity v_0. We first calculated the energy transfer to an electron located at distance b (=impact parameter) to the trajectory of the heavy particle. We will then sum over all electrons, which will be at such an impact parameter in the material and then integrate over all impact parameters within physical limits.

The momentum transfer to an electron is given by:

$$\Delta p_e = \int_{-\infty}^{\infty} F\,dt = e\int_{-\infty}^{\infty} \mathfrak{E}_\perp\,dt = \frac{e}{v_0}\int_{-\infty}^{\infty} \mathfrak{E}_\perp\,ds; \quad \mathfrak{E}_\perp = \textbf{electric field}$$

only the transverse component of the electric field counts because the longitudinal components cancel each other. We use the Gauss theorem to calculate the integral over the electric field:

$$GAUSS: \iiint_V \operatorname{div}\vec{\psi}\,dx\,dy\,dz = \oint_A \vec{\psi}\,d\vec{a}; \ \vec{\psi} = \text{ vector field}$$

$$\iint_A \mathfrak{E}_\perp\,da = \iiint_V \operatorname{div}\vec{\mathfrak{E}}\,dx\,dy\,dz = \frac{1}{\varepsilon_0}\iiint \rho\,dx\,dy\,dz = \frac{ze}{\varepsilon_0}; \quad \operatorname{div}\vec{\mathfrak{E}} = \frac{\rho}{\varepsilon_0}$$

The surface integral on the left hand side in the last line can be simplified for the cylindrical geometry: $da = 2\pi b ds$; $2\pi b\int_{-\infty}^{\infty} \mathfrak{E}_\perp\,ds = \frac{ze}{\varepsilon_0}$

we obtain now for the energy transfer:

$$\Delta p_e = \frac{2}{4\pi\varepsilon_0}\frac{ze^2}{bv_0} = 2k\frac{ze^2}{bv_0}; \qquad \textit{with the abrivation}: \quad k = \frac{1}{4\pi\varepsilon_0}$$

$$\Delta E = -\Delta E_e = -\frac{(\Delta p_e)^2}{2m_e} = -2\frac{z^2 e^4}{b^2 m_e}\left(\frac{k}{v_0}\right)^2$$

Now we sum over all electrons in the material at a distance *b* from the trajectory:

$$-dE(b) = \Delta E(b) n_e dV = 4\pi n_e \frac{z^2 e^4}{m_e}\left(\frac{k}{v_0}\right)^2 \frac{db}{b} ds; \ \left(dV = 2\pi b\, db\, ds\right)$$

which is the energy loss *–dE(b)* (the minus signs is there because we loose energy) of the charged particles to all electrons in the material in the cylindrical shell of radius *b* and thickness *db*. The last thing to do, is to integrate over all impact parameters from 0 to infinity. Both limits are of course un-physical, so we have to invent an intelligent cut off.

$$-\frac{dE}{ds} = -\int_0^\infty \frac{dE}{db} db = 4\pi n_e \frac{z^2 e^4}{m_e}\left(\frac{k}{v_0}\right)^2 \ln\left(\frac{b_{\max}}{b_{\min}}\right)$$

b_{min}: The smaller *b* the larger is the energy transfer to the electron, but of course we know it cannot get infinite as the above formula may suggest. It is classically limited to a maximal value obtained in a head-on collision. We use this value to derive a lower bound for *b*!

$$T_e^{\max} = 2 m_e v_0^2 \gamma^2 = 2\frac{z^2 e^4}{b_{\min}^2 m_e}\left(\frac{k}{v_0}\right)^2$$

$$\Rightarrow\ b_{\min} = \frac{z\cdot e^2 k^2}{\gamma m_e v_0^2}; \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}; \ \beta = \frac{v_0}{c}; \ v_0 = \text{particle speed !}$$

The upper limit for b comes from our assumption that the electron is quasi static with respect to the fast passage of the charge particle. From geometry it is plausible that this assumption is not anymore justified at large distances. We compare the interaction time to the average revolution time of the electron; this depends on the detailed atomic structure of the material and explains why there will be many correction factors.

$$\text{interaction time:} \ \frac{b_{max}}{\gamma v_0} \ll \bar{T} = \ \text{Orbit time;} \qquad \Rightarrow b_{max} = \gamma v_0 \bar{T}$$

Now we can solve the integral and obtain the classical calculation of the energy loss, which contains already all the important features:

- The energy is proportional to the charge squared
- Proportionality to the inverse square of the velocity for non relativistic particles
- A logarithmic rise with gamma for relativistic particles, (This is due to the increase of the transverse component of the electric field for relativistic particle due to the Lorentz contraction, look at your electrodynamics lectures!)
- If you have a decrease at low energy and an increase at high energy you will have a minimum in between, the regime of minimum ionisation for particles, which start to become relativistic.

$$-\frac{dE}{ds} = -\int_0^\infty \frac{dE}{db} db = \frac{4\pi z^2 e^4 k^2}{m_e v_0^2} n_e \ln\frac{\gamma^2 m_e v_0^3 \bar{T}}{z^2 e^2 k^2}$$

2) Kinematics of Compton scattering: $h\nu' = \dfrac{h\nu}{1+\varepsilon(1-\cos\theta_{\gamma'})}$; $\varepsilon = h\nu / m_e c^2$

This formula can be derived by considering longitudinal and transvers momentum and energy conservation.

$$longitudinal: \quad p_\gamma = \frac{h\nu}{c} = \frac{h\nu'}{c}\cos\theta_{\gamma'} + |\vec{p}_e|\cos\theta_{e'}$$

$$transversal: \quad 0 = \frac{h\nu'}{c}\sin\theta_{\gamma'} - |\vec{p}_e|\sin\theta_{e'}$$

$$and \;\; for \;\; the \;\; energy \;\; T_e = h\nu - h\nu'$$

Lets write the formulas with 4-vectors of the photon and electron before and after the scattering:

$$Photon: p_\gamma = \left(\frac{h\nu}{c},\ 0,\ 0,\ \frac{h\nu}{c}\right) \to p'_\gamma = \left(\frac{h\nu'}{c},\ \frac{h\nu'}{c}\sin\theta_{\gamma'},\ 0,\ \frac{h\nu'}{c}\cos\theta_{\gamma'}\right)$$

$$Electron: \ p_e = \left(\frac{m_e c^2}{c},\ 0,\ 0,\ 0\right) \to p'_e = \left(\sqrt{m_e^2 c^2 + \vec{p}_e'^2},\ -|\vec{p}'_e|\sin\theta_{e'},\ 0, +|\vec{p}'_e|\cos\theta_{e'}\ \right)$$

$$p_\gamma + p_e = p'_\gamma + p'_e \ \Rightarrow$$

$$h\nu + m_e c^2 = h\nu' + \sqrt{m_e^2 c^4 + c^2 \vec{p}_e'^2}$$

$$longitudinal: \quad \frac{h\nu}{c} = \frac{h\nu'}{c}\cos\theta_{\gamma'} + |\vec{p}'_e|\cos\theta_{e'}$$

$$transversal: \quad 0 = \frac{h\nu'}{c}\sin\theta_{\gamma'} - |\vec{p}'_e|\sin\theta_{e'}$$

$$\Rightarrow \ \cot\theta_{e'} = \frac{h\nu - h\nu'\cos\theta_{\gamma'}}{h\nu'\sin\theta_{\gamma'}}$$

and

$$c^2|\vec{p}'_e|^2\left[\sin^2\theta_{e'} + \cos^2\theta_{e'}\right] = \left(h\nu'\sin\theta_{\gamma'}\right)^2 + \left(h\nu\right)^2 + \left(h\nu'\cos\theta_{\gamma'}\right)^2 - 2\left(h\nu\right)\left(h\nu'\right)\cos\theta_{\gamma'}$$

Energy of scattered gamma:

$$h\nu + m_e c^2 - h\nu' = \sqrt{m_e^2 c^4 + c^2 \vec{p}_e'^2} \quad \Big| \ square \ \ equation!$$

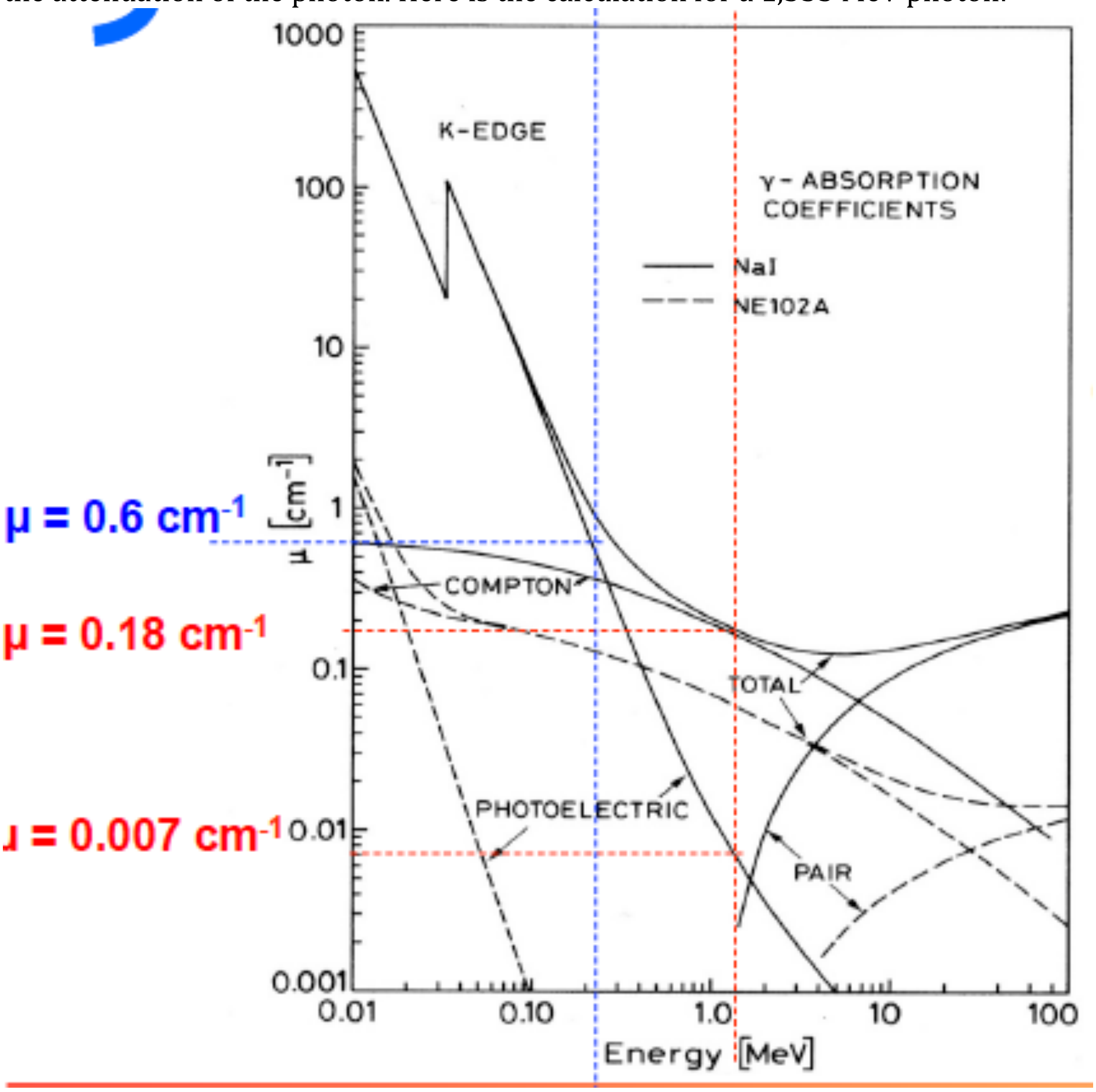
$$\Rightarrow$$

$$\left(h\nu - h\nu'\right) m_e c^2 = \left(h\nu\right)\left(h\nu'\right)\left(1 - \cos\theta_{\gamma'}\right)$$

$$\Rightarrow$$

$$h\nu' = \frac{h\nu}{1+\varepsilon(1-\cos\theta_{\gamma'})}; \ \varepsilon = h\nu / m_e c^2$$

3) Using the photon attenuation curves for NaI given in the lecture to estimate the total efficiency of a NaI standard detector in a given geometry as specified in the lecture II (p74-76), why is your result different from the value quoted?
You have to calculate the geometrical acceptance and the intrinsic efficiency due the the attenuation of the photon. Here is the calculation for a 1,333 MeV photon:



$$\varnothing = 7.62cm\,;\; D = 25cm\,;\; l = 7.62cm\,;$$

$$E_\gamma = 1.333\,MeV:\; \mu_{ph} = 0.007cm^{-1};\; \mu_{compt} = 0.18cm^{-1}$$

$$\varepsilon_{abs}^{photo} = \left(1-\exp\left(-\mu l\right)\right)\frac{\left(\varnothing/2\right)^2\pi}{D^2}\frac{1}{4\pi} = \left(1-\exp\left(-\mu l\right)\right)\left(\frac{\varnothing}{4D}\right)^2$$

$$\varepsilon_{abs}^{photo} = \underbrace{\left(1-\exp\left(-0.007\cdot 7.62\right)\right)}_{0.052}\underbrace{\left(\frac{7.62}{4\cdot 25}\right)^2}_{0.0058} = 3.02\cdot 10^{-4}$$

This result is much smaller because we have NOT taken into account the possibility that a Compton scattered photon can re-interact and deposit its total energy in the detector, which will increase the photo-peak and thus the intrinsic efficiency. For a correct calculation you have to do a complete simulation of all processes.

4) In Water the refractive index is n=1.333. Calculate the minimal energy in keV of an electron to produce Cerenkov light. Where do the electrons come from?
We have to calculate the limiting speed (or beta) of the electron to emit Cherenkov light:

$$\beta > 1/n;\ \ n = 1{,}333 = 4/3\ \Rightarrow \beta > 3/4 = 0{,}75$$

$$p = m_e \gamma \beta c;\quad E_e = m_e c^2 \gamma = 772\ keV;$$

$$\gamma = \left(1-\beta^2\right)^{-1/2} = \frac{4}{\sqrt{7}};\ m_e c^2 = 511 keV$$

5) Derive the expression for the momentum resolution of a tracker in a magnetic field:

$$\frac{dS}{S} = \frac{dp_\perp}{p_\perp} = \frac{80}{3\cdot z}\frac{1}{BL^2}p_\perp dS;\ \ [B] = Tesla;\ \ [L] = m;[p_\perp] = GeV/c$$

$$R^2 = (L/2)^2 + (R-S)^2$$

$$0 = S^2 - 2RS + L^2/4$$

$$S_{1,2} = R\left(1 \pm \sqrt{1-(L/2R)^2}\right)$$

$$p_\perp = p\cdot\sin\theta;$$

$$\theta = \sphericalangle(\vec{p},\vec{B})$$

$$p_\perp\ grand\ R \gg L;\ (1-x)^{1/2} \approx 1 - \tfrac{1}{2}x^2....$$

$$S_1 = 2R - L^2/8R$$

$$S_2 = L^2/8R$$

$$\vec{F} = q\cdot\left(\vec{v}\times\vec{B}\right) = m\vec{a}_R = m\frac{v^2}{R};\ \vec{v}\perp\vec{B}$$

$$R = \frac{m}{q}\frac{v}{B} = \frac{p_\perp}{q\cdot B}$$

$$[p_\perp] = GeV/c;[q] = e\cdot z;[B] = T_{esla} = Vs/m^2$$

$$R = \frac{1}{z\cdot e}\frac{10^9 eV}{c}\frac{1}{Vs/m^2};\ c = 3\cdot 10^8 m/s$$

$$R[m] = \frac{1}{z}\frac{10}{3}\frac{p_\perp[GeV/c]}{B[T]}$$

This is the standard relation between the curvature and momentum in a magnetic field!
The curvature is measured via the sagitta S. We express the radius by the sagitta and derive the expression with respect to p_T

$$S = L^2\frac{z}{8}\frac{3}{10}\frac{B}{p_\perp}$$

$$\left|\frac{dS}{dp_\perp}\right| = L^2\frac{z}{8}\frac{3}{10}\frac{B}{p_\perp}\left(\frac{-1}{p_\perp}\right) = \frac{S}{p_\perp}$$

$$\frac{dS}{S} = \frac{dp_\perp}{p_\perp} = \frac{80}{3\cdot z}\frac{1}{BL^2}p_\perp dS;\ \ [B] = Tesla;\ \ [L] = m;[p_\perp] = GeV/c$$

6) Estimate the nuclear interaction length of protons in Iron (Fe, A=56; density r=7.8 g/cm^3)
   The main equations are the relation between cross-section and the interaction length:

$$\mu = \frac{1}{\lambda} = \sigma_{\text{int}} \cdot \rho \cdot \frac{N_A}{A}; \ Fe: \ A = 56; \ density \ \rho = 7.8 g / cm^3$$

$$\sigma_{\text{int}} \approx \sigma_{geo}$$

$$proton: \ size \approx R_p \approx 1 fm$$

$$Nucleus: \ size \approx \pi R_A^2; \ R_A \approx 1,2 \cdot A^{1/3} \ \Rightarrow R_{Fe} \approx 1,2 \cdot 3.8 = 4.6 fm$$

$$\sigma_{\text{int}} \approx \pi \left(R_p + R_{Fe}\right)^2 = 98 fm^2$$

$$\mu = \frac{1}{\lambda} = \sigma_{\text{int}} \cdot \rho \cdot \frac{N_A}{A} = \frac{98 \cdot 10^{-26} cm^2 \cdot 7.8 g / cm^3 \cdot 6.22 \cdot 10^{23}}{56 g}$$

$$\Rightarrow \lambda_{\text{int}} \approx 11 cm. \quad \text{valeur correct (PDB)}: 17 cm$$

   our estimate for the cross section was a bit to large.

7) The number of particles in an elm shower is proportional to the Energy. If we can measure the number of particles in a shower, how will the energy resolution scale with energy?
   It follows from the Poisson statistics:

$$\frac{dN}{N} = \frac{1}{\sqrt{N}} \ ; \ E \sim N; \quad N = \text{numb. of particles, photons or whatever is measured}$$

$$\Rightarrow \frac{dE}{E} = \frac{1}{\sqrt{E}} \ \text{which is the stochastic term of the energy resolution}$$

   one has to add the contribution of the electronic noise and the non perfection of the calorimeter :

$$\frac{\sigma(E)}{E} = \frac{a}{\sqrt{E}} \oplus b \oplus \frac{c}{E}$$

8) In nuclear physics light ions can be stopped with a 1mmm Silicon detector, after they passed through a thin ΔE counter. Explain why the result of the product of the two counters is proportional to the mass. $\Delta E \times E_{cin} \propto m$
   It just follows from the Bethe Bloch formula:

$$\frac{dE}{dx} \propto \frac{1}{v^2}; \ E_{cin} = \tfrac{1}{2} m v^2; \ \Rightarrow \ \frac{dE}{dx} \times E_{cin} \propto m$$