Logic and Quantum Information

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The goal of this contribution is to explain the relevance of a mathematical theory of relations for quantum mechanics. Why is this formalism relevant? Because quantum mechanics represents a very innovative theory of the relations that fill our physical world. Establishing such connection can improve our understanding of the theory, in particular in connection wit the issue of information.

Classically, we have two different formulations of mutual information:

$$I(J:K) = H(J) + H(K) - H(J,K), \tag{1}$$

and

$$I(J:K) = H(J) - H(J|K). \tag{2}$$

Quantum–mechanically they turn out to be different.

Zurek has proved that when we express entanglement as mutual information it consists of a classical and quantum part (discord) [OLLIVIER/ZUREK 2001] [AULETTA/WANG 2014]:

$$I(S:A) = \mathcal{C}(S:A)_{\{\hat{P}_j^A\}} + \mathcal{Q}(S:A)_{\{\hat{P}_j^A\}}, \tag{3}$$

where the classical part is given by

$$\mathcal{C}(S:A)_{\{\hat{P}_j^A\}} = H_{\mathrm{VN}}(S) - H_{\mathrm{VN}}(S|\{\hat{P}_j^A\}), \tag{4}$$

and

$$H_{\mathrm{VN}}(S|\{\hat{P}_j^A\}) = \sum_j \wp_j H_{\mathrm{VN}}(\hat{\rho}_{S|\hat{P}_j^A}), \tag{5}$$

and this is in accordance with the second form of the classical expressions, while the quantum part is

$$\mathcal{Q}(S:A)_{\{\hat{P}_j^A\}} = H_{\mathrm{VN}}(A) - H_{\mathrm{VN}}(S,A) + H_{\mathrm{VN}}(S|\{\hat{P}_j^A\}). \tag{6}$$

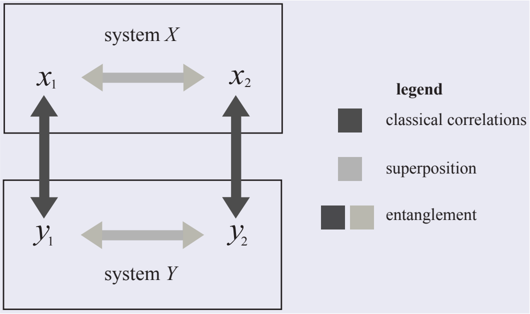

Figure: Superposition is relative to *the basis* while entanglement is intrinsic in *the state* of the system and is therefore independent of the basis used. As a matter of fact, entanglement combines the typical quantum superposition with correlations that have a classical root. *Both* these kind of classical correlations and quantum superposition factors constitute entanglement. Particular case in which the entanglement of two degrees of freedom is among two subsystems and each observable has two eigenvectors only.

We shall build now a logical space of relations [*AULETTA* 2013]. We distinguish monadic, dyadic, and triadic relations. Higher order relations can be logically treated as a combination of some of these basic forms. Let us consider 3 basic relations and their truth value assignment:

$R(X)$	0	0	0	0	1	1	1	1
$R(Y)$	0	0	1	1	0	0	1	1
$R(Z)$	0	1	0	1	0	1	0	1
output	a	b	c	d	e	f	g	h

Table: Inputs and outputs of 3-dimensional logic.

To these monadic relations we need to add expressions for tautologies like $R(X' + X)$, whose truth values are all 1s. The meaning is Everything is related to everything.

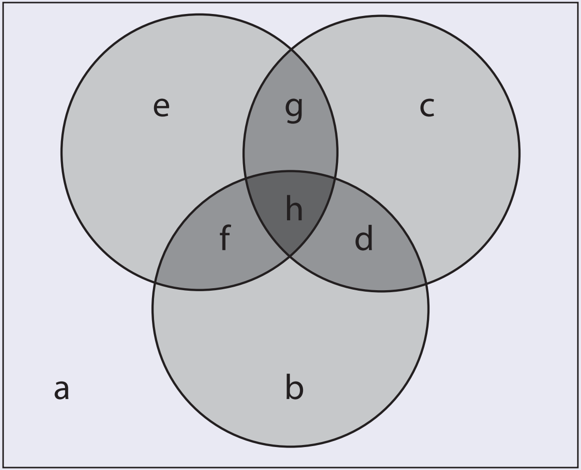

Figure: Venn diagram for the 3D logical space. Note that the grayscale is arranged according to the overlaps among areas: white (0% of black) is empty area, pale gray (25% of black) is no overlap, middle gray (50% of black) two sets overlap, dark gray (75% of black) three sets overlap.

The following list exhausts all the possible binary relations, since we have already dealt with any relation (or absence of relation) between everything and everything:

- Either everything (something) being (not being) member of a certain class is related (not related) to something or everything else or
- These two classes of objects are correlated (not correlated).

Definitions of Relations

Dyadic relations are represented e.g. by

Interpretation	Relation	ID
Everything is related to something	$R(X' + Y)$	11110011
Everything not being like that is related to something	$R(X + Y)$	00111111
Everything is related to nothing	$R(X' + Y')$	11111100
Everything not being like that is related to nothing	$R(X + Y')$	11001111
Something is related to something	$R(X \times Y)$	00000011
Something not being like that is related to something	$R(X' \times Y)$	00110000
Something is related to nothing	$R(X \times Y')$	00001100
Something not being like that iis related to nothing	$R(X' \times Y')$	11000000
Two classes of things are correlated	$R(X \Leftrightarrow Y)$	11000011
Two classes of things are anticorrelated	$R(X \not\Leftrightarrow Y)$	00111100

Table: Some abstract binary relations.

Definitions of Relations

Every binary relation has its order and e.g. $R(X' + Y)$ means that the class X is related in some way with the class Y and not vice versa. Note that I have expressed *Everything is related to something* as e.g. $R(X' + Y)$; *Everything has no relation to anything* as e.g. $R(X' + Y')$; *Something is related to something* as e.g. $R(X \times Y)$, and, finally, *Something is not related to anything* as e.g. $R(X \times Y')$. The reader may wonder whether the expression $R(X' + Y)$ and similar ones does not mean that everything X is related to everything Y. Actually, I assume that in this expression only the first class (X) is taken universally, while in the expression $R(X \times Y')$ only the first class (again X) is taken particularly. This makes the order of the classes inside a single relation quite rigid.

Note also that $R(X \Leftrightarrow Y)$ means a correlation between X and Y while $R(X \not\Leftrightarrow Y)$ absence of correlation. Equivalence is a particular (predicative) case of correlation, and this explains why I have used a different symbol from logical equivalence. In fact, when we say that there is a correlation between two classes of objects, we do not need to assume that all objects e.g. X and Y are correlated but only some of them. In the case in which all X's and Y's are correlated, they constitute a single equivalence class and the notion of correlation becomes logical equivalence. Only correlations are certainly symmetric while most of the relations we deal with are not. Although in ordinary logic it is assumed that the product is symmetric, as we shall see there are kinds of relations that do not fulfill this rule.

We can build a similar table for triadic relations. All of the ternary relations may be summarized as follows:

- Everything (something) being (not being) member of a certain class is related (not related) to everything (something) through everything (something) being (not being) member of another class or
- Either two classes of objects are correlated (not correlated) through a third one being or not being like that or being already correlated (not correlated) are correlated (not correlated) with a third one.

Definitions of Relations

Interpretation	Relation	ID
Everything not being like that is related to Everything not being like that thanks to Everything not being like that	$R(X' + Y' + Z')$ $R(Y' + Z' + X')$ $R(Z' + X' + Y')$	11111110
Something not being like that is related to Something not being like that thanks to Something not being like that	$R(X' \times Y' \times Z')$ $R(Y' \times Z' \times X')$ $R(Z' \times X' \times Y')$	10000000
Something is related to Something thanks to Everything not being like that	$R(X' + (Y \times Z))$ $R(Y' + (Z \times X))$ $R(Z' + (X \times Y))$	11110001 11001101 10101011
Everything not being like that is related to Everything thanks to Something	$R(X \times (Y' + Z))$ $R(Y \times (Z' + X))$ $R(Z \times (X' + Y))$	00001101 00100011 01010001
Two classes of objects are correlated through everything Two classes of objects are not correlated through something	$R(X' + Y \Leftrightarrow Z)$ $R(X \times Y \not\Leftrightarrow Z)$	11111001 00001001

Table: Some abstract ternary relations.

Here an expression like $X' + Y' + Z$ need to be considered as the analogue of $X \rightarrow (Y \rightarrow Z)$, where X is what allows (grounds) the relation (or is the bridge) between the classes Y and Z and therefore takes the first place. Note that ternary relations as a whole always posses a kind of dynamical character that makes them very different relative to binary relations that are always static. In fact, a binary relation is a kind of function that tells us that some classes of objects are related to some other classes. However, this does not imply that these two classes have never interacted (which is a necessary condition for having a dynamical relation). On the other hand, in the case of ternary relations, the ground of the relation (although not necessarily a dynamical factor in itself) works as a mediation between the other two classes of objects bringing them together. This can happen through the dynamism of the ground itself but also through some kind of dynamism of the latter two classes. In sum, ternary relations express the arising of a relation or the bringing together of otherwise unrelated classes of objects, which by definition is a dynamical accomplishment.

Coming back to the issue of quantum mechanics, as mentioned, when we have a superposition, we do not have a correlation among items of two different classes (or among values of two different variables), but among items of the same class (or among values of the same variable describing the same system). In such a case we have a *self–correlation*, a notion that is reminiscent of Dirac's concept of self–interference, although the latter is rather manifestation of the former (it should not be mixed up with the notion of autocorrelation that is used in signal and pattern theory). This kind of relations means (in the case of a full self–correlation) that all items of a class are connected with all items of the same class (or all values of a variable with all values of the same). However, this needs to be expressed as $R(X' + X)$, and we clearly see that is a tautology. This shows that not only for the universal class but for any other class displaying total self–correlation were deal in fact with a tautology. In conclusion, quantum superpositions can be logically represented by tautologies.

Note that this is in full accordance with the idea that any quantum system has zero entropy as well as any of its parts and also the whole universe [FANO 1957]. This result together with the explained non commutativity of relations makes such a calculus particularly suitable for describing quantum information. Moreover, it helps us to overcome a traditional paradox of quantum logic: it has been often said that the latter does not satisfy distributivity. In fact, most of models taken a superposition (like in the two paths of an interferometer) jointly with another statement (e.g. about which detector will click) e build an expression of the form $p \times (q + r)$, where $q + r$ should express the superposition. However this is an incorrect logical formulation of the latter.

Symmetric relations

When we ask about the general forms that relations can take, the first subdivision is among symmetric relations and relations that are not symmetric. Note that only correlations are symmetric relations. In fact, to say that Some X are correlated with some Y is equivalent to say that Some Y are correlated with some X. Symmetric relations can be understood as consisting of a relation e.g. R of both X to Y and Y to X. Negation transforms correlations in absence of correlations and vice versa. A particular instance of correlations is represented by logical equivalence when all X's and Y's are correlated. Another particular case is represented by self–correlations. When we deal with relations in general, we are naturally forced to deal with non-symmetric relations. We may distinguish among 4 forms of non–symmetric relations. In order to understand this problem, it is fundamental to understand the kinds of equivalences that define those relations.

Antisymmetric relations

All X are related to some Y IS EQUIVALENT TO All objects that are not Y are not related to any X. Clearly, classical predicative relations are a particular case of this kind of relation due to the implication (transposition) rule according to which $X \rightarrow Y$ is equivalent to $Y' \rightarrow X'$. This implies that when classical implication among relations is involved the use of the antisymmetric form of inference simplifies the calculus considerably. However, we should also avoid a misunderstanding. In classical logic, the statement $X \times Y$ is also symmetric, included in its quantified forms. Nevertheless, when we deal with relations it could be that a relation $R(X \times Y)$ is not symmetric, and we shall distinguish the case in which some X's have a relation R with some Y's and that in which some Y's have or have not a relation R with some X's. In fact, I recall that commutativity of the classes connected by a relation is not valid. Therefore, the general forms of relations involve a more general and abstract level of consideration.

Negative relations

All X are related to all Y IS EQUIVALENT TO There are non–Y that are related to some X whatsoever. Example: All Soviet spies have spied some common (not spy) occidental citizens can be taken to be equivalent to There are not common (i.e. spy) occidental citizens that have spied some Soviet spies. In other words, we are intending here that Soviet spies have been likely spied by occidental spies and not by common citizens. Note also that this tells us nothing about Soviet common citizens having spied or not spied occidental spies or common citizens. Consider that to negate a whole relations does not necessarily mean that we have a negative relations. In general, it does simply mean that there is no relation of that kind among the involved classes of objects.

Reverse relations

All X are related to some Y IS EQUIVALENT TO Some Y are inversely related to all X. Example: All German people have some preferred kind of bier is equivalent to There is a set of biers that are loved by all German people (in a distributive sense). Unfortunately, there is no traditional logical operator that allows us to reverse a relation, although, at the opposite, the idea of an inverse relation is very common in mathematics. Here, reverse relations correspond in fact to what in mathematics are called inverse relations (I keep the name *reverse relation* for distinguishing this particular case form the fact that for all kinds of relations of X to Y we are asking what happens for the inverse relation of Y to X). For this purpose, I shall introduce a new symbol also for basic mathematical logic that is a monadic operator like negation and reverse the logical ID (while negation interchanges 0s with 1s). Like negation two consecutive reversals bring the expression back to its original status. Moreover, we have a kind of de Morgan's law for reversal:

$$(X^{-1} + Y^{-1})^{-1} = X + Y \quad \text{and} \quad (X^{-1}Y^{-1})^{-1} = XY, \tag{7}$$

where the only difference with the usual de Morgan's laws is that the logical connector (whether sum or product) is not changed.

Neg-reverse relations

All X are related to some Y IS EQUIVALENT TO No Y is inversely related to some X whatsoever. Example: Eavesdropping is such that all good eavesdroppers have a connection with their target but those targets do not have any connection with the eavesdropper. Therefore, we could express the neg-reverse by saying: Some Y are not related to some X, provided that it makes sense to affirm this. The previous considerations about reverse relations are also valid for neg-reverse ones. We can consider also a neg-reverse operator that combines reversal and negation. Note that in such a case $X = X^*$ (due to the properties of both reversal and negation) but

$$(X^* + Y^*)^* = XY \quad \text{and} \quad (X^* Y^*)^* = X + Y, \tag{8}$$

where I have introduced the symbol * for denoting the neg–reverse logical operation.

Kinds of Binary Relations

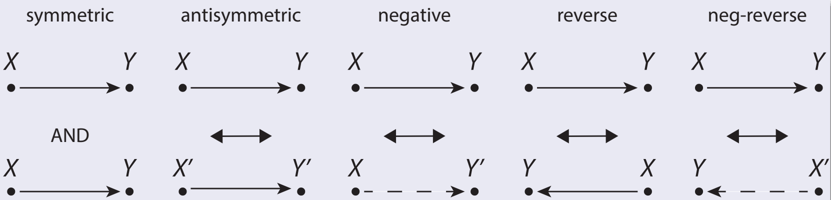

Figure: The five kinds of binary relations. Code: the double arrive means logical equivalence, the single arrive indicates relation and direction of relation, when solid means universal relation when dashed means a particular one. The inverted arrow means a reversed relation.

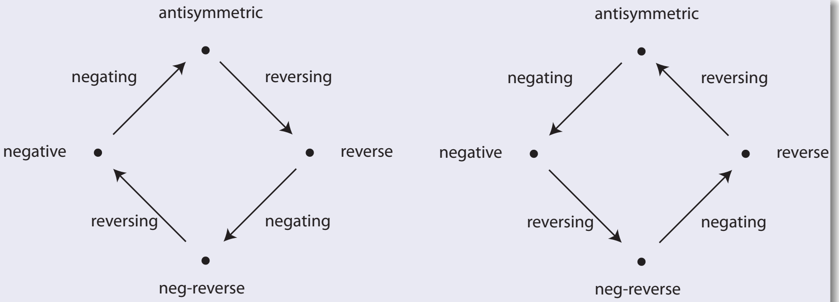

Figure: Circles of relations transformations.

Much more tangled are the ternary relations. Here, we need to distinguish between ground, *relandum*, and target. Let us first consider an example, like: All librarians (X) trade some books (Y) to some library users (Z). Here the librarians are the ground of the relations (since connect books with users: in fact, we may assume that the users get the books only because the librarians have given them to them), the books are the *relandum* that will be connected with users thanks to librarians, and the users are the target of this ternary relation. Let us ask what the inverse would be. Intuitively, this would consist of the library users giving back the books to the librarians. This means that while the class X (librarians) was the ground of the relation between Y and Z, here, in the inverse case, it is the class Z (the library users) to be such a ground. Moreover, to inverse this relation means to affirm that some users *give* some books *back* to all librarians.

Such a structure is that of a *dynamic* relation. In fact, a genuine triadic relation presupposes that the ground is already dyadically related to both the relandum and the target while, as mentioned, the latter two are not beforehand. Now, the ground let the latter two come in contact, that is, interact. Such an interaction is an event, for instance: the act to give a book. As a result of this interaction and of the whole triadic relation there is now a new (binary) relation between relandum and target. For instance, the library users now temporarily possess some books, where the latter binary relation is different from the original ternary relation.

Let us come back to quantum mechanics (but the same mechanism of information acquisition is true for classical mechanics). When measuring, we have an object system, an apparatus and a detector. The apparatus is the ground of the relation and is in fact connected with both the object system (the target) and the detector (the relandum). When the detector clicks (the event here), thanks to these previous binary relations, a new (binary) relation between detector and object system is established such that it allows us to infer what was the value of the variable or observable that we have measured. In such a way we have acquired information (thanks to mutual information between object and apparatus and to information selection at detection).

If we accept distribution as a logical law, it appears that a ternary relation of the kind $R[X + (Y \times Z)]$ could be reduced to the two binary relations $R(X + Y)$ and $R(X + Z)$. This is quite true at a pure formal logical level but it does not express the true nature of a genuine ternary relation (a ternary relation that is irreducible). In fact, a genuine ternary relation $R[X + (Y \times Z)]$ presupposes something more, for instance some relations $A(X + Y)$ and $B(X + Z)$, which in general are different from the relation R. For instance, librarians have a relation A with books and a relation B with library users, which are obviously different from each other but also from R (that expresses the act of assigning books to library users). Also the final relation that connects the two previous unrelated items (now in fact library users temporarily possess books) is again different and can be expressed as $C(Y \times Z)$.

Dynamic relations and symmetries

We can conceive the dynamic relation as a vectorial one, what implies that we formalize this inversion by selecting the complementary class of each involved term, and it is here that the reversal operation becomes helpful fro dealing with our problem. In fact, if in the original relation the librarians were agent and users patient, here this relation is reverted. Now, we need to codify such a reversal in a way that is both logically consistent and semantically satisfactory. Therefore, if we have a relation $A[X' + (Y \times Z)]$, its inverse can be expressed as $B[(Z' \times Y') + X]$. For instance, library users giving back books to librarians is not equivalent to librarians giving books to library users. In other words, in most cases a ternary relation is not the same as its inverse. Indeed, the “book given" in the previous relation meant “book borrowed" but in the inverse relation means “book given back". Therefore, to apply the reversal operator to a ternary relation has rather the significance to restore an original situation, as it is the case when we speak of reversible laws of physics. As we have seen, we need a new ground for being able to reverse (in the physical sense) a relation. For this reason, as already mentioned, it is crucial to distinguish between binary relations as expressing functions, that is, static (and formal) dependencies of some classes on some other classes on the one hand, and ternary relations as expressing transformations. Due to this dynamical character, ternary relations can be associated to causal processes: things modify other things through some connection.

Dynamic relations and symmetries

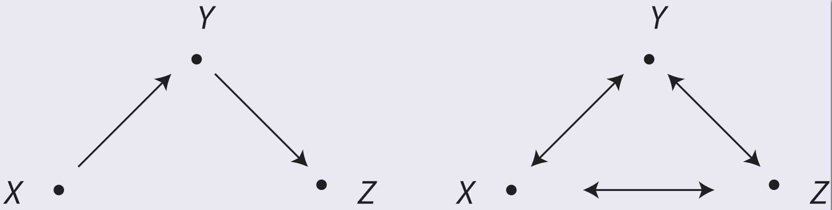

Figure: Dynamic relations. On the left a typical dynamical relation; on the right the possible correlations.

Dynamic relations and symmetries

It could be raised the problem that also binary relations give rise to correlations (and absence of correlations). However, the meaning of binary relations is a (functional) dependence: when Y depends on X we can ask what is the dependence of X on Y, so that, when we have a bidirectional (antisymmetric) dependence as shown on the left of Fig. 3, the correlation expresses precisely this kind of bidirectionally. In fact, it is true that $R(X' + Y) \times R(X + Y')$ gives rise to the correlation $X \Leftrightarrow Y$ and $R(X' + Y') \times R(X + Y)$ to the absence of correlation $X \not\Leftrightarrow Y$ (or $R(X \times Y) + R(X' \times Y')$ and $R(X \times Y') + R(X' \times Y)$ to correlation and absence of correlation, respectively). However, this kind of bidirectionality does no longer exist in the case of ternary relations, as displayed in Fig. 5. Here, although the ternary relations still determine certain correlations, they present a precise directionality that, even when reversed, does not coincide with the possible involved correlations (or absence of correlations), as shown on the right of the same figure (this is why I have spoken of a vectorial nature of this relations). In fact, if the scheme on the right of Fig. 5 would represent any kind of relation but correlations we would find ourselves in a never ending circle that would destroy the very general concept of ternary relation. This is what gives to the graphical representation of relations the character of directed graphs [*PEARL* 1988], obviously also in the case of most binary relations.

Dynamic relations and symmetries

The previous formal expression for ternary relations and their inverses could raise some interpretative problem. In fact, the ground of the inverse relation (Z') does not appear isolated as in the original relation (where it was X) but rather appears in connection with Y'. The formulation can be interpreted as X being now a kind of "passive" ground of the relation between the other two classes, which gives precisely the meaning of an inverted dynamical action, because the notion of passive ground is equivalent to the concept of target. Moreover, the fact that the couple of classes that in the original relation followed the ground is here first, is also quite natural, since a reverse relation need to start by first making reversible the connection between these two classes. Therefore, each time that we find a relation written as $R[(Z' \times Y') + X]$, we need to interpret this as meaning that Some Z are the ground of a relation between some Y and all X. Obviously, there is a degree of arbitrariness in writing the relation in this way and in its interpretation. However, this does not touch the formal nature of the relation and rather deals with its semantic interpretation (and therefore depends on semantic conventions that must be stipulated in advance). What really does matter is that the relation $R[(Z' \times Y') + X]$ and the relation $R[X' + (Y \times Z)]$ (in whatever order may be written) are the inverse of each other.

Formally, sequences of couple of transformations being the inverse of each other that leave the original situation unchanged express *symmetries*, as it is ubiquitous in geometry (for instance the cylindrical symmetry around an axis) or in physics (for instance, time or space translational symmetry). If ternary relations express symmetries, then they must give rise to correlations (or absence of correlations) among some of the involved classes. This is in fact the case, what shows a strict connection between correlations and reversal operation since those symmetric relation are the reversal of themselves, and this can be taken as a formal definition of symmetry.

Dynamic relations and symmetries

Level	Relations	Relation ID	Operation	Symmetry
$R(X' + Y' + Z')$ $R(Z + Y + X)$	11111110 01111111	$\times$	$R(X \not\Leftrightarrow Y \not\Leftrightarrow Z)$	01111110
$R(X' \times Y' \times Z')$ $R(Z \times Y \times X)$	10000000 00000001	$+$	$R(X \Leftrightarrow Y \Leftrightarrow Z)$	10000001
$R(X \times (Y' + Z'))$ $R((Z + Y) \times X')$	00001110 01110000	$+$	$R(X \not\Leftrightarrow Y \not\Leftrightarrow Z)$	01111110
$R(X' + (Y \times Z))$ $R((Z' \times Y') + X)$	11110001 10001111	$\times$	$R(X \Leftrightarrow Y \Leftrightarrow Z)$	10000001

Table: Some symmetries of ternary relations.

Conclusions

Coming back to quantum mechanics,

- Self-correlations and dynamical triadic relations express the basic physical distinction between entanglement (or superposition) and transmission of a signal (classical causality). The former can be non-local only by violating separability, i.e. without any signal transmission, while the latter requires a local interaction.
- In this way we are able to account for the two main operations when dealing with information (apart from information processing): information sharing and information selection. Moreover, dynamical triadic relations express the process of information acquiring (as during measurement) in a quite faithful way.

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