Analytic continuation as a bridge between continuum and bound states

L.D. Blokhintsev

Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, Moscow, Russia

Asymptotic normalization coefficients (ANC) determine the asymptotics of nuclear wave functions in binary channels. ANCs are proportional to vertex constants (VC), which determine the virtual processes A $\Longrightarrow$ B+C.

VCs and ANCs are fundamental nuclear characteristics. They are used actively in analyses of nuclear reactions within various approaches. VCs and ANCs extracted from one process can be used for the prediction of characteristics of other processes. Comparing of empirical values of VCs and ANCs with theoretical ones enables one to evaluate the quality of a model. ANC for the channel A $\Longrightarrow$ B+C determines the probability of the configuration B+C in nucleus A at distances greater than the radius of nuclear interaction.

Thus ANCs arise naturally in cross sections of nuclear reactions between charged particles at low energies, in particular, of astrophysical nuclear reactions.

Note: Coulomb barrier ⇒ cross sections at astrophysical energies are so small that their direct measurement in laboratories is very difficult, or even impossible. At low energies

$$\sigma(E) \propto \frac{1}{k}\frac{2\pi\eta}{e^{2\pi\eta}-1}, \quad \eta = Z_B Z_C e^2 \mu / k -$$ Coulomb parameter

Usually one uses

$$S(E) = E e^{2\pi\eta}\sigma(E) -$$ astrophysical *S* factor

It was shown (A.M.Mukhamedzhanov *et al*.) that the cross section of $B(C,\gamma)A$ at astrophysical energies with a good accuracy is determined by the value of the ANC in the $B+C$ channel. This conclusion made it possible to calculate $S(E=0)$ for a number of radiative capture processes:

^{4}He(d, γ)^{6}Li, ^{4}He(^{3}He, γ)^{7}Be, ^{7,9}Be(p, γ)^{8,10}B, ^{8}B(p, γ)^{9}C,
^{11,13}C(p, γ)^{12,14}N, $^{12\text{-}14}$N(p, γ)$^{13\text{-}15}$O, ^{17}F(p, γ)^{18}Ne, ^{20}Ne(p, γ)^{21}Na

Definition and Properties of ANCs and VCs

$$I_{ABC}(LS;r)\Big|_{r\to\infty} \to C_{ABC}(LS)\, e^{-\kappa r}/r; \qquad \kappa^2 = 2\mu\varepsilon, \quad \varepsilon = m_B + m_C - m_A. \quad (1)$$

$I_{ABC}(LS;r)$ – radial overlap integral of wave functions of A, B, C.

Coulomb interaction ⇒ $e^{-\kappa r} \to W_{-\eta, L+1/2}(2\kappa r)$

$\eta = Z_B Z_C e^2 \mu/\kappa$ - Coulomb parameter.

VC $G_{ABC}(LS)$ is the on-shell matrix element of the virtual $A \leftrightarrow B+C$ process in the given partial-wave state LS.
It is related to the amplitude of elastic BC scattering:

$$\mathrm{res} < LS \mid M^{J_A} \mid LS >|_{E=-\varepsilon} = (-1)^L G^2_{ABC}(LS)$$

G_{ABC} and C_{ABC} are interrelated:

$$G_{ABC}(LS) = -(\pi N_{BC})^{1/2} C_{ABC}(LS)/\mu.$$

N_{BC} arises due to the identity of nucleons.

$$1 \le N_{BC} \le \frac{(A_B + A_C)!}{A_B! A_C!}$$

Often N_{BC} is included into C_{ABC}.

NB! The asymptotics (1) can be rigorously proved for two-body systems only. For three- and more particle systems the asymptotics of overlap integrals may differ from (1) ('anomalous asymptotics'). (L.B. Yad. Fiz. 1981. V.34. P.944).

Considering the Fourier component $I(q)$ of the overlap integral $I(r)$ one gets

$$I(r)\big|_{r\to\infty} = c_0 e^{-\kappa r}/r + c_1 e^{-\kappa_1 r}/r^p, \quad p>1, \qquad (2)$$

where $i\kappa_1$ is the nearest to the origin singular point of the vertex function $G(q)$ for the $a \to b+c$ vertex. If $\kappa_1 < \kappa$, then the second term in (2) dominates at $r \to \infty$.

Consider the diagram of Fig. 1, which contributes to $G(q)$. It results from the Faddeev expansion in the simplest three-body model, in which a consists of d, f and c, b is a bound state of d and f, and e is a bound state of f and c.

For that diagram $p = 2$ and the singular point is:

$$q = i\kappa_1 = i\frac{m_b}{m_d}(\kappa_{ade} + \kappa_{bdf}), \quad \kappa_{ijk} = 2\mu_{jk}\varepsilon_{ijk}, \quad \varepsilon_{ijk} = m_j + m_k - m_i. \qquad (3)$$

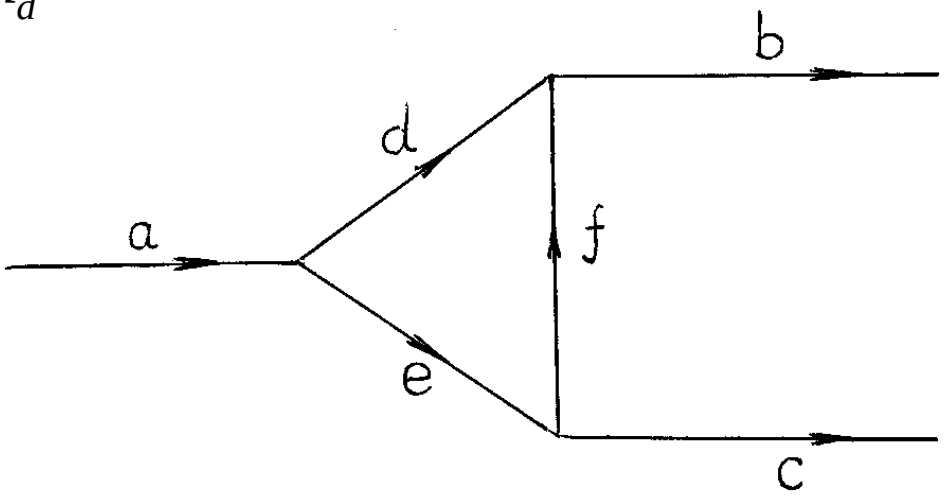


Fig. 1

At any ε_{abc} the 'anomalous' condition $\kappa_1 < \kappa$ could be satisfied if ε_{ade} and ε_{bdf} are sufficiently small.

Methods of determination of ANCs and VCs

1. Microscopic calculations ➡ very tedious

1a. Coordinate representation ➡ asymptotical region ➡ small values of wave functions ➡ low accuracy

1b. Momentum representation ➡ continuation to imaginary values of momenta

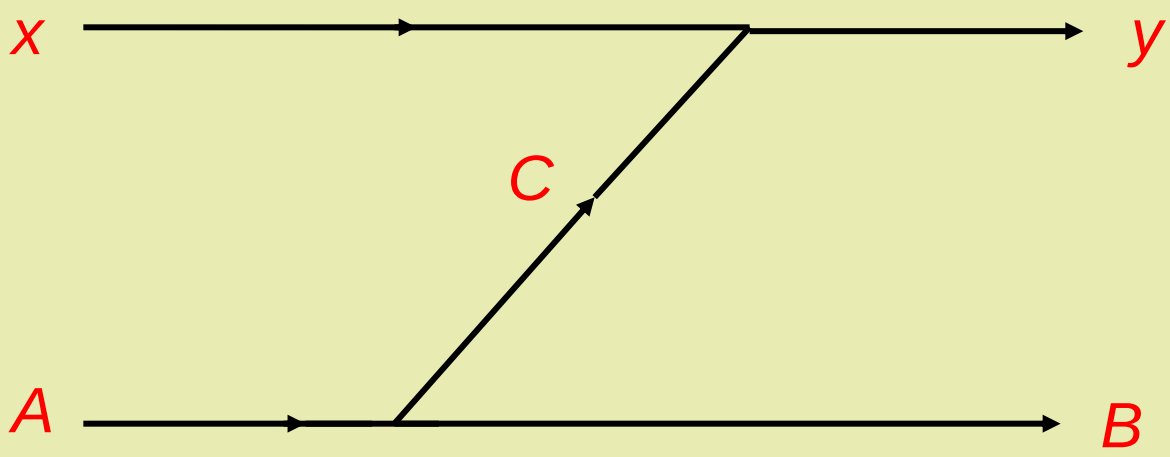
Ab initio calculations

1.M.Viviani, A.Kievsky, M.Rosati. PRC **71**,024006 (2005).
ANCs for an alpha-particle, hyperspherical harmonics.

2. K.M.Nolett, R.B.Wiringa. PRC **83,** 041001 (2011).
ANCs for one-nucleon removals from nuclei with $3 \leq A \leq 9$, GFMC method.

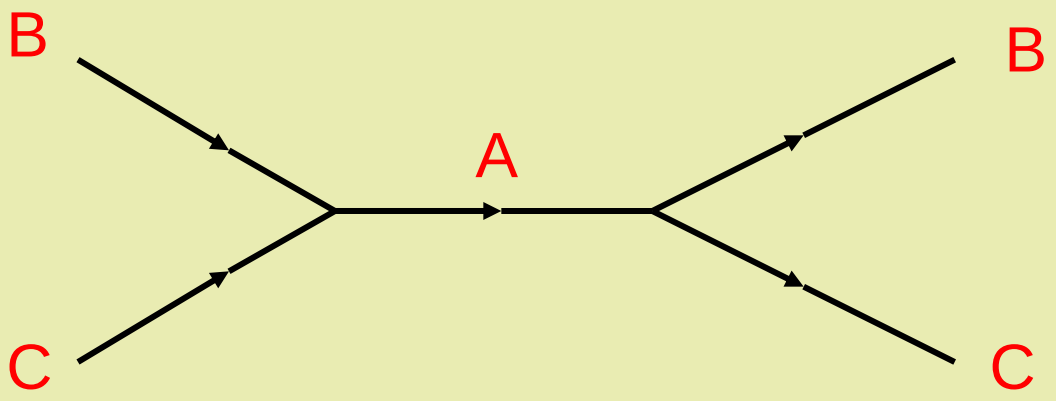
2. Analysis of scattering and reactions

2a. Analysis of data on transfer reactions $A(x,y)B$ proceeding through the pole mechanism



The cross section of this reaction possesses the 2nd order pole at $z = z_0$ ($z = \cos\theta$, $|z_0| > 1$) . If one extrapolates the experimental values of $(z- z_0)^2\ \sigma(z)$ to the pole position ($\sigma(z)$ is the differential cross section), one immediately obtains the value of $|G_{ABC}\ G_{yxC}|^2$.

2b. Extrapolation in energy *E* of the partial-wave amplitude of elastic *BC* scattering (obtained by the phase-shift analysis) to the pole corresponding to the bound state *A*.



2d. For peripheral nuclear reactions VC could be obtained from the DWBA analysis of differential cross sections .

The problem of using continuum-state data to obtain information on bound-state characteristics is not trivial !!!

"Bound-state properties cannot be extracted from the phase shifts of a single partial wave, as a matter of principle".
(R.G.Newton. Scattering theory for waves and particles, 2nd. Ed.Springer-Verlag, New York, 1982; repeated in some original papers)

This assertion is based on the existence of phase-equivalent potentials (PEP). Different PEPs lead to coinciding phase shifts $\delta_L(E)$ but properties of the bound states for a given L are different.

Inverse scattering problem: to restore a local potential one needs:

i) $\delta_L(E)$, $0 \le E < \infty$.
ii) $2N_L$ parameters (N_L – number of bound states for a given L).
$2N_L \Rightarrow N_L$ binding energies and N_L ANCs.

Methods of constructing PEP's

1) Bargmann potentials (R.Newton, Scattering Theory).
2) Supersymmetric transformation (E.Witten (1981)).

Supersymmetric transformation can be used to construct PEP, which differs from the initial potential by any modification of the bound spectrum. A bound state can be added or suppressed; its binding energy and/or ANC can be modified.

Inference: Within the formal potential approach with arbitrary potentials and without any additional conditions, it is impossible to determine unambiguously characteristics of bound states knowing only $\delta_L(E)$.

The way to resolve the ambiguity problem:

The natural requirement that amplitudes of processes are analytic functions of their kinematic variables.

Microcausality principle $\Longrightarrow$ Analyticity

Using analiticity and knowing the partial-wave *BC* scattering amplitude $f_L(E)$ on some segment of the real positive semiaxis, one can continue analytically $f_L(E)$ to the unphysical region $E < 0$ and obtain both the position of the pole $E = -\varepsilon < 0$ and the residue of $f_L(E)$ at that pole, that is, VC and ANC.

Note. We discuss the principal side of the problem and not the practical ways of analytic continuation.

Two-body potential scattering

Knowing ε, ANC C_{ABC}, and $f_L(E)$ at $0 \le E < \infty$, one can construct unambiguously a local potential $V(r)$ using methods of the inverse scattering problem. As a result, the unique "analytic" potential would be selected out of set of PEPs, which leads to the needed analytic properties of the scattering amplitude. That potential describes all bound and continuum states of a given system.

Characteristics of a bound state obtained by the direct analytic continuation of $f_L(E)$ from $E \ge 0$ to $E < 0$ may differ from the characteristics found by solving the bound state problem with the potential which describes correctly $f_L(E)$ at $E \ge 0$. Why?

In potential scattering theory

$$f_L(k) = -\frac{\mu}{2\pi}\int_0^\infty dr\,\varphi_L(kr)V(r)\psi_L(kr)$$

If k is continued to the complex plane, the terms $e^{2|\mathrm{Im}k|r}V(r)$

arise in the integrand, which lead to the divergence of the integral if $V(r)$ does not decrease rapidly enough at $r \to \infty$.

$f_L(k)$ in the above form can be analytically continued to $k = i\kappa$ ($E=-\varepsilon$) if (R.Newton, Scattering Theory):

$$\int_0^\infty dr\, r\left|V(r)\right| e^{2\kappa r} < \infty \qquad (L=0).$$

Other possibilities of analytic continuation of amplitudes

- the explicit form of $f_L(E)$ at $E \geq 0$ is known
- one succeeds in approximating $f_L(E)$ at $E \geq 0$ by a certain analytic expression accurately enough

Consider a trivial example: $$\varphi(z) = \int_0^\infty e^{(a-z)t}\, dt$$

$\varphi(z)$ initially is defined *only* at Re z > Re a since if this inequality is violated, the integral diverges.

On the other hand, the integration can be performed explicitly:

$$\varphi(z) = \frac{1}{z-a}$$

. This expression defines the function, which is analytic in the *whole complex* z *plane* with a pole at $z = a$.

Instructive example of a Bargmann-type potential

(R.Newton, Scattering theory)

$$V_d(r) = -\frac{\kappa}{\mu}\frac{d}{dr}\left[\mathrm{sh}(br)\frac{g_d(\kappa,r)}{g_d(\kappa+b,r)-g_d(\kappa-b,r)}\right], \qquad b>\kappa.$$

$$g_d(x,r) = x^{-1}[e^{-\kappa r} + d\,\mathrm{sh}(xr)].$$

Normalized bound-state wave function ($\varepsilon = \kappa^2/2\mu$):

$$\varphi_d(r) = 2\sqrt{\frac{\kappa d}{b^2-\kappa^2}}\,\frac{\mathrm{sh}(br)}{g_d(\kappa+b,r)-g_d(\kappa-b,r)}$$

S wave scattering amplitude <u>*does not depend*</u> on *d*:

$$f(k) = \frac{e^{2i\delta}-1}{2ik} = \frac{1}{k\,\mathrm{ctg}\,\delta - ik} = \frac{b+\kappa}{-b\kappa+k^2-i(b+\kappa)k}$$

$f(k)$ can be analytically continued to the pole $k = i\kappa$ and ANC is expressed through the residue at that pole:

$$C = \left[\frac{2\kappa(b+\kappa)}{b-\kappa} \right]^{1/2}$$

On the other hand, the expression for ANC depending on the parameter d can be obtained from the explicit form of φ_d:

$$C_d = \left[\frac{4\kappa(b+\kappa)}{d\,(b-\kappa)} \right]^{1/2}$$

$C_d = C$ at $d = 2$ only.

Why $d = 2$ is special? Because the asymptotics of $V_d(r)$ at $d = 2$ is special:

$$V_d(r)_{r\to\infty} \to \begin{cases} -V_1 \cdot e^{-2\kappa r}, & d \neq 2, \\ -V_2 \cdot e^{-2br}, & d = 2. \end{cases}$$

Since $b > \kappa$, the necessary "analyticity" condition:

$$\lim\Big|_{r\to\infty} V_d(r) e^{2\kappa r} = 0$$

is satisfied at $d = 2$ only.

Thus the analytic continuation of the amplitude $f(k)$ to the region of imaginary k allows one to select from the set of PEP $V_d(r)$ the only "analytic" potential corresponding to $d = 2$ and find the correct value of the ANC C.

e sypersymmetric transformation acquire
ype $1/r^2$ at the origin and, as well as $V_d(r)$
y the integral analyticity condition.

$$\int_0^\infty dr\, r \left|V(r)\right| e^{2\kappa r} < \infty.$$

Reactions with composite systems (nuclei)

Description of elastic nucleon-nucleus or nucleus -nucleus scattering in the two-body potential approach $\Longrightarrow$

corresponding potential is complex, nonlocal and energy- and angular momentum dependent.

Nevertheless, one can still use analytic continuation of $f_L(E)$ to $E<0$ to find the binding energy and the VC and ANC.
Analytic continuation can be performed in different ways.

In the work (L.B., V.I.Kukulin *et al.*, PRC 48, 2390 (1993)) VC $G_{^6\mathrm{Li}\alpha d}$ and ANC $C_{^6\mathrm{Li}\alpha d}$ for the *S* state of ^{6}Li were found in two ways.

1. Analytic approximation of experimental values of $k \cot \delta$ using Pade approximants and subsequent continuation to $E<0$.

2. Constructing the effective two-body $d\alpha$ potential $V_{d\alpha}(r)$ describing experimental δ and finding the two-body bound-state wave function for this potential. $V_{d\alpha}(r)$ was written as a sum of gaussians and satisfied the necessary analyticity conditions.

The results of two different methods are in close agreement.

Important comment

In the general case, when B and/or C are composite systems, ANC C_{ABC} corresponds to the overlap integral $I_{ABC}(r)$, which is normalized not to 1 but to the spectroscopic factor $S_{ABC.}$

However, if ANC is found in the two-body model, the corresponding two-body bound-state wave function should be normalized to 1. Normalizing this function to the independently determined spectroscopic factor is incorrect.

Inference

1. Using the fundamental analyticity property of scattering amplitudes and analytic continuation methods allows one to obtain information on characteristics of nuclear bound states (including ANCs) from the phase shift data. Thus the ambiguity related to the existence of phase-equivalent potentials is removed.

2. The most effective method of analytic continuation $\Longrightarrow$ analytic approximation of the experimental values of $k \cot \delta$.

3. If the continuation is performed by fitting a two-body potential, one should use the potential which decreases rapidly enough at $r \to \infty$. One should put the spectroscopic factor equal to 1.

Analytic continuation of effective range expansion (ERE)

One of the most widespread methods is the analytic continuation in energy of the data on the partial wave amplitude of elastic *BC* scattering to the pole corresponding to the bound state *A*. The most effective way of realization of this procedure is the analytic continuation of the effective range function K_L (k_2). This method was used (L.B., V.I.Kukulin et al.) to obtain *S* wave VCs and ANCs for the process ^{6}Li $\rightarrow \alpha$ + d, by Yu.V.Orlov et al. for the systems ^{3}H, $^{2;3;5}$He, ^{5}Li, ^{8}Be, and by J.-M.Sparenberg et al. for the systems ^{16}O + n, ^{16}O + p, and ^{12}C + α.

All above works treated one-channel elastic scattering. However, the description of scattering of particles with nonzero spins even in the absence of inelastic channels often demands account of channel coupling. The most typical situation induced by tensor forces is the case of two coupled channels 1 and 2 with the same J but different L (L_1 and $L_2 = L_1 + 2$). Examples: $d\alpha$ scattering, NN triplet scattering etc. In principle, coupled channels may differ not in L but in channel spins.

In the work [1] (L.B., Yad. Fiz., 2011. V.74. P.1008) it was considered the generalization of the ERE to the case of two coupled channels and using that expansion for the determination of VCs and ANCs. The consideration in [1] was carried out for the short-range interaction, which practically limited using the formalism developed to the reactions induced by neutrons. Then the results of [1] were generalized to account of the Coulomb interaction which radically changes analytic properties of scattering amplitudes and their behavior at low energies.

The formalism developed could be applied to any two-channel nuclear system, for which the results of the phase-shift analysis are known (including the mixing parameter). One of similar important systems is ^{6}Li in the $\alpha + d$ channel. The ANC values for this system determine the cross section of the radiative capture ^{4}He(d,γ)^{6}Li, which is the main process of ^{6}Li formation in the big bang model. Direct measurements of that process at astrophysical energies are absent due to the smallness of the cross section. The data on the values of the VCs and ANCs for $^6\text{Li} \rightarrow \alpha + d$ (L= 0; 2) channel obtained by different methods are characterized by a large spread. In the first place it refers to the D-state constants.

The procedure of analytic continuation of the two-channel ERE described above has been applied to $d\alpha$ scattering using several sets of phase shifts (L.B. and D.A.Savin). The values of the VCs and ANCs for $^6\mathrm{Li} \rightarrow \alpha + d$ ($L = 0; 2$) were extracted.

Using the data from the available phase-shifts analisis results in

$$G_0^2 = 0.4 – 0.5 \text{ fm}; \; C_0 = 2.3 – 2.8 \text{ fm}^{-1/2}.$$

Solution of Faddeev equations (neglecting the Coulomb interaction) gives

$$G_0^2 = 0.3 \text{ fm}; \; C_0 = 2.0 \text{ fm}^{-1/2}$$

Low accuracy of phase-shift analysis at low energies and simplicity of Faddeev equations used did not let to obtain accurate values of VC and ANC for $L = 2$.

$$C_2 = 0.02 – 0.05 \text{ fm}^{-1/2}$$

The sign of C_2 (relative to C_0) turned out to be positive.

Outlook

The method suggested is operable.
To get the reliable values of VCs and ANCs for $^6Li \rightarrow \alpha + d$ it is desirable:

- To measure more accurately the $d\alpha$ scattering differential cross section at low energies

- To perform the thorough phase-shift analysis of the corresponding data

- To perform Faddeev calculations of ^6Li in the 3-body model with realistic pair potentials and account of the Coulomb interaction

Including inelastic channels

The procedure described above considers elastic channels only. On the other hand, low-lying inelastic thresholds might influence the ERE. (The typical example is the deuteron-nucleus scattering). The simplest way to allow for an inelastic channel at $E = E_0$ is to include into the ERE an additional term, which is complex at $E > E_0$.

The form of that term should lead to the correct analytic behavior of scattering amplitudes at the threshold $E = E_0$. According to the general theory of singularities of Feynman diagrams (L.D.Landau), a singular part of a scattering amplitude near a threshold behaves like $(E - E_0)^{(3n-5)/2}$ at n even and like $(E - E_0)^{(3n-5)/2} \ln(E - E_0)$ at n odd, where $n = 2,3,4,\ldots$ is a number of intermediate particles at the threshold.

Adding to the standard ERE singular terms written above, one obtains for the inelasticity coefficient η_n for the most important two-particle ($n = 2$) and three-particle ($n = 3$) thresholds:

$$\eta_2 = \left\{ \frac{K_0^2 + \left(k - d\sqrt{E/E_0 - 1}\right)^2}{K_0^2 + \left(k + d\sqrt{E/E_0 - 1}\right)^2} \right\}^{1/2}, \quad E > E_0,$$

$$\eta_3 = \left\{ \frac{[K_0 + d(E/E_0 - 1)^2 \ln|E/E_0 - 1|]^2 + [k - \pi d(E/E_0 - 1)^2}{[K_0 + d(E/E_0 - 1)^2 \ln|E/E_0 - 1|]^2 + [k + \pi d(E/E_0 - 1)^2} \right\}^{1/2}, \quad E > E_0.$$

K_0 is the standard effective range function and $d > 0$ is an additional fitting constant.

We plan to apply the above formulas to the $d\alpha$ scattering.


Thank U
4 attention