# Applications of dissipative and anisotropic hydrodynamics in description of early stages of relativistic heavy-ion collisions

Wojciech Florkowski

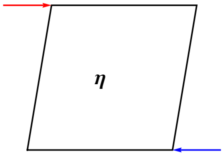
Jan Kochanowski University, Kielce, Poland
&
Institute of Nuclear Physics PAN, Krakow, Poland

in collaboration with
G. Denicol, E. Maksymiuk, R. Ryblewski, M. Strickland, L. Tinti

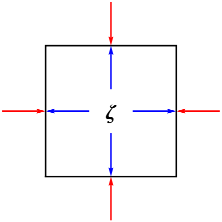
3. International Conference on New Frontiers in Physics
July 28 – August 6, 2014
Kolymbari, Crete, Greece

- relativistic hydrodynamics plays an important role in modeling of relativistic heavy-ion collisions and other physical processes
- strongly-coupled $\mathcal{N} = 4$ SYM theory impose lower bound of the $\eta/\mathcal{S}$, dissipative corrections important (Kovtun, Son, and Starinets, Phys.Rev.Lett. 94, 111601 (2005))
- relativistic viscous hydrodynamics describes experimental data very well

SHEAR VISCOSITY




BULK VISCOSITY




both the shear and bulk viscosities induce corrections to the equilibrium pressure

- canonical treatment based on an expansion of the general distribution function around local equilibrium state (corrections give rise to dissipative currents)

$$f(x,p) = f_{\rm iso}\left(\frac{p^\mu u_\mu}{T(x)}\right) + \delta f(x,p)$$

$\Rightarrow$ early thermalization required

- large anisotropy at early times predicted by microscopic models (CGC, AdS/CFT, ...)
- studied systems are subject to rapid longitudinal expansion

$\Rightarrow$ large viscous corrections to the ideal energy-momentum tensor
$\Rightarrow$ canonical expansion breaks down
$\Rightarrow$ may cause unphysical results

---

- form of relaxation-type equations of motion for the shear-stress tensor and bulk viscous pressure must be derived within a certain framework
- large uncertainties concerning transport coefficients that appear in the equations of motion, many of them not known yet
- at large $T$ the coupling is weak, theory is nearly conformal, the bulk viscosity is expected to be small
- near $T_C$ it can be large enough to affect the time evolution of the matter

- methods to improve early-time dynamics:
  - complete second-order treatments (Denicol, Niemi, Molnar, Rischke)
  - third-order treatments (El, Xu, Greiner, Jaiswal)
  - anisotropic hydrodynamics (Florkowski, Martinez, Nopoush, Ryblewski, Strickland, Tinti, Bazow, Heinz)

W. Florkowski, R. Ryblewski, Phys.Rev. C83, 034907 (2011)
M. Martinez, M. Strickland, Nucl. Phys. A848, 183 (2010)

note/warning: still most commonly used Israel-Stewart formulation

W. Florkowski, R. Ryblewski, M. Strickland, Nucl.Phys. A916 (2013) 249-259
W. Florkowski, R. Ryblewski, M. Strickland Phys.Rev. C88 (2013) 024903

---

- anisotropic hydrodynamics → one expands around an anisotropic background, momentum-space anisotropies are built into the LO

$$f(x,p) = \underbrace{f_{\rm iso}\left(\frac{\sqrt{p^\mu \Xi_{\mu\nu} p^\nu}}{\Lambda(x)}\right)}_{\text{LO}} + \underbrace{\delta\tilde{f}(x,p)}_{\text{NLO}}$$

spheroidal ansatz for $\Xi_{\mu\nu}$ give (LRF) $p^\mu \Xi_{\mu\nu} p^\nu = p_x^2 + p_y^2 + (1+\xi)p_z^2$ (R-S form)

- dynamical equations derived from an underlying classical kinetic theory framework by taking moments of Boltzmann equation
- anisotropic hydrodynamics has various appealing features (no negative pressures, reproduced free-streaming limit, kinetic coefficients included implicitly ...)

- one wants to assess how well these various dissipative relativistic hydrodynamics approaches describe the non-equilibrium evolution of the system
- some exactly solvable cases necessary
- one possibility for doing this is to compare predictions of hydrodynamic models with exact solutions of the underlying kinetic theory
- in general not possible, there are some cases in which this can be done

# Outline

1. Exact solution of the kinetic equation for a massive gas
2. Bulk viscous pressure evolution within viscous hydrodynamics
3. Bulk viscous pressure evolution in anisotropic hydrodynamics
4. Results

- Boltzmann equation in the relaxation time approximation

$$p^\mu \partial_\mu f(x,p) = C[f(x,p)] \qquad C[f] = p^\mu u_\mu \frac{f^{\rm eq} - f}{\tau_{\rm eq}}$$

Bhatnagar, Gross, Krook, Phys. Rev. 94, 511 (1954)

background distribution (Boltzmann statistics)

$$f^{\rm eq} = \frac{g_s}{(2\pi)^3} \exp\left(-\frac{p^\mu u_\mu}{T}\right)$$

- boost-invariant variables (Bialas, Czyz)

$$w = t p_\parallel - zE \qquad v = tE - z p_\parallel$$

---

- for transversely homogeneous boost-invariant system

$$\frac{\partial f}{\partial \tau} = \frac{f^{\rm eq} - f}{\tau_{\rm eq}}$$

$$f^{\rm eq}(\tau, w, p_\perp) = \frac{g_s}{(2\pi)^3} \exp\left(-\frac{\sqrt{w^2 + (m^2 + p_\perp^2)\tau^2}}{T\tau}\right)$$

- particle density, energy density, transverse and longitudinal pressure

$$
\begin{aligned}
n(\tau) &= g_0 \int dP \, \frac{v}{\tau} \, f(\tau, w, p_\perp) \\
\mathcal{E}(\tau) &= g_0 \int dP \, \frac{v^2}{\tau^2} \, f(\tau, w, p_\perp) \\
\mathcal{P}_T(\tau) &= g_0 \int dP \, \frac{p_T^2}{2} \, f(\tau, w, p_\perp) \\
\mathcal{P}_L(\tau) &= g_0 \int dP \, \frac{w^2}{\tau^2} \, f(\tau, w, p_\perp)
\end{aligned}
$$

$$T^{\mu\nu} = (\mathcal{E} + \mathcal{P}_T) u^\mu u^\nu - \mathcal{P}_T g^{\mu\nu} + (\mathcal{P}_L - \mathcal{P}_T) z^\mu z^\nu$$
$$T^{\mu\nu}_{\rm eq} = (\mathcal{E}_{\rm eq} + \mathcal{P}_{\rm eq}) u^\mu u^\nu - \mathcal{P}_{\rm eq} g^{\mu\nu}$$

$$T^{\mu\nu}_{\rm LRF} = \mathrm{diag}(\mathcal{E}, \mathcal{P}_T, \mathcal{P}_T, \mathcal{P}_L)$$
$$T^{\mu\nu}_{\rm eq;LRF} = \mathrm{diag}(\mathcal{E}_{\rm eq}, \mathcal{P}_{\rm eq}, \mathcal{P}_{\rm eq}, \mathcal{P}_{\rm eq})$$

---

- determination of effective temperature

$$u_\mu T^{\mu\nu} = u_\mu T^{\mu\nu}_{\rm eq}$$
$$\mathcal{E}(\tau) = \mathcal{E}^{\rm eq}(\tau)$$
$$= g_0 \int dP \, \frac{v^2}{\tau^2} f^{\rm eq}(\tau, w, p_\perp)$$
$$= \frac{g_0 T m^2}{\pi^2} \left[ 3 T K_2\left(\frac{m}{T}\right) + m K_1\left(\frac{m}{T}\right) \right]$$

- formal solution

$$f(\tau,w,p_\perp) = D(\tau,\tau_0) f_0(w,p_\perp) + \int\limits_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{\rm eq}(\tau')} D(\tau,\tau')\, f^{\rm eq}(\tau',w,p_\perp)$$

$$D(\tau_2,\tau_1) = \exp\left[-\int\limits_{\tau_1}^{\tau_2} \frac{d\tau''}{\tau_{\rm eq}(\tau'')}\right]$$

- initial condition (Romatschke-Strickland form)

$$f_0(w,p_\perp) = \frac{g_s}{(2\pi)^3} \exp\left[-\frac{\sqrt{(1+\xi_0)w^2 + (m^2+p_\perp^2)\tau_0^2}}{\Lambda_0\,\tau_0}\right]$$

$\xi_0 = \xi(\tau_0)$ - initial value of the anisotropy parameter
$\Lambda_0 = \Lambda(\tau_0)$ - initial transverse-momentum scale

$$T m^2 \left[ 3 T K_2\left(\frac{m}{T}\right) + m K_1\left(\frac{m}{T}\right) \right] = \frac{g_s}{4}\left[ D(\tau,\tau_0)\Lambda_0^4 \widetilde{\mathcal{H}}_2\left(\frac{\tau_0}{\tau\sqrt{1+\xi_0}}, \frac{m}{\Lambda_0}\right) + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{\rm eq}(\tau')} D(\tau,\tau') T'^4 \widetilde{\mathcal{H}}_2\left(\frac{\tau'}{\tau}, \frac{m}{T'}\right) \right]$$

- numerical (iterative) method
  1) use a trial function $T' = T(\tau')$ on the RHS of the dynamic equation
  2) the LHS of the dynamic equation determines the new $T = T(\tau)$
  3) use the new $T(\tau)$ as the trial one
  4) repeat steps 1-3 until the stable $T(\tau)$ is found

---

effective bulk pressure in the kinetic theory

$$T^{\mu}_{\ \mu\,\rm LRF} = T^{\mu}_{\ \mu\,\rm visc;LRF} \qquad \to \qquad \Pi^k_\zeta = \frac{1}{3}\left[\mathcal{P}_\parallel(\tau) + 2\mathcal{P}_\perp(\tau) - 3\mathcal{P}_{\rm eq}(\tau)\right]$$

$$T^{\mu\nu}_{\rm visc} = \varepsilon u^\mu u^\nu - \Delta^{\mu\nu}(P + \Pi) + \pi^{\mu\nu}$$

$$T^{\mu\nu}_{\rm visc;LRF} = {\rm diag}(\varepsilon, P + \Pi + \pi/2, P + \Pi + \pi/2, P + \Pi - \pi)$$

energy and momentum continuity equation (zero net charge, no charge diffusion)

$$\partial_\mu T^{\mu\nu}_{\rm visc} = 0 \qquad \to \qquad \partial_\tau \varepsilon = -\frac{\varepsilon + P + \Pi - \pi}{\tau}$$

- at first order (Fourier–Navier–Stokes, acausal)

$$\pi = \frac{4\eta}{3\tau} \qquad\qquad \Pi = -\frac{\zeta}{\tau}$$

correct bulk viscosity coefficient in the relaxation time approximation

$$\zeta(T) \quad = \quad \tau_{\rm eq} P_{\rm eq} \frac{\mu^2}{3}\left[-\frac{\mu K_2}{3(3K_3+\mu K_2)} + \frac{\mu}{3}\left(\frac{K_1}{K_2} - \frac{K_{i,1}}{K_2}\right)\right]$$

Redlich, Sasaki, Phys.Rev.C 79 055207 (2009); Bozek, Phys.Rev.C 81 034909 (2010)

$$\cancel{\zeta(T) = \tau_{\rm eq} P_{\rm eq} \frac{\mu}{3}\left[\frac{3\left(G^2\mu - 5G - \mu\right)}{\mu^2 + 5G\mu - G^2\mu^2 - 1} + \frac{\mu^2}{3}\left(\frac{3G-\mu}{\mu^2} + \frac{K_1 - K_{i,1}}{K_2}\right)\right]}$$

Anderson, Witting, Physica 74, 466 (1974)

Cercignani, Kremer, *The Relativistic Boltzmann Equation: Theory and Applications*)

---

- at second order different methods employed:
  - → Grad's 14-moment approximation and second moment of Boltzmann equation
  - → Chapman-Enskog like expansion for distribution function close to equilibrium
- transport coefficients should be extracted by matching fluid dynamics to the underlying microscopic theory, most of them still not written in a convenient form to be implemented

- shear-stress evolution (no terms $\propto \lambda_{\pi\Pi}$ coupling)

$$\tau_\pi \dot{\pi} + \pi = \frac{4}{3\tau}\eta - \left(\frac{1}{3}\tau_{\pi\pi} + \delta_{\pi\pi}\right)\frac{\pi}{\tau}$$

$\frac{1}{3}\tau_{\pi\pi} + \delta_{\pi\pi} = 4/3\tau_\pi$
W. Israel, J. M. Stewart, Ann. Phys. (N.Y.) 118, 341 (1979)
$\frac{1}{3}\tau_{\pi\pi} + \delta_{\pi\pi} = 38/21\tau_\pi$
G. S. Denicol, H. Niemi, E. Molnar, and D. H. Rischke, Phys. Rev. D 85, 114047 (2012)
A. Jaiswal, Phys. Rev. C87, 051901 (2013)

- bulk pressure evolution (no terms $\propto \lambda_{\Pi\pi}$ coupling)

$$\tau_\Pi \dot{\Pi} + \Pi = -\frac{\zeta}{\tau} - \frac{1}{2}\tau_\Pi \Pi \left[\frac{1}{\tau} - \left(\frac{\dot{\zeta}}{\zeta} + \frac{\dot{T}}{T}\right)\right] \qquad (A)$$

Muronga, Phys.Rev.C 69, 034903 (2004);  Heinz, Song, Chaudhuri, Phys.Rev.C 73, 034904 (2006)

$$\tau_\Pi \dot{\Pi} + \Pi = -\frac{\zeta}{\tau} - \frac{4}{3}\tau_\Pi \Pi \frac{1}{\tau} \qquad (B)$$

Jaiswal, Bhalerao, Pal, Phys.Rev.C 87, 021901 (2013)

$$\tau_\Pi \dot{\Pi} + \Pi = -\frac{\zeta}{\tau} \qquad (C)$$

Heinz, Song, Chaudhuri, Phys.Rev.C 73, 034904 (2006)

M. Nopoush, R. Ryblewski, M.Strickland, Phys. Rev. C 90, 014908 (2014)

anisotropic one-particle distribution function

$$f(x,p) = f_{\rm iso}\left(\frac{\sqrt{p^\mu \Xi_{\mu\nu} p^\nu}}{\lambda}\right)$$

anisotropy tensor decomposition

$$\Xi^{\mu\nu} u^\mu u^\nu + \xi^{\mu\nu} - \Delta^{\mu\nu}\Phi$$

$$u_\mu \xi^{\mu\nu} = 0 \qquad u_\mu \Delta^{\mu\nu} = 0 \qquad \xi^\mu_\mu = 0 \qquad \Delta^\mu_\mu = 3$$

$$\xi^{\mu\nu} = \mathrm{diag}(0, \boldsymbol{\xi}) \qquad \boldsymbol{\xi} \equiv (\xi_x, \xi_y, \xi_z)$$

L. Tinti, W. Florkowski, Phys. Rev. C89 034907 (2014)
M. Nopoush, R. Ryblewski, M.Strickland, Phys. Rev. C 90, 014908 (2014)

equations of motion for $\xi_z$, $\Phi$, $\lambda$, $T$ for (0+1)d case are obtained by taking moments of the Boltzmann equation in the relaxation time approximation

$$p^\mu \partial_\mu f = p^\mu \frac{u_\mu}{\tau_{\rm eq}}(f^{\rm eq} - f) \qquad \rightarrow \qquad \partial_{\mu_1} \int dP\, p^{\mu_1} ... p^{\mu_{n+1}} f = u_{\mu_1} \int dP\, p^{\mu_1} ... p^{\mu_n} \frac{1}{\tau_{\rm eq}}(f^{\rm eq} - f)$$

0th moment (1 eq.)

$$\partial_\mu N^\mu = \frac{u_\mu}{\tau_{\rm eq}}(N^\mu_{\rm eq} - N^\mu)$$

1st moment (2 eq.)

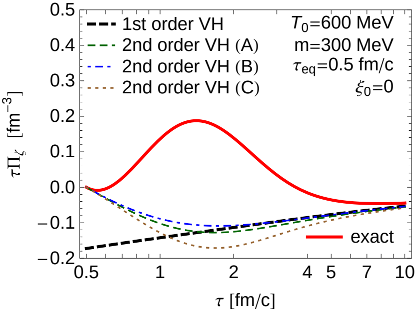
$$\begin{aligned} u_\nu \partial_\mu T^{\mu\nu} &= u_\nu \frac{u_\mu}{\tau_{\rm eq}}(T^{\mu\nu}_{\rm eq} - T^{\mu\nu}) \\ u_\mu T^{\mu\nu}_{\rm eq} &= u_\mu T^{\mu\nu} \end{aligned}$$

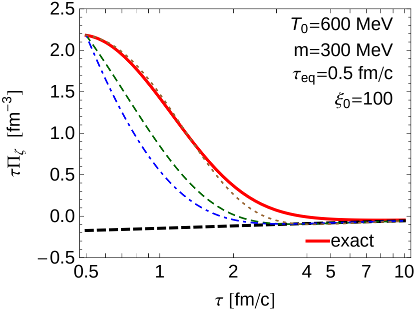
2nd moment (1 eq.)

$$\begin{aligned} X^i_\mu X^i_\nu \partial_\lambda \Theta^{\lambda\mu\nu} &= X^i_\mu X^i_\nu \frac{u_\lambda}{\tau_{\rm eq}}\left(\Theta^{\lambda\mu\nu}_{\rm eq} - \Theta^{\lambda\mu\nu}\right) \\ i &= 0, 1, 2, 3 \end{aligned}$$

...

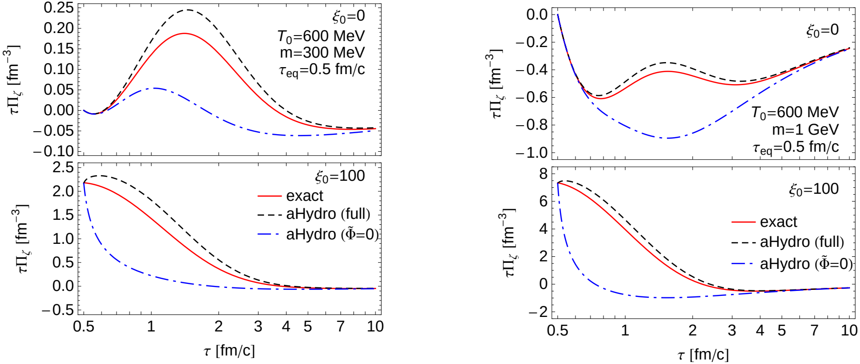
W. Florkowski, E. Maksymiuk, R. Ryblewski, M.Strickland, Phys.Rev. **C 89** (2014) 054908




- exact solution and all 2nd order viscous hydrodynamics variations tend toward the 1st order solution at late times
- none of the 2nd order viscous hydrodynamics variations seems to qualitatively describe the early-time evolution of the bulk viscous pressure in all cases
- there is something incomplete in the manner in which 2nd order viscous hydrodynamics treats the bulk pressure (neglected shear–bulk coupling)

M. Nopoush, R. Ryblewski, M.Strickland, Phys. Rev. C 90, 014908 (2014)

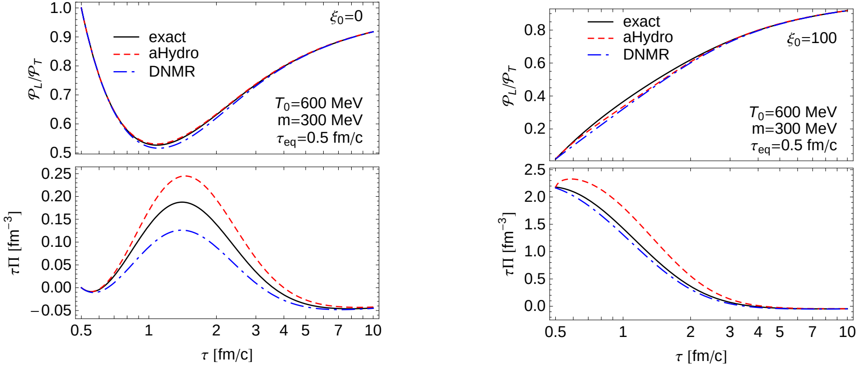



allowing for the bulk degree of freedom significantly improves agreement between anisotropic hydrodynamics and the exact solution

kinetic coefficients implicitly included

G. Denicol, S. Jeon, C. Gale, arXiv:1403.0962
G.Denicol, R. Ryblewski, W. Florkowski, M.Strickland, arXiv:1407.4767

SHEAR-BULK COUPLING




the shear-bulk couplings are extreemly important for correct description of the bulk viscous correction

# Conclusions

- exact solution of kinetic equation allows for testing various approximation schemes
- commonly used 2nd order viscous hydrodynamics equations do not describe early-time evolution of bulk viscous pressure correctly (shear–bulk coupling missing!)
- shear–bulk coupling is crucial for understanding the bulk viscous pressure evolution in 2nd order viscous hydrodynamics
- forms and values of the kinetic coefficients entering hydrodynamic equations are extreemly important
- caution: some references provide incorrect formulas for bulk viscosity
- explicit inclusion of parameter accounting for bulk viscous correction within anisotropic hydrodynamics helps to improve agreement with exact solution