Multipartite nonlocality in quantum mechanics

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Abstract:

All pure entangled states exhibit standard nonlocality. It is an open problem whether all pure genuine(full) multipartite entangled states are genuine nonlocal. We propose a set of conditions on the joint probabilities as a test of genuine multipartite nonlocality without inequality. A pass of our test by a state therefore indicates that this state cannot be simulated by any nonsignaling local models i.e. the state exhibits genuine multipartite nonlocality. It turns out that all entangled symmetric n-qubit ($n \geq 3$) states pass our test and therefore are n-way nonlocal.

The talk is based on the papers: Phys. Rev. Lett. **109**, 120402 (2012); Phys. Rev. Lett. **112**, 140404 (2014).

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Outline

Introduction

All entangled pure states violate Hardy's inequality

A new Hardy-type test to detect genuine multipartite nonlocality

All pure entangled symmetric n-qubit (n >2) states are genuine multipartite nonlocal.

Entangled states

- If a state ρ of quantum system $H_{A_1} \otimes H_{A_2} \otimes \ldots \otimes H_{A_n}$ can be written as

$$\rho = \sum_i p_i \rho_i^{A_1} \otimes \rho_i^{A_2} \otimes \ldots \otimes \rho_i^{A_n},$$

it is separable. Otherwise it is entangled.

- Entangled state is a very important resource in quantum information and computation.
  - teleportation
  - dense coding
  - quantum cryptography
  - ……

Standard local realistic models

Each observer cannot have nonlocal correlations, with any other distant observers, the **joint probability distribution can be fully factorized** and written as, e g, for 3-particle system,

$$p(abc\,|\,xyz) = \int P_\lambda(a\,|\,x)P_\lambda(b\,|\,y)P_\lambda(c\,|\,z)\rho_\lambda d\lambda.$$

- *xyz: measurement settings (input)*
- *abc: the outcomes of the measurements (output)*

The definition of nonlocality

For "standard" nonlocal correlations, the joint probability distribution **cannot** be written as

$$p(abc \mid xyz) = \int P_\lambda(a \mid x) P_\lambda(b \mid y) P_\lambda(c \mid z) \rho_\lambda d\lambda.$$

however as **partial factorization is not excluded**, e.g.,

$$p(abc \mid xyz) = \int P_\lambda(ab \mid xy) P_\lambda(c \mid z) \rho_\lambda d\lambda.$$

- *xyz: measurement settings (input)*
- *abc: the outcomes of the measurements (output)*

Genuine multipartite nonlocality

– "standard" nonlocal correlations, the joint probability distribution **cannot** be written as

$$p(abc\,|\,xyz) = \int P_\lambda(a\,|\,x) P_\lambda(b\,|\,y) P_\lambda(c\,|\,z) \rho_\lambda d\lambda$$

– genuine multipartite nonlocal correlations, **cannot** be written as [Svetlichny, PRD87]

$$p(abc\,|\,xyz) = \int P_\lambda(a\,|\,x) P_\lambda(bc\,|\,yz) \rho_\lambda d\lambda +$$

$$\int P_u(b\,|\,y) P_u(ac\,|\,xz) \rho_u du + \int P_v(c\,|\,z) P_v(ab\,|\,xy) \rho_v dv$$

- *xyz: measurement settings (input)*
- *abc: the outcomes of the measurements (output)*

Hardy's nonlocality without Inequality

- Consider *n* particles labeled with index set $I=\{1,2,\ldots,n\}$. For each subsystem $k\in I$ we choose two observables $\{a_k, b_k\}$ taking binary values {0,1}. The following relations cannot hold simultaneously in any LHV models:

$$\langle a_I\rangle_{LHV}=\int\prod_{i\in I}a_i(\lambda)\rho(\lambda)d\lambda>0,$$

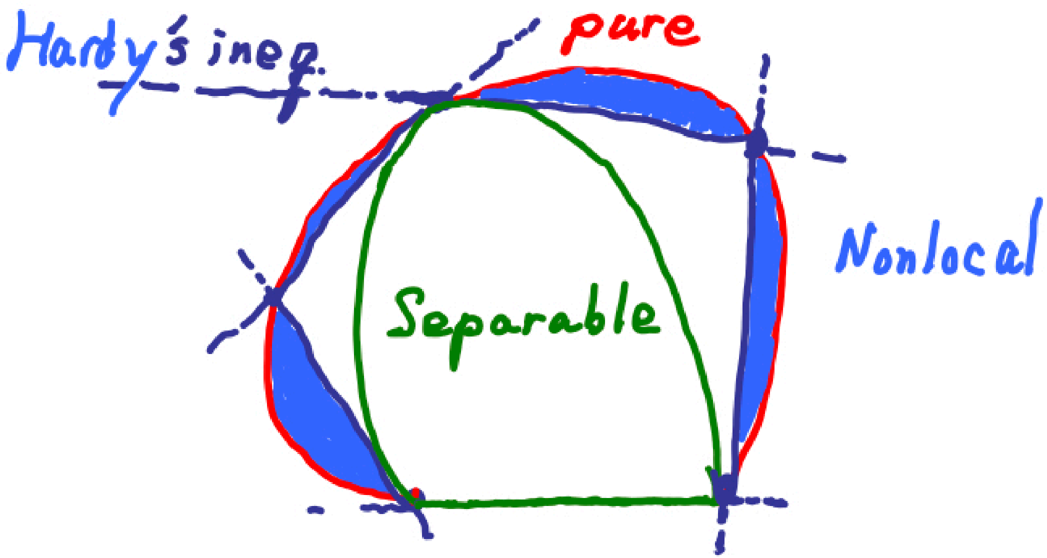
$$\langle\overline{b}_I\rangle=\left\langle\prod_{i\in I}(1-b_i)\right\rangle=0,$$

$$\langle b_k a_{\overline{k}}\rangle=\left\langle b_k\prod_{i\in I\setminus k}a_i\right\rangle=0,\ \forall k\in I.$$

- Hardy [PRL92,93] showed that all 2-qubit pure states **except** maximally entangled states satisfy all the relations above.
- Cereceda [PLA04] proved that all entangled generalized n-particle GHZ states (n>2) satisfy Hardy type nonlocality without inequality (**NLWI**).

Gisin's theorem for multiparticles

We prove the strongest possible extension:

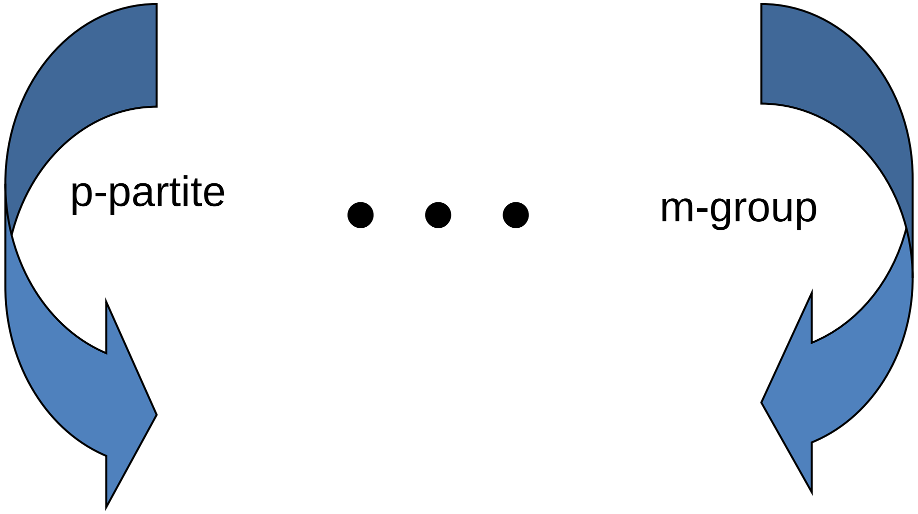


All entangled pure states of a given number of particles violate a single Bell's inequality, namely Hardy's inequality, with each observer measuring two alternative observables.

[Yu-Chen-Zhang-Lai-Oh, Phys. Rev. Lett. **109**, 120402 (2012)].

Different classes of nonlocality and entanglement

- standard nonlocality (entanglement) -- weakest



- genuine multipartite nonlocality (entanglement) -- strongest

Genuine multipartite nonlocality

– "standard" nonlocal correlations, the joint probability distribution **cannot** be written as

$$p(abc \mid xyz) = \int P_\lambda(a \mid x) P_\lambda(b \mid y) P_\lambda(c \mid z) \rho_\lambda d\lambda$$

– genuine multipartite nonlocal correlations, **cannot** be written as [Svetlichny, PRD87]

$$p(abc \mid xyz) = \int P_\lambda(a \mid x) P_\lambda(bc \mid yz) \rho_\lambda d\lambda +$$

$$\int P_u(b \mid y) P_u(ac \mid xz) \rho_u du + \int P_v(c \mid z) P_v(ab \mid xy) \rho_v dv$$

- *xyz: measurement settings (input)*
- *abc: the outcomes of the measurements (output)*

Svetlichny inequalities

- Define recursively the Svetlichny polynomials as

$$S_n = \frac{1}{2}(S_{n-1}A'_n + S'_{n-1}A_n) \text{ for } n \geq 3,$$

$$S_2 = (A_1A_2 + A_1A'_2 + A'_1A_2 - A'_1A'_2)/2 = CHSH,$$

$$S'_2 = (A_1'A'_2 + A'_1A_2 + A_1A'_2 - A_1A_2)/2$$

$$Examples: S_3 = (-A_1A_2A_3 + A_1A_2A'_3 + A_1A'_2A_3 + A'_1A_2A_3 + A_1A_2'A'_3 + A_1'A'_2A_3 + A'_1A_2A'_3 - A'_2A'_2A'_3)/4$$

- Svetlichny inequality $\langle S_n \rangle \leq 1$ holds for any bipartite probability distribution.
- For GHZ state: $\langle S_n \rangle_{GHZ} = \sqrt{2}$ by proper choosing measurement settings.
- There only exist limited states that violate Svetlichny inequalities. [S PRD 87, CGPRS PRL02, SS PRL02.]

New test

$$P(0_I \mid a_I) > 0 \qquad (1a)$$

$$P(0_I \mid b_k a_{\bar{k}}) = 0, \forall k \in I \qquad (1b)$$

$$P(1_{k'} 1_k 0_{I\setminus\{k',k\}} \mid b_{k'} b_k a_{I\setminus\{k',k\}}) = 0, \forall k \in I \setminus \{k'\} \qquad (1c)$$

- Proposition. Any probability distribution that satisfies Eq. (1) is genuine multipartite nonlocal.

[Chen-Yu-Zhang-Lai-Oh, Phys. Rev. Lett. **112**, 140404 (2014)]

Outline of the proof

- via **reductio ad absurdum**

- The most general hybrid local-nonlocal probability distribution can be written as

$$P(r_I \mid M_I) = \sum_{\alpha \neq \phi, \alpha \subset I} \int \rho_{\alpha,\lambda} P_\alpha (r_\alpha \mid M_\alpha, \lambda) P_{\overline{\alpha}} (r_{\overline{\alpha}} \mid M_{\overline{\alpha}}, \lambda) d\lambda \quad \text{(p0)}$$

  - α : nonempty proper subset of I
  - $r_\alpha, r_{\overline{\alpha}}$: outcomes, $M_\alpha, M_{\overline{\alpha}}$: observables

- Suppose a probability distribution satisfies Eq.(1) BUT is not genuine multipartite nonlocal, i.e., it has the form of Eq. (p0).

Outline of the proof

- From Eq. (1a) there must exist some α_0 and λ_0

$$P_{\alpha_0}(0_{\alpha_0} \mid a_{\alpha_0}, \lambda_0) > 0 \text{ and } P_{\overline{\alpha}_0}(0_{\overline{\alpha}_0} \mid a_{\overline{\alpha}_0}, \lambda_0) > 0 \quad \text{(p1)}$$

- From Eq. (1b)

$$P_{\alpha_0}(0_k 0_{\alpha_0 \setminus k} \mid b_k a_{\alpha_0 \setminus k}, \lambda_0)\, P_{\overline{\alpha}_0}(0_{\overline{\alpha}_0} \mid a_{\overline{\alpha}_0}, \lambda_0) = 0, \forall k \in \alpha_0$$

$$P_{\overline{\alpha}_0}(0_k 0_{\overline{\alpha}_0 \setminus k} \mid b_k a_{\overline{\alpha}_0 \setminus k}, \lambda_0)\, P_{\alpha_0}(0_{\alpha_0} \mid a_{\alpha_0}, \lambda_0) = 0, \forall k \in \overline{\alpha}_0 \quad \text{(p2)}$$

- By combining (p1) and (p2) we get

$$P_{\alpha_0}(0_k 0_{\alpha_0 \setminus k} \mid b_k a_{\alpha_0 \setminus k}, \lambda_0) = 0, \forall k \in \alpha_0 \quad \text{(p3a)}$$

$$P_{\overline{\alpha}_0}(0_k 0_{\overline{\alpha}_0 \setminus k} \mid b_k a_{\overline{\alpha}_0 \setminus k}, \lambda_0) = 0, \forall k \in \overline{\alpha}_0 \quad \text{(p3b)}$$

Outline of the proof

- Suppose $k' \in \overline{\alpha}_0,\ j \in \alpha_0$, from Eq. (1c) we get

$$P_{\alpha_0}(1_j 0_{\alpha_0 \backslash j} \mid b_j a_{\alpha_0 \backslash j}, \lambda_0)\, P_{\overline{\alpha}_0}(1_{k'} 0_{\overline{\alpha}_0 \backslash k'} \mid b_{k'} a_{\overline{\alpha}_0 \backslash k'}, \lambda_0) = 0$$

- If $P_{\alpha_0}(1_j 0_{\alpha_0 \backslash j} \mid b_j a_{\alpha_0 \backslash j}, \lambda_0) = 0$, by combining Eq.(p3a) we find

$$P_{\alpha_0 \backslash j}(0_{\alpha_0 \backslash j} \mid a_{\alpha_0 \backslash j}, \lambda_0)$$
$$= P_{\alpha_0}(0_j 0_{\alpha_0 \backslash j} \mid b_j a_{\alpha_0 \backslash j}, \lambda_0) + P_{\alpha_0}(1_j 0_{\alpha_0 \backslash j} \mid b_j a_{\alpha_0 \backslash j}, \lambda_0) = 0$$

  However, $P_{\alpha_0 \backslash j}(0_{\alpha_0 \backslash j} \mid a_{\alpha_0 \backslash j}, \lambda_0) \geq P_{\alpha_0}(0_{\alpha_0} \mid a_{\alpha_0}, \lambda_0) > 0$

  $\Rightarrow$ a contradiction!

- If $P_{\overline{\alpha}_0}(1_{k'} 0_{\overline{\alpha}_0 \backslash k'} \mid b_{k'} a_{\overline{\alpha}_0 \backslash k'}, \lambda_0) = 0$, by combining Eq. (p3b) a similar contradiction can also be made.

Permutation symmetric states

Any n-qubit pure symmetric state can be written as

$$|\psi\rangle = \sum_{j=0}^{n} h_j |S(n,j)\rangle, \quad |S(n,j)\rangle = \sum_{\alpha \subseteq I, |\alpha| = j} |0_{\overline{\alpha}} 1_{\alpha}\rangle$$

Example

$$n = 2: \ |\psi\rangle = h_0|00\rangle + h_1(|01\rangle + |10\rangle) + h_2|11\rangle$$

- Proposition. All pure multipartite entangled permutation symmetric n-qubit ($n \geq 3$) states are genuine multipartite nonlocal.

New test

$$\langle a_I \rangle > 0,$$

$$\langle b_k a_{\bar{k}} \rangle = 0,\ \forall k \in I,$$

$$\langle \overline{b}_k \overline{b}_{k'} a_{I \setminus \{k',k\}} \rangle = 0, \forall k \in I \setminus \{k'\}.$$

Examples

n = 2

$$\langle a_1 a_2 \rangle > 0,$$

$$\langle b_1 a_2 \rangle = 0,\ \langle a_1 b_2 \rangle = 0,$$

$$\langle \overline{b}_1 \overline{b}_2 \rangle = 0.$$

n = 3

$$\langle a_1 a_2 a_3 \rangle > 0,$$

$$\langle b_1 a_2 a_3 \rangle = 0,\ \langle a_1 b_2 a_3 \rangle = 0, \langle a_1 a_2 b_3 \rangle = 0,$$

$$\langle \overline{b}_1 \overline{b}_2 a_3 \rangle = 0, \langle \overline{b}_1 a_2 \overline{b}_3 \rangle = 0.$$

Outline of the proof

We prove the proposition by showing that all pure entangled **symmetric** states pass our test by choosing the measurement setting appropriately

Outline of the proof

For a given state $|\psi\rangle$ to satisfy out test, we need to find two measurement settings $\{|a_k\rangle, |b_k\rangle\}$ for each particle such that

$$\begin{aligned} &\langle a_I | \psi \rangle \neq 0, \\ &\langle a_{\bar{k}} b_k | \psi \rangle = 0, \forall k \in I, \qquad (2) \\ &\langle \overline{b_1} \overline{b_k} a_{I \setminus \{1,k\}} | \psi \rangle = 0, \forall k \in I \setminus \{1\} \end{aligned}$$

➢ We suppose that $|0_I\rangle$ is already the closest product state of $|\psi\rangle$. The computational basis determined by the closest product state is a magic basis. $\Rightarrow h_0 \neq 0, h_1 = 0$

➢ We choose the measurement settings on the qubits from 2 to n to be the same.

The measurement settings

- Choosing local projection as

$$|a_1\rangle = |0\rangle + x_1^*|1\rangle, |b\rangle = |0\rangle + y_1^*|1\rangle,$$

$$|a_i\rangle = |a\rangle = |0\rangle + x^*|1\rangle, |b_i\rangle = |0\rangle + y^*|1\rangle \text{ for } 2 \le i \le n.$$

- Eq. (2) becomes

$$\langle a_1 a_2 | \psi_{12} \rangle \neq 0, \qquad (3a)$$

$$\langle \overline{b}_1 \overline{b}_2 | \psi_{12} \rangle = \langle a_1 b_2 | \psi_{12} \rangle = \langle b_1 a_2 | \psi_{12} \rangle = 0. \qquad (3b)$$

where $|\psi_{12}\rangle = \langle a_2|^{\otimes(n-2)} |\psi\rangle = c_0|00\rangle + c_1(|01\rangle + |10\rangle) + c_2|11\rangle$

$$c_i = \sum_{k=0}^{n-2} h_{k+i} x^k C_{n-2}^k, \; 0 \le i \le 2.$$

Towards the Gisin's theorem for genuine multipartite nonlocality

- Our numerical results show that our test Eq.(1) can be satisfied by all pure genuine multipartite entangled states of three and four qubit systems.

- Conjecture: Our test Eq.(1) can be satisfied by all pure genuine multipartite entangled states.

Summary

i. For the standard nonlocality case, we have proved Gisin's theorem in its most general form: *every entangled pure state of a given number of particles, each of which may have a different number of energy levels, violates one single Bell's inequality, namely, Hardy's inequality.*

ii. For the genuine multipartite nonlocality case, we have proposed a new test, which can be regarded as a natural generalization of Hardy's test. By using such test we have shown that all pure symmetric entangled states are genuinely multipartite nonlocal. Moreover, our numerical results suggest that all pure genuine multipartite entangled states satisfy our test.

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Thank you

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