in many physical systems of interest can’t use standard perturbation theory
- need non-perturbative techniques

different approaches (for example):
- hard thermal loop (HTL) effective theory
- Schwinger-Dyson equations
- $n$-particle irreducible ($n$PI) effective theories
- exact renormalization group (eRG)

issues:
- physics
- symmetries (gauge invariance)
- renormalization
- computational advantages
I consider \((\text{symmetric})\) scalar \(\phi^4\) theory
2PI for Scalar Theories:

generating functional with local and bi-local sources

\[ Z[J, K] = e^{iW[J,K]} = \int \mathcal{D}\varphi e^{i(S[\varphi]+J_i\varphi_i+\frac{1}{2}B_{ij}\varphi_i\varphi_j)} \]

short-hand notation:

\[ \int dx \int dy \varphi(x)B(x,y)\varphi(y) \rightarrow \varphi_i B_{ij} \varphi_j \rightarrow B\varphi^2 \]
Legendre transform:

\[ \Gamma[\phi, G] = W[J, K] - J_i \phi_i - \frac{1}{2} B_{ij} \phi_i \phi_j \]

\[ = S_{cl}[\phi] + \frac{i}{2} \text{Tr} \ln G^{-1} + \frac{i}{2} \text{Tr} G_0^{-1}(G - G_0) + \Gamma_2[\phi, G] \]

\( \Gamma[\phi, G] \) is a functional of the 1- and 2-point functions

\( \phi \) and \( G \) are determined self-consistently from the equations of motion variational principle (in the absence of sources)

\[ \frac{\delta \Gamma}{\delta \phi} = \frac{\delta \Gamma}{\delta G} = 0 \]

\[ \Rightarrow G^{-1} = G_0^{-1} - \Sigma[\phi, G], \quad \Sigma[\phi, G] := 2 \frac{\delta \Phi}{\delta G}, \quad \Phi = i \Gamma_2 \]
Compare to $\Gamma[\phi] = \text{1PI effective action}$:

- $\Gamma[\phi, G]$ depends on the self consistent propagator
  $\to$ truncated $\Gamma[\phi, G]$ includes an infinite resummation of diagrams
  $\to$ non-perturbative

- $\Gamma[\phi, G]$ is $\text{2PI}$ - no double counting

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{diagram.png}
\caption{\text{2PI} and \text{2PR}}
\end{figure}
nPI effective action

nPI $\Gamma$ is a functional of $n$-point functions

3PI $\Gamma[\phi, G, U]$, 4PI $\Gamma[\phi, G, U, V] \cdots$

$n$-point functions determined self-consistently from the equations of motion

$\Rightarrow$ hierarchy of coupled equations

- no exact solution method is available

- use approximation techniques: truncate the effective action
FIRST CALCULATION:

compare the 2pi and 4pi calculations in 3 dimensions

\[
\Phi_2^{2\pi-3\text{loop}} = -\frac{1}{2} \quad + \frac{1}{8} \quad + \frac{1}{8} \quad \text{EIGHT} \quad + \frac{1}{8} \quad \text{EIGHT}_{ct} \quad + \frac{1}{48} \quad \text{BBALL}_0
\]

\[
\Phi_2^{4\pi-4\text{loop}} = -\frac{1}{2} \quad + \frac{1}{8} \quad + \frac{1}{8} \quad \text{EIGHT} \quad + \frac{1}{24} \quad \text{EIGHT}_{ct} \quad + \frac{1}{24} \quad \text{BBALL}_0 \quad + \frac{1}{24} \quad \text{BBALL}_{ct} \quad \frac{1}{48} \quad \text{BBALL} \quad + \frac{1}{48} \quad \text{LOOPY}
\]
eom for the 2-point function: $\Sigma = 2\delta\Phi/\delta G$

\[ \Sigma_{2\pi} = -\frac{1}{\delta G} + \frac{1}{2} \quad + \frac{1}{2} \quad + \frac{1}{6} \]

\[ \Sigma_{4\pi} = -\frac{1}{\delta G} \quad + \frac{1}{2} \quad + \frac{1}{2} \quad + (2)\frac{1}{6} \quad + (2)\frac{1}{6} \quad - \frac{1}{6} \quad + \frac{1}{4} \]

\[= -\frac{1}{\delta G} \quad + \frac{1}{2} \quad + \frac{1}{2} \quad + \frac{1}{6} \quad + \frac{1}{6} \quad + \frac{1}{6} \]
eom for the 4-point function:

2π: Bethe-Salpeter (BS) int eqn for self-consistent 4-vertex \( M \) from \( \frac{\delta^2 \Phi}{\delta R_2 \delta G} \)

the kernel is the 4-vertex \( \Lambda = 4 \frac{\delta^2 \Phi}{\delta G^2} \)

\[
\Lambda = \begin{array}{c}
\text{diagram 1} \\
\text{diagram 2} \\
\text{diagram 3} \\
\text{diagram 4}
\end{array}
= \begin{array}{c}
\text{diagram 5} \\
\text{diagram 6} \\
\text{diagram 7} \\
\text{diagram 8}
\end{array}
+ \begin{array}{c}
\text{diagram 9} \\
\text{diagram 10} \\
\text{diagram 11} \\
\text{diagram 12}
\end{array}
\]

\[
M = \begin{array}{c}
\text{diagram 13} \\
\text{diagram 14} \\
\text{diagram 15} \\
\text{diagram 16}
\end{array}
= \begin{array}{c}
\text{diagram 17} \\
\text{diagram 18} \\
\text{diagram 19} \\
\text{diagram 20}
\end{array}
+ \begin{array}{c}
\text{diagram 21} \\
\text{diagram 22} \\
\text{diagram 23} \\
\text{diagram 24}
\end{array}
\]

4π: eom from \( \frac{\delta \Phi}{\delta V} = 0 \)

\[
\begin{array}{c}
\text{diagram 25} \\
\text{diagram 26} \\
\text{diagram 27} \\
\text{diagram 28}
\end{array}
= \begin{array}{c}
\text{diagram 29} \\
\text{diagram 30} \\
\text{diagram 31} \\
\text{diagram 32}
\end{array}
+ \begin{array}{c}
\text{diagram 33} \\
\text{diagram 34} \\
\text{diagram 35} \\
\text{diagram 36}
\end{array}
+ \begin{array}{c}
\text{diagram 37} \\
\text{diagram 38} \\
\text{diagram 39} \\
\text{diagram 40}
\end{array}
\]
Solutions:

we have (coupled) self-consistent eom’s for the 2- and 4-point functions
i work in 2- and 3-dimensions – no vertex counter-terms
solve using a numerical lattice method


- rotate to Euclidean space
- use an \( N^d \) symmetric lattice - in 2D \( N_{\text{max}} = 16 \); in 3D \( N_{\text{max}} = 12 \)
the lattice spacing is \( a = \frac{2\pi}{(Nm)} \)
each momentum component is discretized:

\[
Q_i = \frac{2\pi}{aN} n_i = mn_i, \quad n_i = -\frac{N}{2} + 1, \ldots, \frac{N}{2}
\]
- indices outside \( \{-N/2 + 1, N/2\} \) wrapped inside using periodic b.c.
- use a numerical iterative method to solve set of self-consistent equations
  - search for fixed points
Memory constraints:

# points in phase space of a vertex is $N^{l \times d}$
- $l$ is the number of independent momenta and $d$ is the dimension
for $V$: $l = 3$, $d = 3$, $N_{\text{max}} = 12 \implies 5.16 \times 10^9$ points

trick: reduce the phase space of $V$ using the symmetries of the vertex
- $V$ is symmetric under interchange of legs and directions in momentum space
- don’t need to calculate all points

Table: size of phase space and number of needed representative points

<table>
<thead>
<tr>
<th>$N$</th>
<th>$N^{3 \cdot (d=3)}$</th>
<th># of reprs</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>10,077,696</td>
<td>11,424</td>
</tr>
<tr>
<td>8</td>
<td>134,217,728</td>
<td>129,502</td>
</tr>
<tr>
<td>10</td>
<td>1,000,000,000</td>
<td>913,661</td>
</tr>
<tr>
<td>12</td>
<td>5,159,780,352</td>
<td>4,608,136</td>
</tr>
</tbody>
</table>

** the function generates the uncalculated points must be FAST
for $\lambda$ large, the perturbative, 2pi and 4pi vertices are different

for certain momentum configurations $M$ and $V$ are close together

- this happens when $s$-channel contributions are big
if you expand nPI eom’s you get an infinite set of diagrams
⇒ infinite sets of embedded sub-divergences and counter-terms
need renormalization conditions (RCs) to determine counter-terms
that cancel sub-divergences

compare to perturbative expansion at $L$ loops:
• some diagrams are missing
• some which are present have different coefficients


recall: goal of nPI is to (?) resum the physically important contributions
2π in 4-dimensions: we know how to renormalize


trick is to determine vertex counter-term using a RC on the BS 4-point fcn
what to do with 4π is unclear (more definitions of the 4-point functions)
I want to try a different strategy
will show that one can do the 2pi calculation without using counter-terms. The next step will be to apply the same method to higher npi calculations.
add to the action a non-local regulator term

\[ S_\kappa[\varphi] = S[\varphi] + \Delta S_\kappa[\varphi], \quad \Delta S_\kappa[\varphi] = -\frac{1}{2} \hat{R}_\kappa \varphi^2 \]

\[ \hat{R}_\kappa(q) = \frac{q^2}{e^{q^2/\kappa^2} - 1} = \begin{cases} 0 & \text{for } q \geq \kappa \\ \kappa^2 & \text{for } q < \kappa \end{cases} \sim (\text{unaffected}) \sim (\text{suppressed}) \]

family of theories indexed by the continuous parameter \( \kappa \)
fluctuations are smoothly taken into account as \( \kappa \) is lowered to zero

\( \kappa \to \infty \) regulated action \( \to \) classical action

\( \kappa \to 0 \) (include all fluctuations) regulated action \( \to \) full quantum action

generating functionals defined in the usual way:

$$Z_{\kappa}[J, J_2] = \int [d\varphi] \exp \{i(S[\varphi] - \frac{1}{2}\hat{R}_\kappa \varphi^2 + J\varphi + \frac{1}{2}B\varphi^2 + \cdots)\}$$

calculate $1\pi$, $2\pi$, $\cdots$ effective action

the flow equation gives the dependence of the action on $\kappa$


$$\partial_\kappa \Phi_\kappa = \frac{1}{2} \partial_\kappa R_\kappa (\langle \varphi^2 \rangle - \phi^2)$$

- same form for any $n\text{PI}$ effective action
- definition of the expectation values different for different actions

$$1\pi : \partial_\kappa \Phi_{1\text{PI}.\kappa} = -\frac{1}{2} \partial_\kappa R_\kappa \left[ \frac{\delta^2 \Phi_{1\text{PI}.\kappa}}{\delta \phi^2} + R_\kappa \right]^{-1}$$

functional derivatives wrt $\phi \rightarrow$ infinite coupled hierarchy of eRG equations

- practical calculations require truncation
regulated 2pi effective action:
- also gives infinite coupled hierarchy of eRG equations
- but they truncate naturally when the action is truncated

advantages to truncating at the level of the action:
- straightforward to systematically extend the order of the approximation
- we expect that the truncation respects the symmetries of the original theory
to the order of the approximation

the 1st two flow equations from the regulated 2pi effective action:

\[ \partial_\kappa \Sigma_\kappa(P) = \frac{1}{2} \int dQ \partial_\kappa (\Sigma_\kappa(Q) + R_\kappa(Q)) G^2_\kappa(Q) \Lambda_\kappa(Q, P) \]

\[ \partial_\kappa \Lambda_\kappa(P, K) = \frac{1}{2} \int dQ \partial_\kappa [R_\kappa(Q) + \Sigma_\kappa(Q)] G^2_\kappa(Q) \Lambda^{03}_\kappa(Q, P, K) \]
\( \Lambda_{03}^{\kappa} \) has its own flow equation of the form \( \partial_{\kappa} \Lambda_{03}^{\kappa} \sim \int dQ \partial_{\kappa} G_{\kappa} \Lambda_{04}^{\kappa} \) \ldots

BUT: hierarchy of flow eqns truncates when the action is truncated

- at the level of our approximation (3-loop 2\text{pi}) \( \Lambda_{04}^{\kappa} \) is a constant
- right side of eqn for \( \partial_{\kappa} \Lambda_{03}^{\kappa} \) is an exact differential - can integrate directly
  (integration constant = zero because no 6-vertex in the Lagrangian)

equivalent:

we can simply obtain \( \Lambda_{03}^{\kappa} \) directly from the effective action:

\[
\Lambda_{03}^{\kappa}(Q, P, K) = -\lambda^2(G_{\kappa}(Q + P + K) + G_{\kappa}(Q + P - K) + G_{\kappa}(Q - P + K) + G_{\kappa}(Q - P - K))
\]

note momentum integral in the \( \Lambda \) flow equation finite
- more in a second
boundary conditions

to solve flow equations must specify bc’s from which flow starts at $\kappa = \mu$
idea is to choose $\mu$ large and use the (known) classical solutions as bc’s
⇒ solve the equations to obtain the quantum solutions at $\kappa = 0$
must show bc’s consistent with RC’s (defns of physical parameters) $\kappa = 0$
the solution for the 2-point function will look like

$$G_{\kappa}^{-1} = P^2 + m^2 + \Sigma_{\kappa}(P) + C$$

$C$ is any $\kappa$ independent constant

to compare with the standard 2pi calculation we use the same RC’s

$$G_0^{-1}(0) = m^2, \quad \frac{d}{dP^2}G_0^{-1}|_{P=0} = 1, \quad M_0(0) = -\lambda$$

caution: subscripts 0 indicate $\kappa = 0$ not bare/non-interacting quantities

impose RC on the 2-point function $\rightarrow$ choose $C = -(\Sigma_0(0) + P^2\Sigma'_0(0))$
can show that with this choice of $C$, the limit $\mu \rightarrow \infty$ gives

$$G_{\mu}^{-1} = Z_{\mu}(P^2 + m_{\mu}^2)$$

with $Z_{\mu}$ and $m_{\mu}$ momentum independent

for the 4-point function can show:

bc $\Lambda_{\mu}(P, Q) = -\lambda_{\mu}$ is consistent with the RC on the 4-point function

equivalently:

momentum integrals in flow eqns are all either finite or $\vec{p}$-independent

$\rightarrow$ all divergent contributions can be absorbed into defns of $m_{\mu}, \lambda_{\mu}$
a fundamental technical difficulty with the eRG formalism

the RC’s (which define the physical parameters)
- are defined in terms of the quantum ($\kappa = 0$) $n$-point fcns
- these are obtained only after the calculation is finished

we want to specify chosen values for the physical mass and coupling
- but required input is the bare parameters

an arbitrary choice of bare parameters $\not\Rightarrow$ the chosen physical ones
- we do not know in advance which choice of bare parameters will

must “tune” the bare parameters to produce physical mass and coupling

calculations at finite temperature are done using these bare parameters
- increase $T$ by decreasing size of euclidean box in the temporal direction

23
eRG method reproduces results of the standard 2pi calculation
- - - without using counter-terms - - -
to test the calculation:
reducing $a_s$ while holding the 3-length of the box $L = a_s N_s$ fixed
compare: 2pi calculation with cts ($\lambda \rightarrow \lambda + \delta \lambda$) on basketball diagram
we plot $M(0, 0)$ versus log $1/a_s$ at $T = 2m$
- in the incorrect calculation $M(0, 0)$ increases when $a_s$ is reduced
- in the correct 2pi and eRG calculations the curve is flat
→ shows that the renormalization is done correctly
Conclusions

• in 3 dimensions the 4pi 2- and 4-point fcns are significantly different from the corresponding 2pi ones when the coupling is large

• the eRG can be used to do 4d 2pi calculations without counter-terms

   eRG is a promising method to do higher order 4d $n$pi calculations