Phase space dynamics and control of the quantum particles associated to hypergraph states

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Overview

- Problem definition in terms of Fractal distribution and Complexity
- Connection to Graph and Hypergraph description
- Fractional covering number as entropy parameter
- Fractional entropy descriptor of a hypergraph
- Topological order relation to nonlocality and quantum states
- Multilevel hypergraph partitioning algorithms
- Conclusion

Complexity measures

- Algorithmic complexity (length of the shortest code).
- Fractional dimension.
- Shannon information (entropy).
- Correlation dimension (topologic dimension of an attractor). Topological entropy.
- Functional clustering.

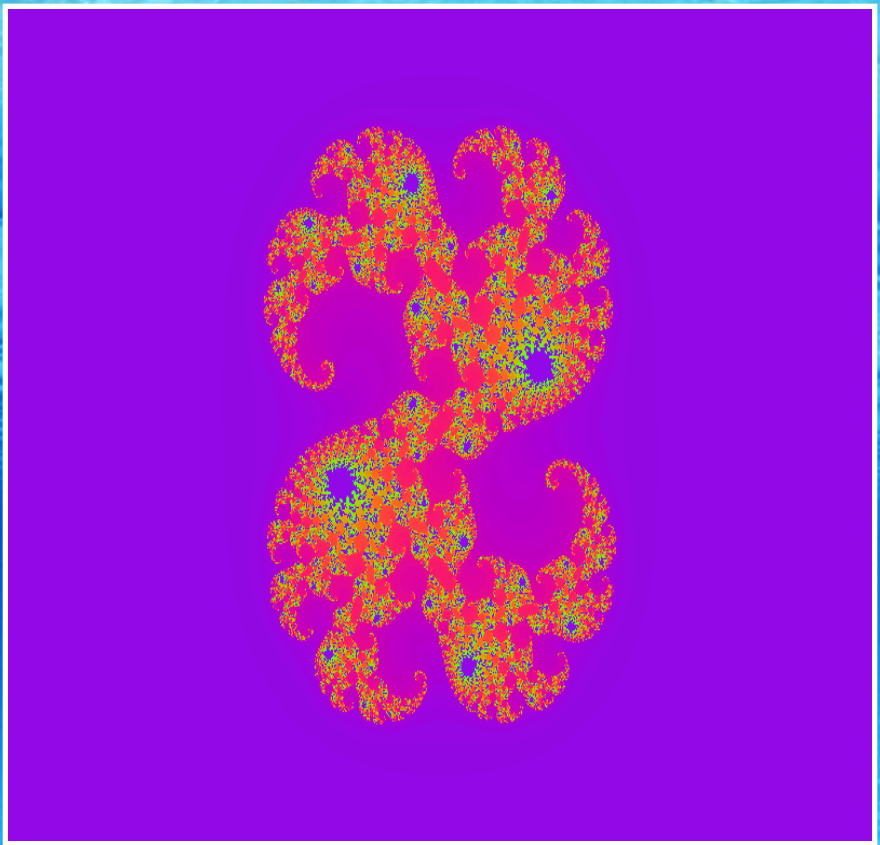
Fractals as complex systems

- A fractal is a mathematical object that is both self-similar and chaotic.

  - self-similar: As you magnify, you see the object over and over again in its parts.

  - chaotic: Fractals are infinitely complex.

  - Noninteger fractal dimension.

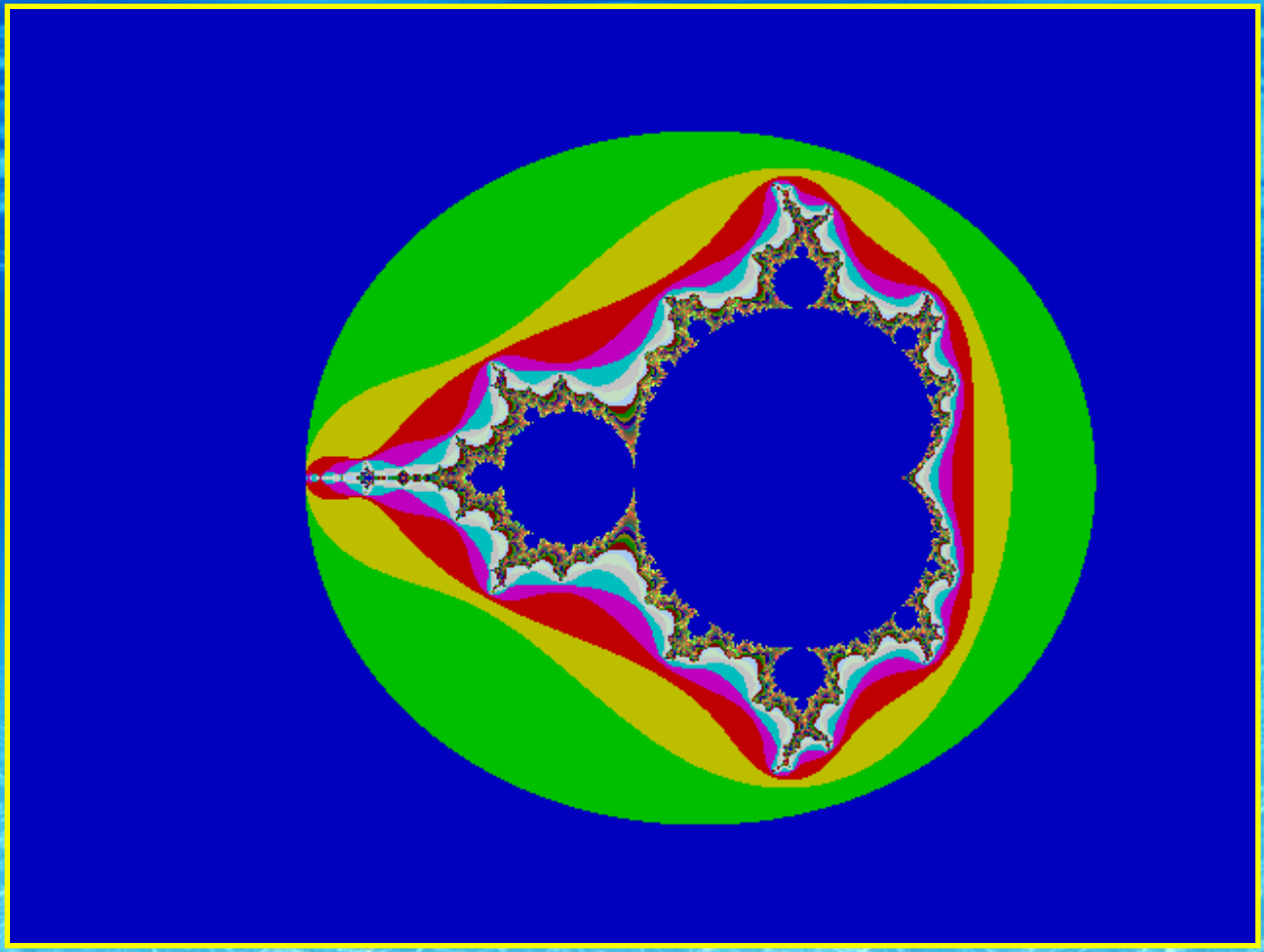


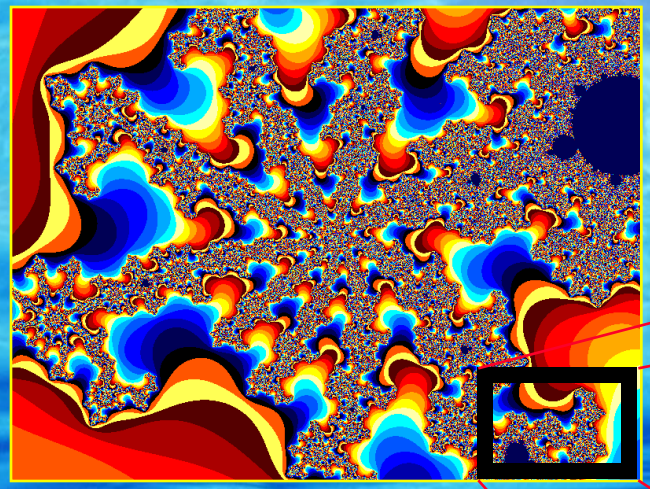
- **Fractal dimensions are used to characterize a broad spectrum of objects ranging from the abstract to practical phenomena, including turbulence, river networks, urban growth, human physiology, medicine, and market trends.**

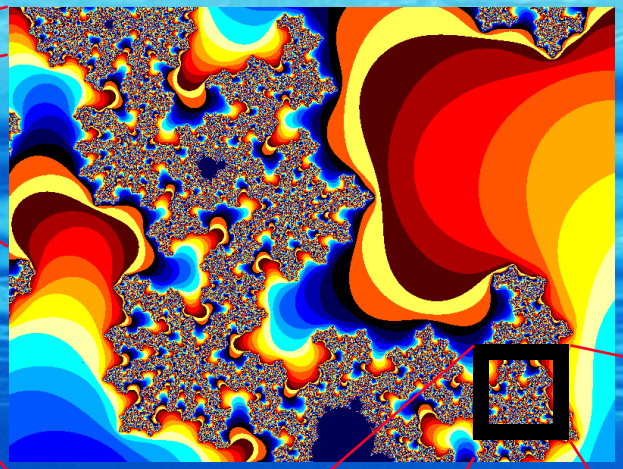
The first fractals were discovered by Gaston Julia who discovered them decades before the advent of computer graphics.

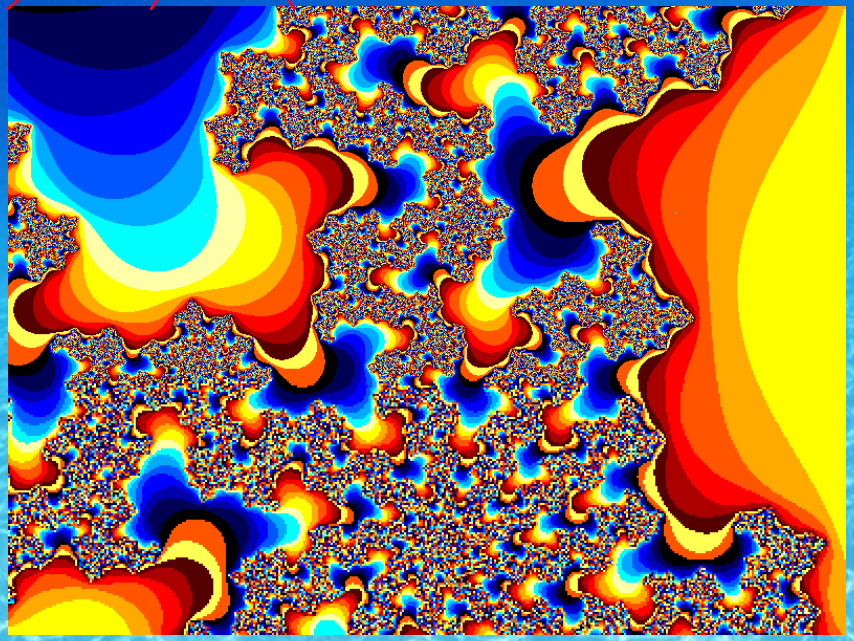
Benoit Mandelbrot rediscovered Julia's work.

Result: the most famous of all fractals is the Mandelbrot set.

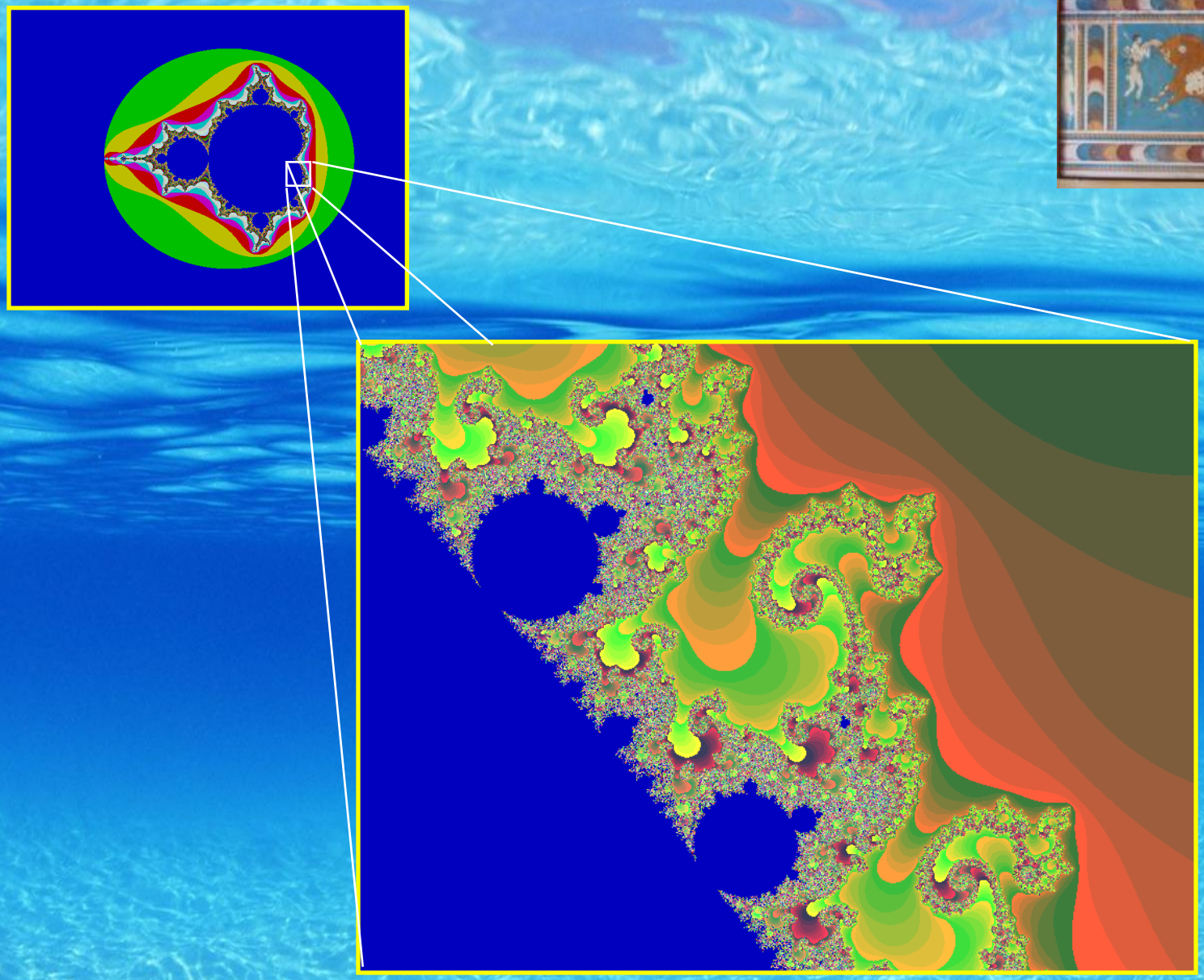




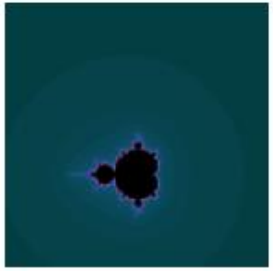




As we magnify the object, we see the same pattern over and over again.....This is Self Similarity

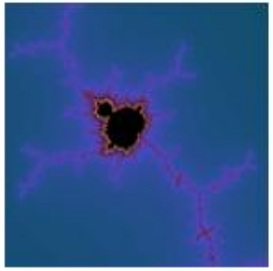


In fractal analysis, complexity is a change in detail with change in scale.

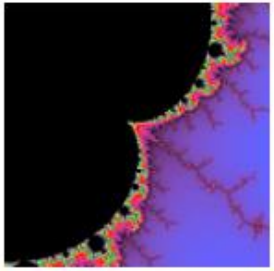


$M = 10^{-1}$

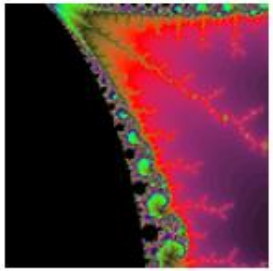
$M = 10^{0}$



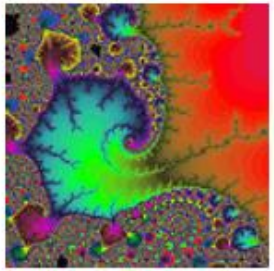
$M = 10^{1}$



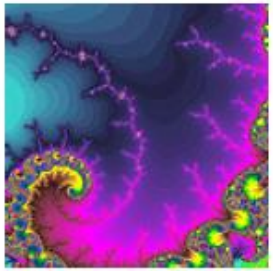
$M = 10^{2}$



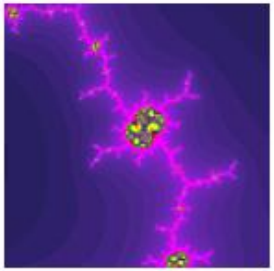
$M = 10^{3}$



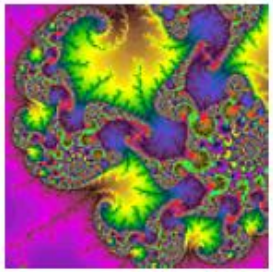
$M = 10^{4}$



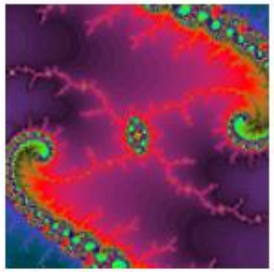
$M = 10^{5}$



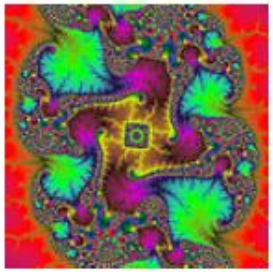
$M = 10^{6}$



$M = 10^{7}$

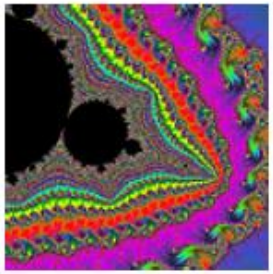


$M = 10^{8}$

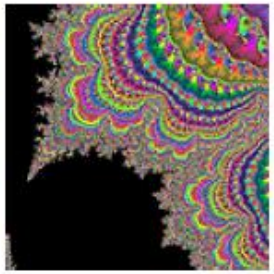


$M = 10^{9}$

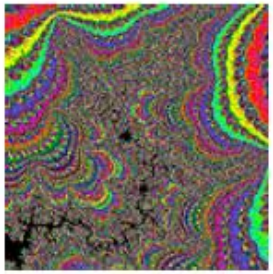
$M = 10^{10}$



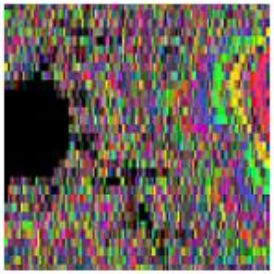
$M = 10^{11}$



$M = 10^{12}$



$M = 10^{13}$



$M = 10^{14}$

Nature possesses Fractal Geometry

Why is geometry often described as cold and dry? One reason lies in its inability to describe the shape of a cloud, a mountain, a coastline, or a tree. Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line...

...Nature exhibits not simply a higher degree but an altogether different level of complexity. The number of distinct scales of length of patterns is for all purposes infinite. ..Benoit Mandelbrot

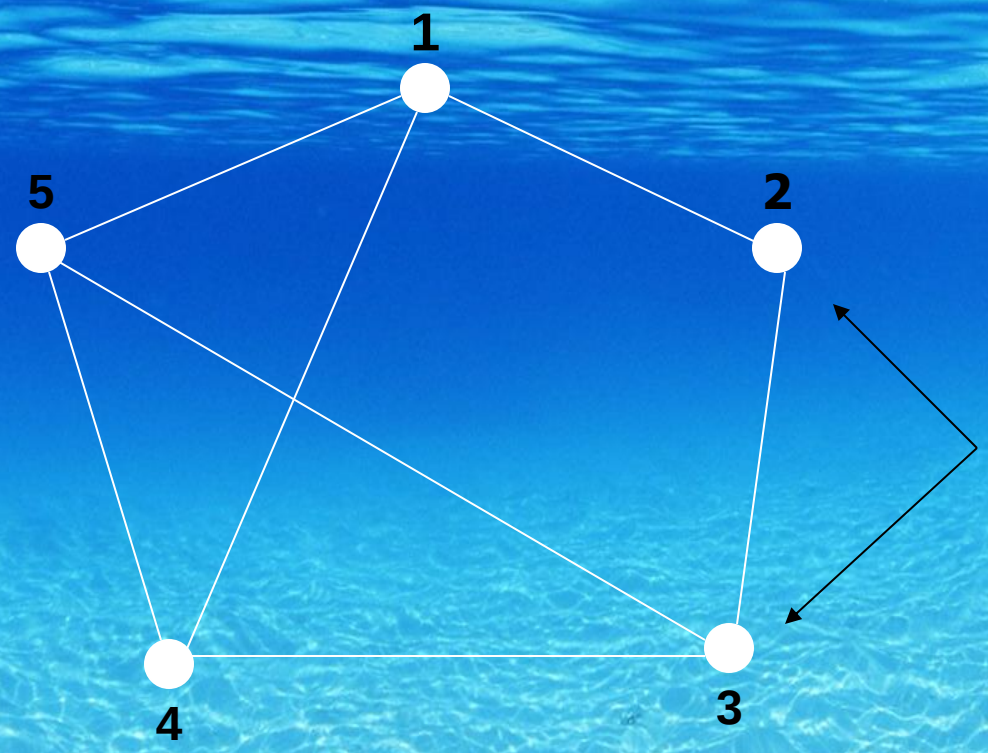
What about describing the shape of nature via

Graph
or…
Hypergraph

A graph $G=(V,E)$:

***V*={1,2,3,4,5}**

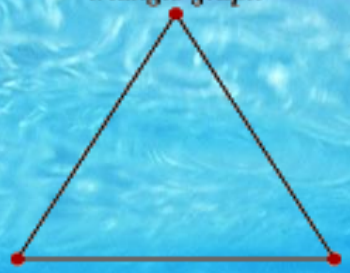
***E*={{1,2},{2,3},{3,4},{4,5},{5,1},{1,4},{3,5}}**



adjacent vertices

(2, 1)
2–path graph

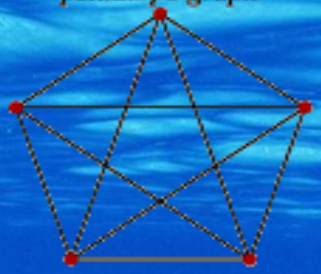
(3, 1)
triangle graph



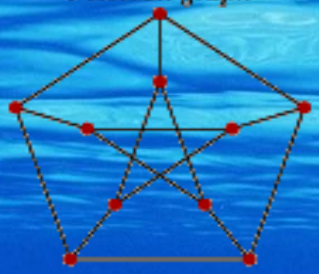
(4, 1)
tetrahedral graph

(4, 2)
2–ladder rung graph

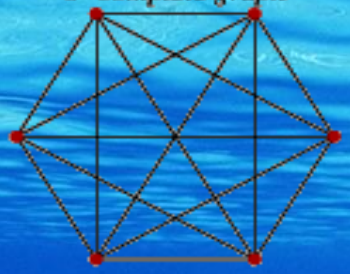
(5, 1)
pentatope graph



(5, 2)
Petersen graph



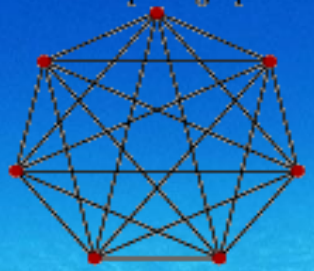
(6, 1)
6–complete graph



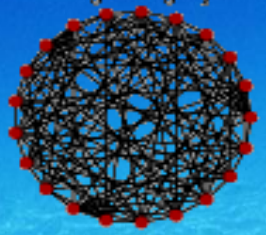
(6, 2)
generalized quadrangle (2,2)

(6, 3)
10–ladder rung graph

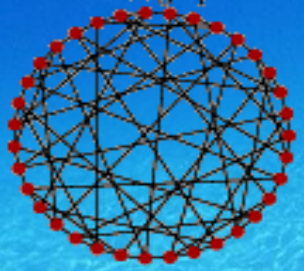
(7, 1)
7–complete graph



(7, 2)
complement of 7–triangular graph

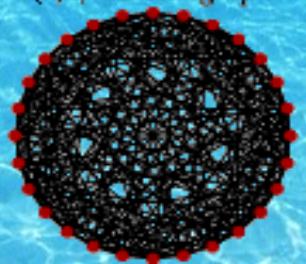


(7, 3)
4–cold graph

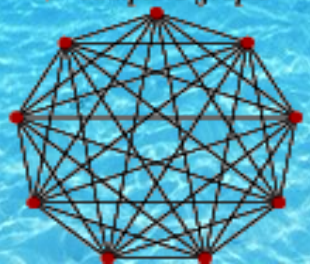


(8, 1)
8–complete graph

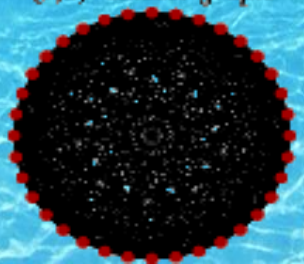
(8, 2)
(8,2)–Kneser graph



(9, 1)
9–complete graph



(9, 2)
(9,2)–Kneser graph

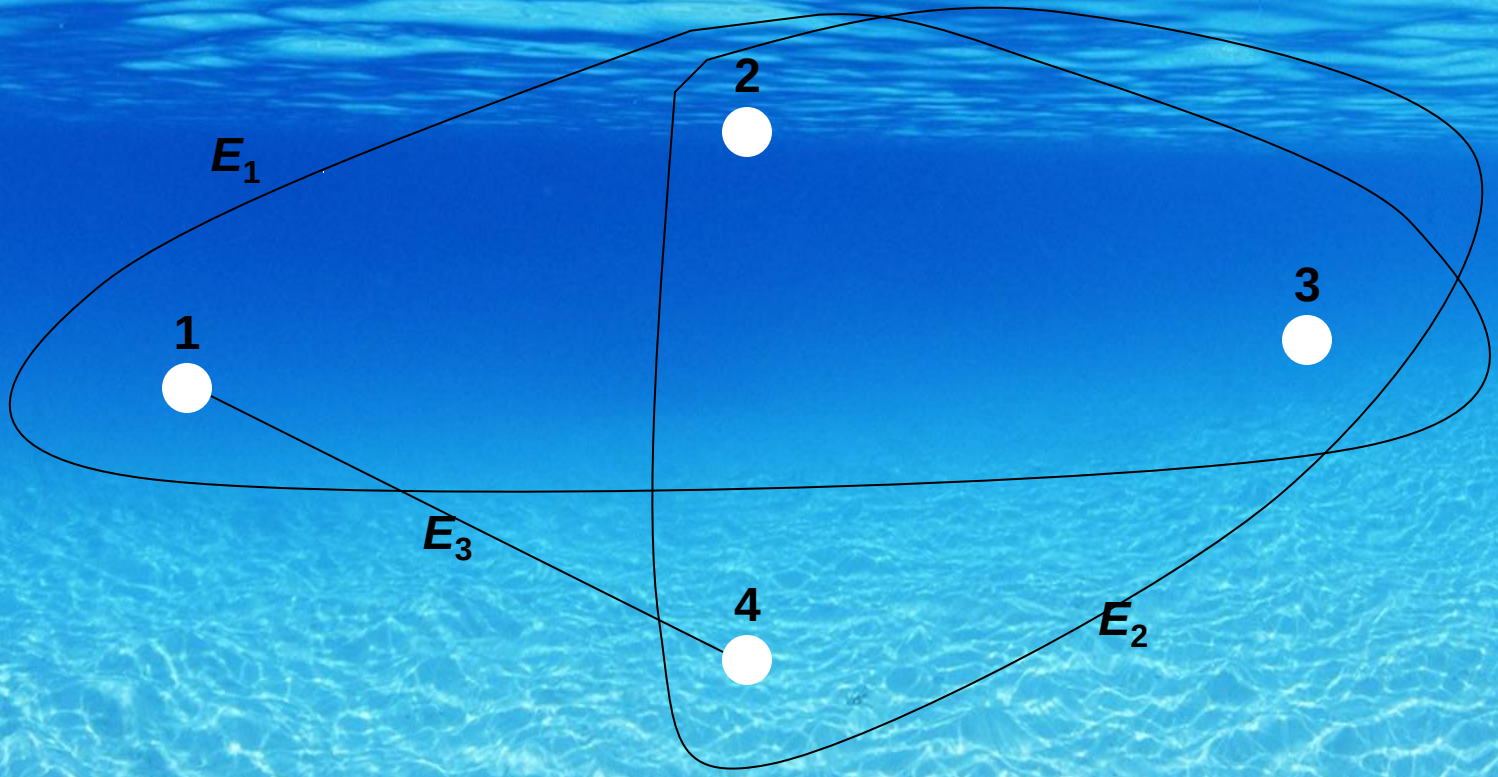


- **A** hypergraph ***H*** may be drawn as a set of points representing the vertices.
- The edge $\boldsymbol{E_j}$ is represented by a continuous curve joining the two elements if $|\boldsymbol{E_j}| = \mathbf{2}$, by a loop if $|\boldsymbol{E_j}| = \mathbf{1}$, and by a simple closed curve enclosing the elements if $|\boldsymbol{E_j}| \geq \mathbf{3}$

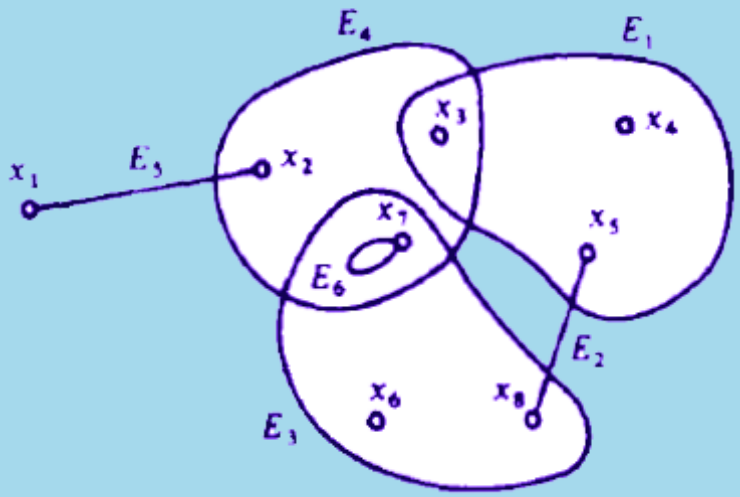
…A hypergraph

$H=(X,E)$, $X=\{1,2,3,4\}$,

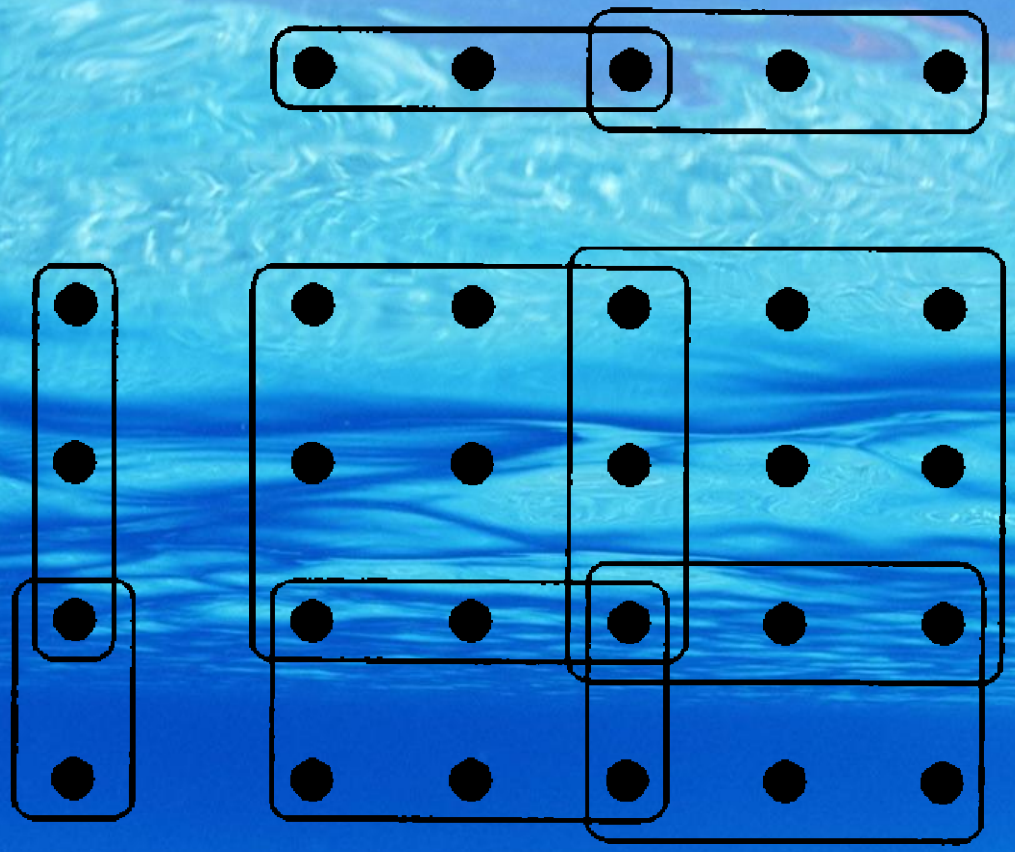
$E=\{\{1,2,3\},\{2,3,4\},\{1,4\}\}=\{E_1,E_2,E_3,\}$



A hypergraph can be defined by its incidence matrix *A* with columns representing the edges E_l ,E_2.,. .,E_m and rows representing the vertices x_1,x_2,. ..,x_n.



$$A = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{array} \begin{array}{c} \begin{array}{cccccc} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$



The 20-vertex hypergraph is the product of the 4-vertex hypergraph (left) and the 5-vertex hypergraph (top). The vertex set of the product is the Cartesian product of the vertex sets of the factors. Hyperedges in the product are the Cartesian products of hyperedges of the factors.

Both graphs and hypergraphs may be partitioned to optimize some objective.

A hypergraph is used to represent the connectivity information from the circuit specification. Each vertex in the hypergraph represents a cell in the circuit and each hyperedge represents a net from the circuit's netlist.

Hypergraph covering

A hypergraph *H* is a pair (*X, E*), where *X* is a finite set and *E* is a family of subsets of *X*.
The set *X* is called the *ground set* or the *vertex set* of the hypergraph, and so we sometimes write *V* (*H*) for *X*.
The elements of *E* are called *hyperedges* or sometimes just *edges*.

A *covering* (alternatively, an *edge covering*) of *H* is a collection of hyperedges $E_1, E_2, \ldots, E_j$ so that $X \in E_1 \cup \cdots \cup E_j$.

The least *j* for which this is possible (the smallest size of a covering) is called the *covering number* (or the *edge covering number*) of *H* and is denoted *k*(*H*).

- The fractional covering number of a graph can be computed in polynomial time.
- The constraint matrix in the fractional covering number problem has size $|X| \times |E|$ and therefore polynomial-time LP solutions for this problem exist.
- A more efficient approach, however, is to find the matching number of $H(X, E)$ using a standard bipartite matching algorithm.

Fractional Hamiltonicity can also be tested in polynomial time.

Methods:

formulation of fractional Hamiltonicity as a multicommodity flow problem.

There are two types of variables in this formulation:

capacities and flows.

To begin, one can arbitrarily assign a direction to every edge. To each edge *E* we assign a "capacity" variable in the interval [0, 1].

Fractional entropy descriptor of a hypergraph

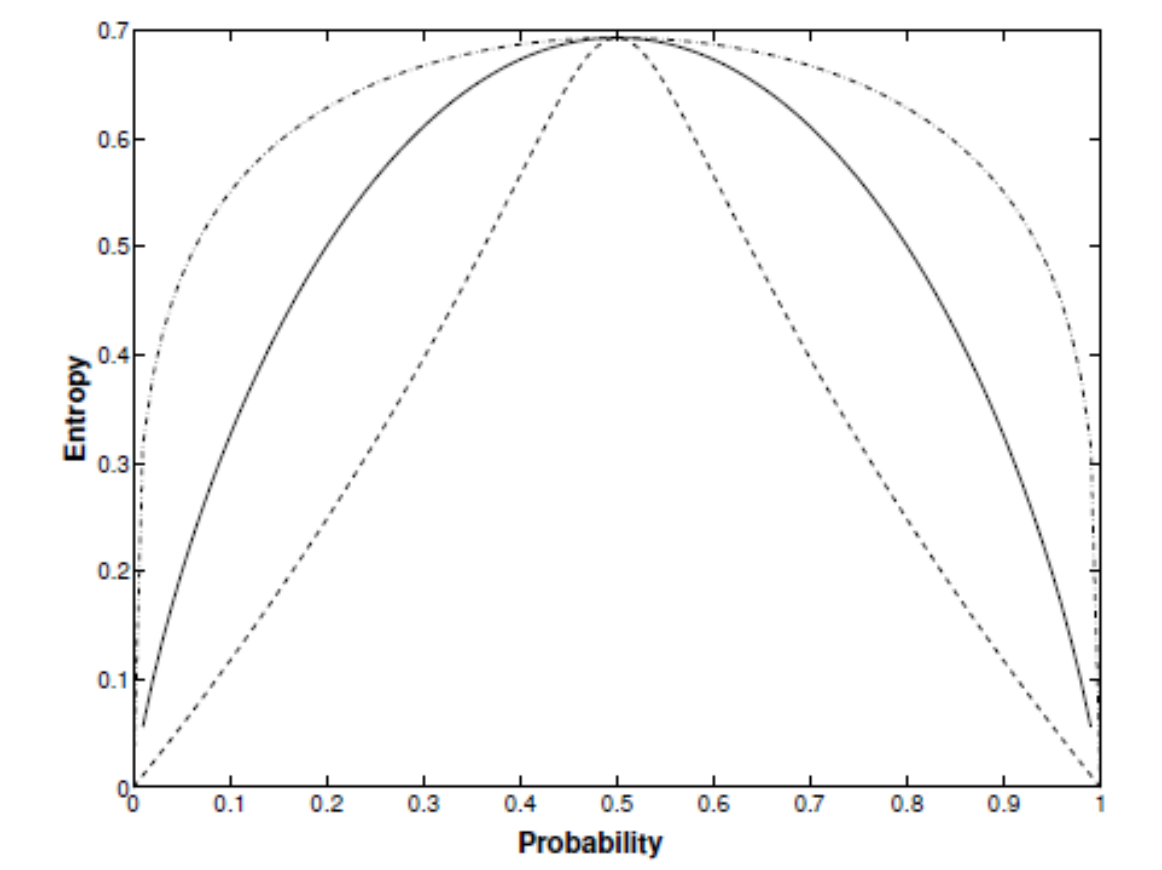
Fractional entropy descriptor as the Rényi's entropy of topological order *a*

$$H_R\left(H_G=(X,E)\right)=\frac{1}{1-a}\log\int_{H=(X,E)}p\left(I(x),H_G=(X,E)\right)^a dx$$

Using L'Hôpital, in the limit $a\to 1$ **Renyi's entropy converges to the Shannon's entropy.**

For any value of *a greater or equal to* 0, Rényi's entropy is nonnegative;

and for $a\in[0,1]$ **Rényi's entropy is concave showing an additional parameter which can be used to make it more or less sensitive to the shape of PDF, *p*.**



Topological Rényi entropy H_R as a function of the probability *p* of a binary source (*p*, 1 - *p*) (Bernoulli's distribution), for three values of the order *a* = 0.4 (dash-dotted line), *a* = 10 (dashed line), and *a* = 1 identified by plain line corresponding to the Shannon's entropy.

$H\left(H_G=(X,E)\right)$ **denotes an integral entropy estimation associated to a particular hypergraph region**

$$H\left(H_G=(X,E)\right)=\int_{H_G=(X,E)}\varphi\left(p\left(I(x),H_G=(X,E)\right)\right)dx$$

with φ **function and its derivative given by**

$$\varphi(r)=\varphi_a(r)=-\log\left(r^a\right)$$

$$\varphi'_a(r)=-\frac{a}{r}$$

Using φ_a function, from topological Rényi's entropy of order *a* we obtain an integral entropic measure integrating a fractional parameter.

Moreover, let's note that at the limit *a* = 1, we obtain the Ahmad-Lin estimator of Shannon's entropy

$$\varphi_a(r) = -\ln(p)$$

Remainder:

The **Rényi's** entropy of order *a* is a generalization of the von Neumann entanglement entropy that characterizes the quantum entanglement between two complementary subsystems A and B. It is defined as

$$S_a^{AB} = \ln \mathrm{Tr}\left[\rho_A^a\right] / (1-a)$$

Example for φ_a function:

Fractional covering number of *H*:

$$\boxed{k_f\left(H_G\right)=\lim_{t\to\infty}\frac{k_t\left(H_G\right)}{t}=\inf\frac{k_t\left(H_G\right)}{t}}$$

Convexity **Subadditivity** **Invariance to LUO**

Properties of $k_f(H)$

1. Convexity!

The matching polytope is defined as the convex hull of the incidence vectors of all the matchings of *G* and *H*. Any convex combination of optimal fractional packings is also an optimal fractional packing.

2. Subadditivity!

If H is any hypergraph, then there exist positive integers s and N such that, $k_s(H) + k_t(H) = k_{s+t}(H)$.

3. Invariance with respect to local unitary operations!

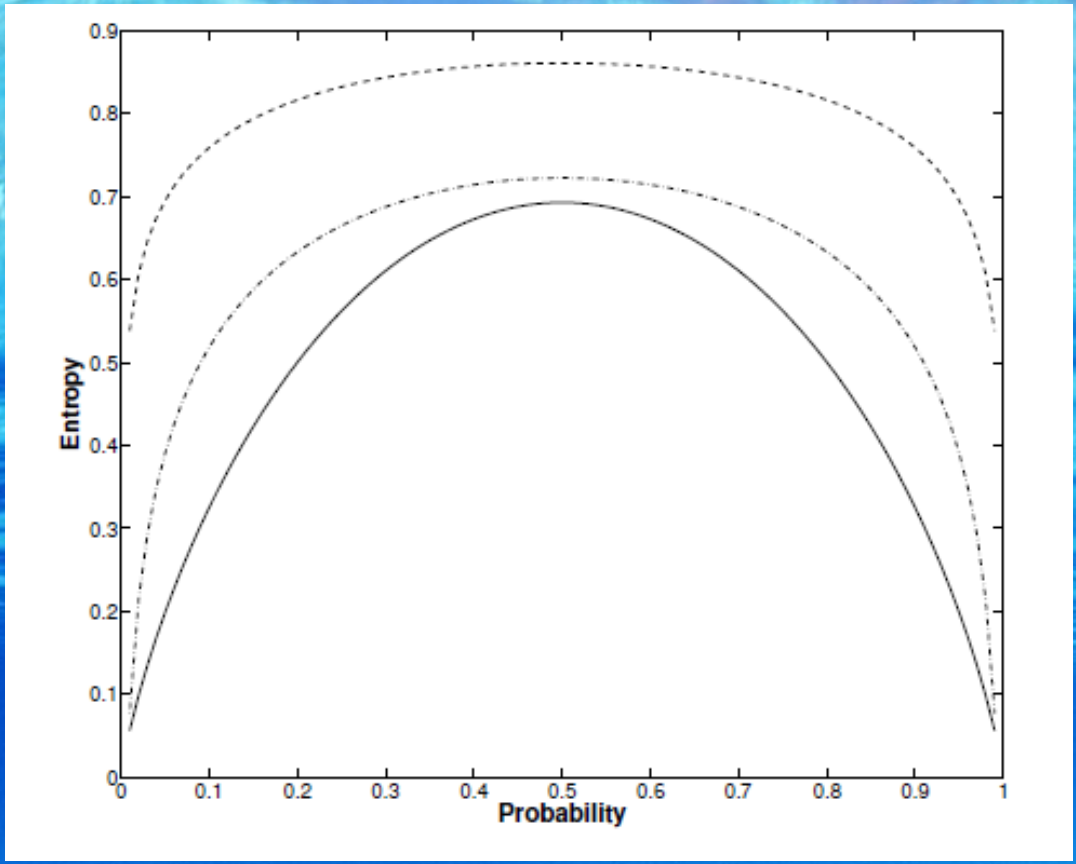
The nonlocal properties do not change under local transformations, i.e., unitary operations.

Consequence:

Fractional entropy descriptor of a hypergraph serves for quantification of nonclassical correlation + imply nonlocality

- **Nonlocality signifies that the statistical behaviour of a system cannot be described by a local realistic theory.**

- **For nonlocality it is essential that the correlation probabilities of such theories obey so-called Bell inequalities, which are violated for certain quantum states.**



Fractional entropy measure of a hypergraph as a function of the probability *p* of a binary source (*p*, 1 - *p*) (Bernoulli's law), for two values of the order *a* = 0.1 (dash-dotted line), *a* = 0.2 (dashed line). Plain line corresponds to the Shannon's entropy.

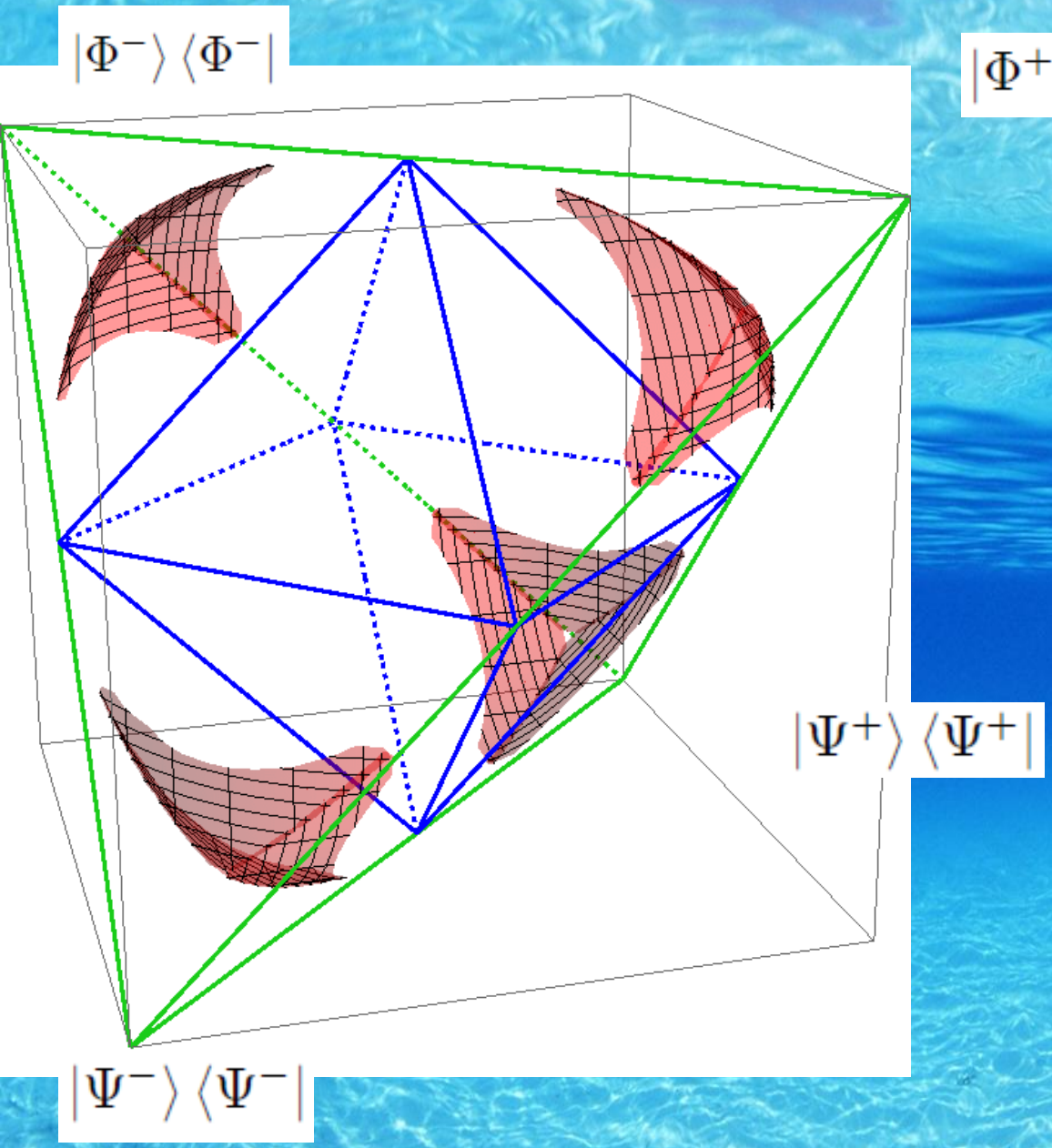


Illustration of the tetrahedron and the borders of nonlocality. States ρ beyond the red/meshed surfaces implies entanglement correlation

Entanglement vs separability

$$\left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}} = \left(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle\right) = \frac{1}{\sqrt{2}}\left(\left|01\right\rangle - \left|10\right\rangle\right)$$

$$\left|\Psi^+\right\rangle = \frac{1}{\sqrt{2}} = \left(\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle\right) = \frac{1}{\sqrt{2}}\left(\left|01\right\rangle + \left|10\right\rangle\right)$$

$$\left|\Phi^+\right\rangle = \frac{1}{\sqrt{2}} = \left(\left|\uparrow\uparrow\right\rangle + \left|\downarrow\downarrow\right\rangle\right)\frac{1}{\sqrt{2}} = \left(\left|00\right\rangle + \left|11\right\rangle\right)$$

$$\left|\Phi^-\right\rangle = \frac{1}{\sqrt{2}} = \left(\left|\uparrow\uparrow\right\rangle - \left|\downarrow\downarrow\right\rangle\right)\frac{1}{\sqrt{2}} = \left(\left|00\right\rangle - \left|11\right\rangle\right)$$

Pure maximally entangled states

Werner - mixed entangled state $\longrightarrow$ $\rho^W = p\left|singlet\right\rangle\left\langle singlet\right| + \frac{1}{4}(1-p)\mathbb{I}$

$\longrightarrow$ **it does not violate the Bell inequality for** $p > 1/3$

when it is entangled

If a quantum state violates the Bell inequality then we know that entanglement is present. The reverse is not true!!!

From standard (logical) basis → to Bell basis

With the standard product basis and the standard vectors $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|10\rangle = |1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |11\rangle = |1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

we can construct a basis with the Bell vectors

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad |\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad |\phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

Numerical procedures

Coarsening **Partitioning** **Entropy Refinement**

- **Successive coarsening is the first step towards finding a good partition.**
  - **Uniform vertex weights.**
  - **<u>Exposed edge-weight must decrease rapidly</u>.**
- **The *`how to coarsen'* computation must be fast.**
- **The size of successive coarse graphs must decrease relatively fast**

Computational mapping f projects linearly into computational logical states:

$$f(|\psi\rangle) := \begin{cases} |0\rangle & \text{if } |\psi\rangle = |\uparrow\rangle \\ |1\rangle & \text{if } |\psi\rangle = |\downarrow\rangle \end{cases}$$

The computational mapping *f* projects the 4-level (S, T_0, $T_{+/-}$) system to a two-level subspace. In the case of a two- ½ spin qubit, *f* sends two degrees of freedom to zero computational meaning. Over the remaining subspace it is a linear mapping that maps the two basis states to computational states:

$$f(|\psi\rangle) := \begin{cases} 0 \text{ if } |\psi\rangle' = span\{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle\} \\ |0\rangle \text{ if } |\psi\rangle' = |S\rangle = \dfrac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} \\ |1\rangle \text{ if } |\psi\rangle' = |T\rangle = \dfrac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} \end{cases}$$

Conclusions

- **We proposed an original fractional entropy measure inspired from Rényi's topological order making possible description of the complex systems and strong variations of the shapes of the non parametrically estimated related PDF.**
- **The main motivation was to overcome the limitations of Shannon's entropy which appeared not adapted to partition problem.**
- **Method is proposed for hypergraph structures which reflect nonlocal characteristics of correlations between separate objects and can be used for description of entanglement resources.**

Thank you for attention!