N2 SUGRA BPS MULTICENTER BHs.
Mass Formulas, Jordan Algebras and FTS

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Based on:

–JHEP 1405(2014) 081 (JJ,ET)
–N2 BPS Mass formulas, Sub. PRL (JJ,ET)
–Freudenthal and Jordan transformations, (JJ, A. Marrani,ET)

INTRO: BPS states. Central Charge.

- BPS STATES ($N > 1$): Short multiplets invariant under a fermionic (1/(2,4,8)) SUSY subalg.
  -M =| Z |.
  -QM-stable: the mass formula is exact.

- EXTREMAL SUGRA BHs: a realization of BPS states by the SUGRA superalgebra.
  $\longrightarrow$ Killing Spinor **(1st order)** eqs+**(2nd order)** eqs. of motion.

- MASS FORMULAS: -necc. conditions for BPS states (BHs).
  -Lower bounds on M
  -$M > M_{extr} > 0$ Witten81,Gibbons,Hawking,83

- String Connection: Extremal BHs: bound states of d-branes in compactified space,$\longrightarrow$ microstates,$\longrightarrow$ BH entropy.

INTRO: BHs Mass formulas. EXAMPLES.

- 4d Pure SUGRA (no scalars) $\sim$ **E-MAXWELL GRAV.**

$\longrightarrow$ REISSNER-NORD: $\mathbf{M^2 = P_m^2 + Q_e^2}$.

$\longrightarrow$ (Static) Majumdar-Papatreou:

$$ds^2 = e^{2U(\mathbf{x})}dt^2 - e^{-2U(\mathbf{x})}d\mathbf{x}^2, \quad e^{-U} = 1 + \sum_i \frac{M_i}{|x - x_i|},$$

BPS: $\mathbf{M_i > 0,\ M_i^2 = q_i^2,\ M_{ADM} = \sum M_i = f(q_i)}$

Extremal BH M: equilibrium eq of state. $\sim$ Balanced Grav./EM.

- **GENERAL SUGRA:** scalars= extra attractive long range force

$\longrightarrow$ modified relations

SUGRA dilaton model: $\mathbf{M^2 = Q^2 \exp(2\phi_\infty)/2}$ Garfinkle:1990qj

- **OBJECTIVE:** $\mathbf{M_{ADM}, n_c, n_v}$

THE MODEL: $N2$ $D4$ SUGRA $+n_v$ vector multiplets

- Bosonic content (essentially) $(\mathbf{g}_{\mu\nu}, \mathbf{A}_{\mu}{}^{\mathbf{I}}, \mathbf{z}^{\alpha})$, $\quad I = n_v + 1, \alpha = n_v$.

$$\mathbf{S_{bos}} = \int_{\mathbf{M(4d)}} \mathbf{R} \star \mathbf{1} + \mathcal{G}_{\alpha\bar{\beta}}(\mathbf{z})\mathbf{d}\mathbf{z}^{\alpha} \wedge \star \mathbf{d}\bar{\mathbf{z}}^{\bar{\beta}} + \mathbf{F}^{\mathbf{I}} \wedge \mathbf{G}_{\mathbf{I}}$$

$$F^I = dA^I, \qquad G_I = a_{IJ}(z)F^J + b_{IJ}(z) \star F^J$$

- $dF^I = 0, dG_I = 0 \longrightarrow Sp(2n_v + 2, \mathbb{R})$ Simplectic Duality

$\longrightarrow$ sympletic vectors: $\begin{pmatrix} F^I \\ G_I \end{pmatrix} \longrightarrow \mathbf{q} = \begin{pmatrix} p^I \\ q_I \end{pmatrix}$

- (A^I, z^{α}) same super-multiplet $\longrightarrow$ Special Kahler-Hodge geometry of the scalar Manifold $\longrightarrow$ G_{ab}, a_{IJ}, b_{IJ} :

Special Kahler geometry =∼

• A Complex Kahler manifold plus a symplectic space on top it. :A $(2n_v + 2\ )$ vector complex space **W**.

ANTISIM: $\langle \mathbf{X} \mid \mathbf{Y} \rangle$, SIM: $< \mathbf{S(z)X, Y} >$ ($\longrightarrow$ FTS(J), J. Algebras)

• **PREPOTENTIAL**: $F = F(z)$, Holomorphic, 2-homogenous: $\mathbf{F_{IJ} \equiv \partial_I \partial_J F}$, $\mathbf{F_I \equiv \partial_I F}$.

$$\begin{aligned} \mathcal{G}_{\alpha\bar{\beta}}(\mathbf{z}) &= \partial_\alpha \partial_{\bar{\beta}} \mathcal{K} = \left\langle \mathbf{D}_\alpha \mathbf{V} \,\middle|\, \bar{\mathbf{D}}_{\bar{\beta}} \bar{\mathbf{V}} \right\rangle , \\ \mathbf{N_{IJ} = a_{IJ} + i b_{IJ}} &= \mathbf{F_{IJ} + (F_{IJ}, F_I) \ldots} \\ \mathbf{S} &= \mathbf{S(F_{IJ})} \end{aligned}$$

BPS stationary solutions.

SUSY+BPS CONDITIONS+FIELD EQUATIONS (+stationary +asympt. at ∞):

- Most general 4d BPS stationary: IWP metric, Tod83,IWP72 ($\omega = \omega_i dx^i$)

$$\mathbf{ds^2 = e^{2U(x)}(dt+\omega)^2 - e^{-2U(x)}dx^2}$$

- $dF^I = dG_I = 0$: $\longrightarrow$ $\mathbf{(F^I)_{mn} = \star_3 d_3 I^I, (G_I)_{mn} = \star_3 d_3 I_I}$
- BPS condition: relations A, U, ω fields. Gauntlett02,Ortin'03 $\longrightarrow$ U, ω, z^α solutions in terms of

$$\mathcal{I}(\mathbf{x}) \equiv \begin{pmatrix} \mathbf{I^I} \\ \mathbf{I_I} \end{pmatrix}, \ \mathbf{R^3 - harmonic}$$

n_c-center: TAKE $\mathcal{I}(\mathbf{x}) \equiv (I^I, I_I)$

- $\mathcal{I} = \mathcal{I}_\infty + \sum_a \frac{\mathbf{q_a}}{|\mathbf{x} - \mathbf{x_a}|}$ Hawking,Hartle'72

- TOY MODEL $n_c = 2$: q_1, q_2, 1-scalar z, 2 $A_\mu^{0,1}$ (Particular example:Axion-dilaton).

- $\mathcal{I} = \mathcal{I}_\infty + \frac{\mathbf{q_1}}{|\ \mathbf{x} - \mathbf{x_1}\ |} + \frac{\mathbf{q_2}}{|\ \mathbf{x} - \mathbf{x_2}\ |}$.

$2n_c = dim\ W$:

$$q_{1,2} = \begin{pmatrix} p^0 \\ p^1 \\ q_0 \\ q_1 \end{pmatrix}_{1,2} , \mathcal{I}_\infty = \begin{pmatrix} I^0_\infty \\ I^1_\infty \\ I_{\infty 0} \\ I_{\infty 1} \end{pmatrix} \quad (1)$$

- **PARAMS:** $\mathbf{M_{ADM}, M_i, S_{BH}, r_{12}, z_\infty = f(I_\infty, q_i, x_i)}$.

BPS Solutions: $\mathcal{I}(\mathbf{x}), \mathcal{S} \longrightarrow :(U, \omega), z^\alpha, A^I_\mu$

SOME PROPERTIES:

- $d\omega = 2\,\langle \mathcal{I} \mid \star_3 d\mathcal{I}\rangle$, $\omega_\infty \to 0$, $(\omega \sim J d\phi)$:
  $\mathbf{r_{ab}}$ are restricted (linear eqs in $\mathbf{1/r_{ab}}$).

- Scalar stabilization equations:

  $z^\alpha_\infty \sim$: moduli. $\quad \mathbf{z^\alpha_\infty = (P_-\mathcal{I}_\infty)^\alpha/(P_-\mathcal{I}_\infty)^\alpha}$
  z^α_h: BPS attractor mechanism $\longleftarrow q's \quad (\longrightarrow S_{BH})$

- $e^{-2U} = \langle \mathcal{S}\mathcal{I} \mid \mathcal{I}\rangle$, $\quad \mathbf{e^{-2U_\infty} = \langle \mathcal{S}_\infty \mathcal{I}_\infty \mid \mathcal{I}_\infty\rangle \to 1}$

- $\mathbf{\mid Z(z^\alpha_\infty, Q) \mid^2 = \mid \langle \mathcal{S}_\infty \mathcal{I}_\infty \mid Q\rangle \mid^2 + \mid \langle \mathcal{I}_\infty \mid Q\rangle \mid^2}$

ADM MASS:BEHAVIOUR AT INFINITY

REQUEST MINK SPACE AT INFINITY F-Melgarejo, Torrente, JHEP2014

- $e^{-2U} = \langle \mathcal{S}\mathcal{I} \mid \mathcal{I} \rangle \underset{x \to \infty}{\longrightarrow}$

$$1 + \frac{2 \sum_b \langle \mathcal{S}_\infty \mathcal{I}_\infty \mid q_b \rangle}{r} + \frac{\sum_{ab} \langle \mathcal{S}_\infty q_a \mid q_b \rangle}{r^2} + O(\frac{1}{r^3})$$

$$\equiv 1 + 2\frac{M_{ADM}}{r} + \frac{A_\infty}{r^2} + O(\frac{1}{r^3}).$$

With: $M_{ADM} = \langle \mathcal{S}_\infty \mathcal{I}_\infty \mid Q \rangle$, $A_\infty = \langle \mathcal{S}_\infty Q \mid Q \rangle$.

$$M_a = \langle \mathcal{S}_\infty \mathcal{I}_\infty \mid q_a \rangle, \; m_a = M_a / M_{ADM}.$$

- BPS condition: $\mid Z(z^\alpha_\infty, Q) \mid^2 = \mid \langle \mathcal{S}_\infty \mathcal{I}_\infty \mid Q \rangle \mid^2 = M^2_{ADM}$

BPSMass formulas...

Fdez-Melgarejo, Torrente2014

USING asympt. flatness +special geometry dim, WE SHOW ($dimW = 2n_c, n_v + 1 = n_c,$):

$$\mathbf{M^2_{ADM} = a\, M_{ADM} + b} \quad (2)$$
$$\mathbf{z^{\alpha}_{\infty} = = f^{\alpha}(M_i, r_{ij}, z_{\infty}).} \quad (3)$$

a, b: **depend ONLY on physical parameters:**

- SYMPL. products of q's ($\sim$ BH J_{ij}, $\langle q_i \mid q_j \rangle$ $\langle Sq_i \mid q_j \rangle$),
- Moduli at infinity, z^{α}_{∞} ($\mathcal{S}_{\infty}$ in $\langle \mathcal{S}_{\infty} q_i \mid q_j \rangle$),
- r_{ab}, $m_i = M_i / M_{ADM}$.

• QUAD. RELATION in $M, 1/r_{ij}, J_{ij}$: **(M > 0)$\longrightarrow$ restrictions on coeffs.** $\longrightarrow$ BH existence neccesary conditions

EXPLICIT CASE: $n_c = 2$, 1 scalar

BPS Mass Eq:

$$M^2_{ADM} \quad = \quad M^2_0 + \alpha J^2 \left(1 + \frac{2M_{ADM}}{r} + \frac{A_\infty}{r^2}\right).$$

$A_\infty = \langle \mathcal{S}_\infty Q \mid Q \rangle, J = \langle q_1 \mid q_2 \rangle \; \alpha > 0, A_\infty, M_0^2 : f(q_i, m_i, z_\infty)$

Characteristics:

- 1 real, positive solution: ($A_\infty, \alpha > 0$), for any r and J^2.
- $r \to \infty$: $M^2_\infty = M^2_0 + \alpha J^2$, $r \to 0$: $M \sim A/r + Br$
- **Lower Bound :$\mathbf{M^2 > M^2_\infty}$**

$n_c = 2$, 1 sclar. Minimal Mass Formula

- MINIMIZE $M_{ADM}(m_i)$ $\qquad (M_0 = f(m_i))$

$$M_{ADM}^2 \;=\; A_\infty + \alpha J^2\left(1 + \frac{2M_{ADM}}{r} + \frac{A_\infty}{r^2}\right).$$

$$m_{i,\min} = \frac{\langle \mathcal{S}_\infty q_i \mid Q\rangle}{\langle \mathcal{S}_\infty Q \mid Q\rangle}$$

- $r \to \infty$: $\mathbf{M_{ADM}^2 = A_\infty + \alpha J^2}$. $\qquad A_\infty = \langle \mathcal{S}_\infty Q \mid Q\rangle$
- $\mathbf{M_{ADM}^2(nc = 2) > \langle \mathcal{S}_\infty Q \mid Q\rangle = M_{ADM}^2(nc = 1)}$
- Relative $\mathbf{m_1, m_2}$ **RESTRICTED:**

$$M_0^2 > 0 \longrightarrow m_i \in m_{i,\min} \pm \sqrt{\alpha}/A_\infty$$

Static Effective "force" a la Smarr

SMARR Formulas: Smarr,PRL72

$M_{ADM}(\sim E) \sim 1/r$:

- Smarr-like expressions, From $M_{ADM}(J, z_\infty, q, r)$

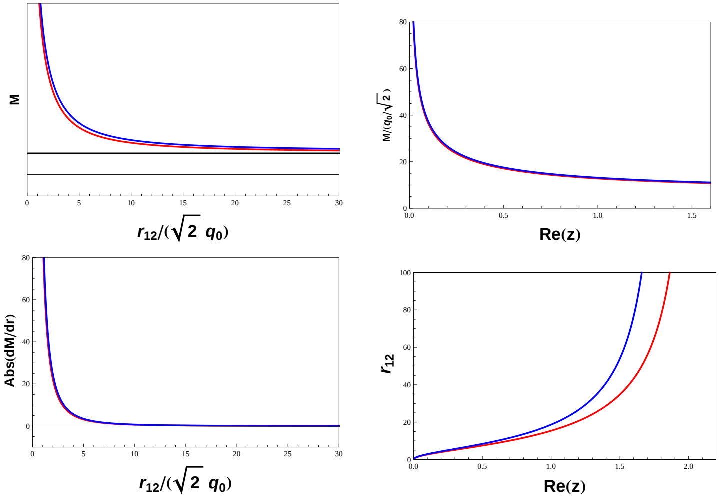
$$dM = \Omega dJ + \Sigma dz_\infty + \Phi_i dq_i - F dr$$

- dz_∞: eliminated from scalar eqs.$\longrightarrow$ (at q_i cte.)

$$dM \,|_{q=cte} = -F dr,$$

- F:Effective "force" between centers. TIPICALLY:
  - F :$\sim$ repulsive (for viable M_{ADM}).
  - $F \sim 1/r^2$, $r \to \infty$.
  - $F \sim f_0 + f_1/r^2$, $r \to 0$.

EXAMPLE:AXION-DILATON, 2 center.



FIX: $q_1 = (1, 8, 0, -1)q_0$,$q_2 = (1, 8, -4, 1)q_0$,
$J^2 > 0$. $A_\infty > 0, -\det(S) > 0$. $m_i^{min} = (9/16, 7/16)$.

BHs, Jordan Algebras, FTS, entanglement..

- 4d: W: $\langle X \mid Y \rangle$, $< SX, Y >$ **+Int. Oper., $\mathbf{T(X), \Delta_4(X, Y, Z, W)}$**

$\longrightarrow$ FTS(J), "Freudenthal Tripe System", J=JORDAN Algebra

$\longrightarrow$ J: Jordan Algebra , 5d BH's Duff,Ferrara,...

- $\tilde{q} = \mathcal{S}_\infty q$, $q \to aq + b\tilde{q}$ F-Melgarejo, Torrente, JHEP2014

- **FTS(J), J**: ONLY known explicit reps of Lie alg. $\mathbf{E_{(7)}, E_{(6)}}$

- **BH-Q.Infomation LINK**: $S^2_{BH} = \Delta_4(X) \sim S^2_{entangl}(3 - qubit)$.

Duff,Marrani,Borsten,Ferrara..

Classification BHs $\longleftrightarrow$ class. entangled qbits

Conclusions

- Multicenter BHs in $N2$ $D4$ SUGRA $+n_v$ vector multip.

$\longrightarrow$ new BPS mass formulas

–Properties, Construction and existence of BH multicenter solutions

–Counting of microstates (in stringy inspired models)

- EXAMPLE: minimal M_{ADM}, 2 center BPS solutions

$$\mid \mathbf{Z} \mid^2 = \mathbf{M}^2 \;=\; \mathbf{A}_\infty \left(1 + \alpha \mathbf{J}^2 \left(1 + \frac{2\mathbf{M}}{\mathbf{r}} + \frac{\mathbf{A}_\infty}{\mathbf{r}^2}\right)\right)$$

- Continuos family, for q_i, $r \in (0, \infty)$,$M_{ADM} \in (\infty, M_\infty)$ and z_∞
- Minimal mass (Mass Gap)
- Range of values of m_i allowed. etc..

- N2 d4 SUGRA: specially suitable: many restrictions. $\mathcal{M}_z =$ Sp. Kahler.

BUT SIMPLE ESSENTIAL INGREDIENTS: BPS condition, asymptotical flatness... $\longrightarrow$ generalizable other contexts.

Multicenter ANSATZ.

- n_c MULTICENTER: NEED $\mathcal{I}(\mathbf{x}) \equiv (I^I, I_I)$
  TAKE: $\boldsymbol{\mathcal{I} = \mathcal{I}_\infty + \sum_a \frac{q_a}{|x - x_a|}}$,
  $\mathcal{I}_\infty, q_a \in (2n_v + 2)$-dim real $\in W$

- • r_{ab} : restricted,
  • TAUB-NUT charge $N = 0 = \langle \mathcal{I}_\infty \mid Q \rangle = \sum_a \langle \mathcal{I}_\infty \mid q_a \rangle = 0$:

$$\mathbf{dd\omega = 0} \quad \longrightarrow \quad \boldsymbol{\langle \mathcal{I} \mid \Delta \mathcal{I} \rangle = 0} \tag{4}$$

$$\longrightarrow \quad \boldsymbol{\langle \mathcal{I}_\infty \mid q_b \rangle + \sum_a \frac{\langle q_a \mid q_b \rangle}{r_{ab}} = 0} \tag{5}$$

INTRO: Multicenter. Existence conditions

Existence and Construction of extremal BHs: a rather trivial problem in a wide number of well known theories.

- 1- center, static n-center solutions: given parameters (and a suitable metric ansatz) $\mathbf{M, Q, z^a_\infty, J, \Sigma^a}$..
  (or a subset of them) satisfiying simple local relations,
  $\longrightarrow$ Build explicitly solutions from them.

- General Stationary Multicenter solutions:

$$\mathbf{M_{ADM}, M_i, Q, q_i, J_{ij}, r_{ij}, z_\infty}$$

$\longrightarrow$ BPS solutions: not a trivial problem for given params.

$\longrightarrow$ **Desiderable**: neccesary and sufficient BPS existence conditions in terms of these "macroscopical" params.
but **this is not known.** $\longrightarrow$ OUR OBJECTIVE HERE.

GENERAL PROPERTIES: $r_{ij} \to \infty$, Minimal Mass.

M_{ADM} decre. with incre. r_{ij}: $\mathbf{1 = aM^2 + b_j N_j M + c_{ij} N_i N_j}$

- If $r_{ij} \to \infty$. Then $N_i = \sum c_{ij}/r_{ij} \to 0$. Mass EQ:

$$\mathbf{1 \;\; = \;\; a(m_i) M_\infty^2}$$

$\longrightarrow$ (if $M_\infty^2 > 0$) Minimal ADM mass for a given q_i, m_i configuration.

- MINIMIZE $M_\infty^2 = M^2(m_i, q = cte)$
  $\mathbf{m_{i,min} \simeq \langle \mathcal{S}_\infty Q \mid q_i \rangle \,/\, \langle \mathcal{S}_\infty Q \mid Q \rangle + O(J^2)}$,

- in this case $M_{ADM}(r_{ij} \to \infty)$

$$\mathbf{(M_\infty^2)_{min} = \frac{1}{Im \sum_{ij} \; g^{i\bar{\jmath}}}}$$

$\longrightarrow$ **Minimal ADM mass for a given $\mathbf{q_i}$ configuration**.

Simplectic expansions

Well known useful expansion: Bossard,Katmadas'13

- $(2n_v + 2)$-W simplectic space, a basis: $\{\mathbf{V}, \mathbf{D_\alpha V}, \bar{\mathbf{V}}, \bar{\mathbf{D}}_{\bar{\alpha}} \bar{\mathbf{V}}\}$

By special geometry:

$\langle V \mid \bar{V} \rangle = -i, \langle \bar{V} \mid D_\alpha V \rangle = \langle V \mid D_{\bar{\beta}} V \rangle = 0, \langle D_\alpha V \mid D_{\bar{\beta}} V \rangle = G_{\alpha\bar{\beta}}$.

Eigenvectors of $\mathcal{S}(N)$: $\mathcal{S}(N) V = iV, \mathcal{S}(N) D_\alpha V = -iD_\alpha V$.

- Any real $X \in W$

$$X = x_0 V + x^{\bar{0}} \bar{V} + x^{\bar{\beta}} \bar{D}_{\bar{\beta}} \bar{V} + x^\beta D_\beta V$$

- **ALTERNATIVE: MULTICENTER BHs: MANY SYMPLECTIC VECTORS: $\mathcal{I}_\infty, \mathbf{q_a} \longrightarrow$ TO DEFINE A BASIS IN W ($\mathbf{W^\pm}$)**

MASS RELATIONS:CHARGE SPACE EXPANSIONS

Fernandez-Melgarejo:2013ksa

- W Basis: $\mathcal{S}_\infty \equiv \mathcal{S}(z^a_\infty)$ eigenvectors, $\mathbf{(P_+w_k, P_-w_k)}$,
$\mathbf{w_k = (q_n, s_a)}$, $2n_a = 2n_V + 2 - 2n_c$, $\mathrm{P}_\pm = (1 \pm i\mathcal{S}_\infty)/2$.

- Define "metric": $\mathbf{g_{k\bar{k}} \equiv \langle P_+w_k \mid P_-w_k \rangle}$, $\mathbf{g_{i\bar{\jmath}}g^{\bar{\jmath}k} = \delta_i^k}$

- Any real X, (i.e. $\mathcal{I}_\infty$) , in "covariant/contravariant" coordinates,

$$\begin{aligned}
\mathbf{X} &\equiv \alpha^{\mathbf{k}}\mathbf{P_+w_k} + \alpha^{\bar{\mathbf{k}}}\mathbf{P_-w_k} \\
&\equiv \alpha_{\mathbf{l}}\mathbf{g^{lk}P_+w_k} + \alpha_{\bar{\mathbf{l}}}\mathbf{g^{\bar{l}k}P_-w_k}
\end{aligned}$$

With:$a_j = \langle X \mid \mathrm{P}_+ w_j \rangle = g_{\bar{\imath}j}\alpha^{\bar{\imath}}$.

- "scalar" product: $\langle \mathrm{P}_+ X \mid \mathrm{P}_- Y \rangle = \alpha_i \beta^{\bar{\imath}} = \alpha_i \beta_{\bar{\jmath}} g^{i\bar{\jmath}}$

General Mass formulas: INGREDIENTS

- EXPAND $\mathcal{I}_\infty = \alpha_i g^{ik} \mathsf{P}_+ w_k + \alpha_{\bar{\jmath}} g^{\bar{l}\bar{k}} \mathsf{P}_- w_k$
- ADVANTAGE: contravariant components are Phys. params.

$$
\begin{aligned}
a_j(\mathcal{I}_\infty) &= (N_j - iM_j)/2, \quad j = 1, n_c \\
&= \left(\sum \langle q_j \mid q_i \rangle / r_{ij} - iM_j \right) /2
\end{aligned}
$$

- ASYMPT. flatness: $e^{-2U} \to \langle \mathcal{S}_\infty \mathcal{I}_\infty \mid \mathcal{I}_\infty \rangle \to 1$,

$$\mathbf{1 = -2i \left\langle P_+ \mathcal{I}_\infty \mid P_- \mathcal{I}_\infty \right\rangle = -2i\alpha_i \alpha^{\bar{\imath}} = -2i\alpha_i \alpha_{\bar{\jmath}} g^{i\bar{\jmath}}}$$

with $\alpha_i = \alpha(q_i, M_i, r_{ij})$, $g^{ij} = g^{ij}(q_i, z_\infty)$.

- **GENERAL MASS FORMULA**: $M_{ADM} = \sum_i M_i$, $m_i = M_i / M_{ADM}$

$$\mathbf{1 \quad = \quad aM_{ADM}^2 + bM_{ADM} + c}$$

2 CENTER MASS RELATIONS ($n_c = 2, n_v = 1$)

ASSUME:
- One scalar $(n_v = 1)$, $dimW = 2n_v + 2 = 4$.

2 centers: $\mathbf{q_1, q_2} \longrightarrow$ W-Basis $= (\mathbf{P_\pm q_{1,2}})$
Parms(5): $M_1, M_2, r_{12}, z_\infty$. $(q = \text{fixed})$
Eqs(3): $1M{+}2\ z$: $\longrightarrow$ $dof = 5 - 3 = 2$.

- Metric matrices $g_{ij} = (A + iS)/2, g^{ij} = X + iY$

$$A = \langle q_1 \mid q_2 \rangle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} \langle \mathcal{S}_\infty q_1 \mid q_1 \rangle & \langle \mathcal{S}_\infty q_2 \mid q_1 \rangle \\ \langle \mathcal{S}_\infty q_2 \mid q_1 \rangle & \langle \mathcal{S}_\infty q_2 \mid q_2 \rangle \end{pmatrix}.$$

- $\omega-$ compatibility eqs: $\mathbf{-N_2 = N_1 = \langle q_1 \mid q_2 \rangle / r \equiv J/r}$

- $\mathbf{a_{1,2}(\mathcal{I}_\infty) = (J/r - iM_{1,2})/2, \quad -2ia_i a_j g^{ij} = 1}$

The 2 CENTER MASS RELATION

- BPS 2-c Mass Equation: $A_\infty = \langle \mathcal{S}_\infty Q \mid Q \rangle$, $M_0^2 = 1/S_{ij}^{-1} m_i m_j$.

$$M_{ADM}^2 \quad = \quad M_0^2 \left(1 + \frac{J^2}{-\det(S)} \left(1 + \frac{2M_{ADM}}{r} + \frac{A_\infty}{r^2}\right)\right).$$

$det(S), A_\infty, M_0^2 : f(q_i, m_i, z_\infty)$

Characteristics:

- 1 real, positive solution: ($A_\infty > 0$) only if $\det(S) < 0$.
- Unique solution for any r and J^2.
- ($r \to \infty$): $M_\infty^2 = M_0^2 \left(1 + \frac{J^2}{|\det(S)|}\right)$,
- $M \sim M_\infty + J^2/(|\det(S)|r)$ $(r \to \infty)$. $M \sim A/r + Br$ $(r \to 0)$.
- m_i: restricted.

Minimal Mass Formula

- MINIMIZE $M_{ADM}(m_i)$

$$M^2_{ADM} \quad = \quad A_\infty \left(1 + \frac{J^2}{|\det(S)|}\left(1 + \frac{2M_{ADM}}{r} + \frac{A_\infty}{r^2}\right)\right).$$

$$m_{i,\min} = \frac{\langle \mathcal{S}_\infty q_i \mid Q\rangle}{\langle \mathcal{S}_\infty Q \mid Q\rangle}, \qquad A_\infty = \langle \mathcal{S} Q \mid Q\rangle \,, J = \langle q_1 \mid q_2\rangle$$

- At large r: $\mathbf{M^2_{ADM} = A_\infty \left(1 + \frac{J^2}{|\det(S)|}\right)}$.

  MINIMAL MASS FOR A 2 CENTER BH (N2,1scalar)

- Relative $\mathbf{m_1, m_2}$ **RESTRICTED:**

$$M^2_0 > 0 \longrightarrow m_i \in m_{i,\min} \pm \sqrt{|\det(S)|}/A_\infty$$

EXAMPLE(1): Pure 4d SUGRA. $F = \frac{-i}{2}(X^0)^2$

- Pure N2 SUGRA (e^a_μ, A_μ, ψ): ($\sim$ Einstein-Maxwell) .

Dim $W = 2n_v + 2 = 2$.

$$F = \frac{-i}{2}(X^0)^2, \quad N_{00} = F_{00} = -i, S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

- Static BPS solutions $g_{\mu\nu}, A_\mu$

$$\begin{aligned} ds^2 &= e^{2U(\mathbf{x})} dt^2 - e^{-2U(\mathbf{x})} dx^2 \\ A_t &= e^U \qquad (Killing\ Spinor\ eqs.) \\ \nabla^2 e^{-U} &= 0 \qquad (Maxwell\ eqs.) \end{aligned}$$

- TAKE: (n_c arbitrary): $q_i = (P, Q)_i$.

$$\mathbf{e^{-2U} = \langle \mathcal{S}\mathcal{I} \mid \mathcal{I} \rangle}, \quad \mathcal{I} = \mathcal{I}_\infty + \sum_{\mathbf{nc}} \frac{\mathbf{q_i}}{\mid \mathbf{x} - \mathbf{x_i} \mid}$$

RN: $\longrightarrow$ TAKE $n_c = 1$

Pure SUGRA: 1-center: RN Solution

- RN: $\longrightarrow$ TAKE $n_c = 1$. Q. Basis W$= (P_\pm Q)$.
  $\mathcal{I} = \mathcal{I}_\infty + \frac{Q}{r}$. Expand: $\mathcal{I}_\infty = \alpha^0 P_+ Q + \alpha^{\bar{0}} P_- Q$.

- $g_{0\bar{0}} = (i/2) \langle Sq \mid q \rangle$, $g^{0\bar{0}} = 1/g^{\star}_{0\bar{0}}$.

- BPS MASS FORMULA $(\langle S\mathcal{I}_\infty \mid \mathcal{I}_\infty \rangle = 1)$:

$$1 \quad = \quad \frac{M_{ADM}}{\langle SQ \mid Q \rangle}, \quad M^2_{ADM} = \langle SQ \mid Q \rangle = p_m^2 + q_e^2$$

Pure SUGRA: 2 center: Majumd.-Papapetrou

- $n_c = 2$:q_1, q_2,$2n_c >$dim W. Many Basis W$=(P_{\pm}q_i)$,$(P_{\pm}Q)$...

$\boldsymbol{\mathcal{I} = \mathcal{I}_\infty + \frac{q_1}{|x-x_1|} + \frac{q_2}{|x-x_2|}}$.

- Expand $\mathcal{I}_\infty$: ($\longrightarrow$ 3 simultaneous Mass EQS).

$$\begin{aligned}
\mathcal{I}_\infty &= \alpha^0 P_+ q_1 + \alpha^{\bar{0}} P_- q_1, \\
&= \alpha^0 P_+ q_2 + \alpha^{\bar{0}} P_- q_2, \\
&= \alpha^0 P_+ Q + \alpha^{\bar{0}} P_- Q,
\end{aligned}$$

- M Formulas: **($\mathbf{N_i = \langle \mathcal{I}_\infty \mid q_i \rangle}$, $\mathbf{N = N_1 + N_2 = 0}$, $\mathbf{N_1 = J/r_{12}}$)**

$$\begin{aligned}
1 &= \frac{M_i^2 + N_i^2}{\langle S q_i \mid q_i \rangle}, \quad \longrightarrow M_i^2 + \frac{J^2}{r^2} = \langle S q_i \mid q_i \rangle \\
1 &= \frac{M_{ADM}^2 + N^2}{\langle S Q \mid Q \rangle}, \quad \longrightarrow M_{ADM}^2 = \langle S Q \mid Q \rangle
\end{aligned}$$

$\longrightarrow$ CASES: STATIC ($J = 0$), STATIONARY ($J \neq 0$)

MAJUMDAR-PAPAPETROU (Revisited)

- STATIC (Well-Known): $M_i = \langle Sq_i \mid q_i \rangle$, $M^2_{ADM} = \langle SQ \mid Q \rangle$,

$$M_{ADM} = M_1 + M_2 \longrightarrow 1 \quad = \quad \frac{\langle Sq_1 \mid q_2 \rangle^2}{\langle Sq_1 \mid q_1 \rangle \langle Sq_2 \mid q_2 \rangle}$$

$\longrightarrow$ $q_1 = \lambda q_2$: (in particular q_1, q_2 only elec. or only magn.)
$\longrightarrow$ r_{12} : unrestricted

- STATIONARY (Not so well Known): M_1, M_2, r_{12} fixed by 3 EQs.

$$M_i^2 = \frac{(\langle Sq_i \mid q_i \rangle + \langle Sq_i \mid q_j \rangle)^2}{\langle SQ \mid Q \rangle}$$

$$r_{12}^2 = M^2_{ADM} = \langle SQ \mid Q \rangle = p_m^2 + q_e^2$$

EXAMPLE(2): AXION-DILATON MODEL, $F = -iX_0X_1$.

TOY MODEL: 1-scalar, 2 $A_\mu^{0,1}$: Bellorin:2006xr,F-Melgarejo:2013ksa.

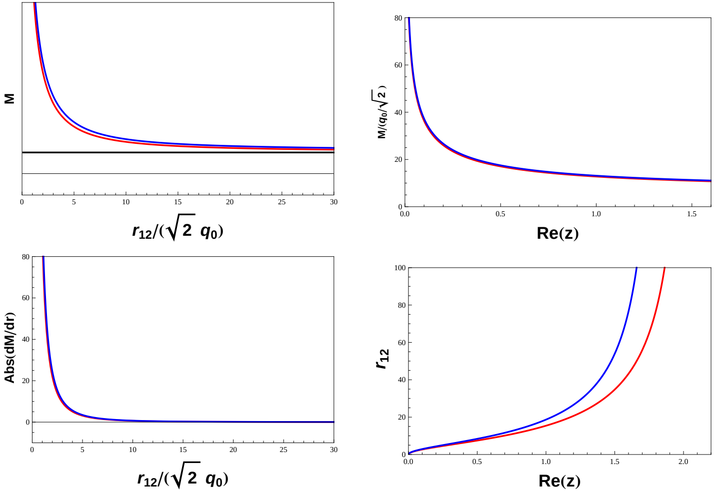
- **$\mathbf{F = -iX_0X_1}$, S: scalar independent.**
  $\chi + ie^{-\phi} \equiv -iz$.
  $\mathcal{K} = -\log \mathrm{Re}\,(z)\,, \mathcal{G}_{z\bar{z}} = (2\mathrm{Re}\,(z))^{-2}$. $\mathrm{Re}\,(z) > 0$.

- EXAMPLE: $n_c = 2$. AT FIXED CHARGES:

  Params:5 (M_1, M_2, r, z_∞). EQS$=3$: (M_{ADM}, z_∞).
  $\longrightarrow$ dof$=2$

  Minimize $M_{ADM}(m_i)$: Eqs.$+1$, dof$=1$
  $\longrightarrow$ Choose: $r_{12}, \mathrm{Re}\,(z)\,, \mathrm{Im}\,(z)\,....$

AXION-DILATON, 2 center.



FIX: $q_1 = (1, 8, 0, -1)q_0$,$q_2 = (1, 8, -4, 1)q_0$,
$J^2 > 0$. $A_\infty > 0, -\det(S) > 0$. $m_i^{min} = (9/16, 7/16)$.

EXAMPLE(3): Stringy models. STU Models

Stringy models: cubic $\mathbf{F = d_{ABC} X^A X^B X^C / X^0}$,
Simplest, STU model: $\mathbf{F = -\frac{X^1 X^2 X^3}{X^0}}$.
$\longrightarrow$ 3 complex scalars: $dimW = 2n_v + 2 = 8$. $S(F)$

- **CASE 2 centers. Basis $\mathbf{W = (P_{\pm} q_{1,2}, P_{\pm} s_{1,2})}$**
  ♯Parms $= 13$: $M_1, M_2, r_{12} : 3, \lambda^i : 4, z^{\alpha}_{\infty} : 6$.
  ♯Eqs(7): Mass:1,z : 6. dof=6. ($\longrightarrow$ 3 z^{α}_{∞}).
  Minimal Mass(1): dof=5.

- **CASE 4 centers. Basis $\mathbf{W = (P_{\pm} q_{1,2,3,4})}$**
  ♯Parms $= 16$: $M_i, r_{ij} : 10, z^{\alpha}_{\infty} : 6$.
  ♯Eqs(7): Mass:1,z : 6. dof=9 ($\longrightarrow$ ...+ 3 z^{α}_{∞}).
  Minimal Mass(3): dof=6.

Conclusions

- Multicenter BHs in $N2$ $D4$ SUGRA $+n_v$ vector multip.

$\longrightarrow$ new BPS mass formulas

–Properties, Construction and existence of BH multicenter solutions

–Counting of microstates (in stringy inspired models)

- EXAMPLE: minimal M_{ADM}, 2 center BPS solutions

$$\mathbf{| Z |^2 = M^2 \;\; = \;\; A_\infty \left(1 + \alpha J^2 \left(1 + \frac{2M}{r} + \frac{A_\infty}{r^2}\right)\right)}$$

- Continuos family, for q_i, $r \in (0, \infty)$,$M_{ADM} \in (\infty, M_\infty)$ and z_∞
- Minimal mass (Mass Gap)
- Range of values of m_i allowed. etc..

- N2 d4 SUGRA: specially suitable: many restrictions. $\mathcal{M}_z =$ Sp. Kahler.

BUT SIMPLE ESSENTIAL INGREDIENTS: BPS condition, asymptotical flatness... $\longrightarrow$ generalizable other contexts.

Special Kahler geometry =A Kahler manifold plus...

- A projective embedding, define $X^I/z^\alpha = X^I/X^0$.
- A $(2n_v + 2\ )$ vector complex space $\mathbf{W}$. $\langle \mathbf{X} \mid \mathbf{Y} \rangle$
- An (almost) complex structure on W def by $\boldsymbol{\mathcal{S}}(\mathbf{X})$,compatible with the SYMP. prod.:
  $\mathcal{S}^{-1} = -1,\ \mathrm{P}_\pm := \frac{1}{2}(\mathbb{1} \pm i\mathcal{S}) \Rightarrow \quad W = W^+ \oplus W^-$
  DEF: $\mathbf{g}(\mathbf{X}, \mathbf{Y}) \equiv \langle \mathcal{S}\mathbf{X} \mid \mathbf{Y} \rangle = \left\langle \mathbf{Y} \,\middle|\, \mathcal{S}^\dagger \mathbf{X} \right\rangle$
- DEF: $\mathbf{D}_\alpha = \partial_\alpha + \mathbf{p}\mathbf{Q}_\alpha,\ \mathbf{Q}_\alpha = \partial_\alpha \mathcal{K}$
- W sympl. sections $\mathbf{V} = (\mathbf{V}^\mathbf{I}, \mathbf{V}_\mathbf{I}), \boldsymbol{\Omega} \equiv (\mathbf{X}^\mathbf{I}, \mathbf{F}_\mathbf{I})$.
  $\mathbf{D}_{\bar\alpha}\mathbf{V} = \mathbf{0}.\ \partial_\alpha \boldsymbol{\Omega} = \mathbf{0}.\ \left\langle \mathbf{V} \,\middle|\, \bar{\mathbf{V}} \right\rangle = -\mathbf{i}.\ \boldsymbol{\Omega} = \mathbf{e}^{\mathcal{K}/2}\mathbf{V}.$
- Then

$$\mathcal{G}_{\alpha\bar\beta} = \partial_\alpha \partial_{\bar\beta} \mathcal{K} = \left\langle \mathbf{D}_\alpha \mathbf{V} \,\middle|\, \bar{\mathbf{D}}_{\bar\beta} \bar{\mathbf{V}} \right\rangle,$$

$$\mathbf{N}_{\mathbf{IJ}} = \mathbf{a}_{\mathbf{IJ}} + \mathbf{i}\mathbf{b}_{\mathbf{IJ}}\mathbf{i} = \mathbf{F}_{\mathbf{IJ}} + \ldots$$

Where $\mathbf{F}_{\mathbf{IJ}} = \partial_\mathbf{I}\partial_\mathbf{J}\mathbf{F}$, $\mathbf{F}_\mathbf{I} = \partial_\mathbf{I}\mathbf{F}$. $\mathbf{S}(\mathbf{F}) = \mathbf{S}(\mathbf{F}_{\mathbf{IJ}})$

$\mathcal{S}$ and Freudenthal transformations

- $\mathcal{S}$-transformations

$$X \quad \longrightarrow \quad X' = f(\mathcal{S})X, \qquad\qquad f(\mathcal{S}) = a + b\mathcal{S} \equiv \lambda e^{\theta \mathcal{S}}$$

  - $\langle X' \mid Y' \rangle = \lambda^2 \langle X \mid Y \rangle$

- Freudenthal dual

Borsten,Dahanayake,Duff,Ruben'09
Ferrara,Marrani,Yeranyan'11

$$X \quad \longrightarrow \quad \tilde{X} = T(X)|\Delta_4(X)|^{-1/2}, \qquad\qquad \tilde{\tilde{X}} = -X$$

such that $\Delta_4(\tilde{X}) = \Delta_4(X)$, where $S_{4,bh} = \pi\sqrt{|\Delta_4(x)|}$

- Matching Freudenthal/$\mathcal{S}$ transformations

$$\tilde{X} \equiv \exp\left(\tfrac{\pi}{2}\mathcal{S}\right) X$$

$\mathcal{S}$-transformations

- Scalings

$$X \longrightarrow \lambda e^{\theta \mathcal{S}} X \quad \Rightarrow \quad \left\{ \begin{array}{rcl} S & \to & \lambda^2 S \\ M_{ADM} & \to & \lambda M_{ADM} \\ r_{ab} & \to & \lambda r_{ab} \end{array} \right.$$

- Quartic invariant

$$\Delta_4(X) = \Delta_4(\tilde{X}) = \frac{1}{4} \langle X \mid X \rangle^2$$

- Generalized Freudenthal transformation

$$\Delta_4(aX + b\tilde{X}) = (a^2 + b^2)^2 \Delta_4(X)$$

Freudenthal triple system

- Definition. A Freudenthal Triple System (FTS) is axiomatically defined as a finite dimensional vector space $\mathfrak{F}$ over a field $\mathbb{F}$, such that:

  1. $\mathfrak{F}$ possesses an antisymmetric bilinear form $\{x, y\}$
  2. $\mathfrak{F}$ possesses a symmetric four-linear form $\Delta(x, y, z, w)$
  3. $\mathfrak{F}$ possesses a ternary product $T(x, y, z)$ defined by $\{T(x, y, z), w\} = \Delta(x, y, z, w)$. Then $3\{T(x, x, y), T(y, y, y)\} = \{x, y\}\Delta\{x, y, y, y\}$

- Freudenthal dual: $\tilde{X} = T(X)|\Delta(X)|^{-1/2}$, $\tilde{\tilde{X}} = -X$

- $\Delta_4(X)$: example of Δ, where $\Delta_4(X) \equiv \Delta(X, X, X, X)$

Jordan algebras

- Definition. A Jordan algebra $\mathfrak{J}$ is a vector space over a ground field $\mathbb{F}$ equipped with a bilinear product such that $\forall X, Y \in \mathfrak{J}$,

$$X \bullet Y = Y \bullet X$$
$$X \bullet (X \bullet Y) = X \bullet (X^2 \bullet Y)$$

- Integral cubic Jordan algebra, cubic form $N : \mathfrak{J} \longrightarrow \mathbb{F}$

$$N(X, Y, Z) := \tfrac{1}{6}[N(X+Y+Z) - N(X+Y) - N(X+Z) - N(Y+Z) + N(X) + N(Y) + N(Z)]$$

- Define a quadratic map $^\sharp : \mathfrak{J} \longrightarrow \mathfrak{J}, \quad Tr(X^\sharp, Y) = 3N(X, X, Y)$

- Jordan dual: $X^* = X^\sharp N(X)^{-1/3}, \qquad X^{**} = X$

- $S_{5,\text{black string}} = 2\pi\sqrt{N(X)} \qquad S_{5,\text{black hole}} = 2\pi\sqrt{N(Y)}$

4D/5D uplift

- Given $(\mathfrak{J}, N)$, one can construct an integral FTS,

$$\mathfrak{F}(\mathfrak{J}) = \mathbb{F} \oplus \mathbb{F} \oplus \mathfrak{J} \oplus \mathfrak{J}$$

- Duality scheme

Borsten,Dahanayake,Duff,Ruben'09
Gaiotto,Strominger,Yin'05

$$\begin{array}{ccc}
\text{4D black hole } x & \xrightarrow{\text{4D/5D lift}} & \text{5D black string } A \sim \tilde{B}^{\star} \\
\text{Freudenthal dual} \downarrow & & \downarrow \text{Jordan dual} \\
\text{dual 4D black hole } \tilde{x} & \xrightarrow[\text{4D/5D lift}]{} & \text{dual 5D black hole } \tilde{B} \sim A^{\star}
\end{array}$$

4D/5D uplift

- Given $(\mathfrak{J}, N)$, one can construct an integral FTS,

$$\mathfrak{F}(\mathfrak{J}) = \mathbb{F} \oplus \mathbb{F} \oplus \mathfrak{J} \oplus \mathfrak{J}$$

- Duality scheme

Borsten,Dahanayake,Duff,Ruben'09

Gaiotto,Strominger,Yin'05

$$\begin{array}{ccc}
\text{4D black hole } x & \xrightarrow{\text{4D/5D lift}} & \text{5D black string } A \sim \tilde{B}^{\star} \\
\text{Generalised} \Big\downarrow \text{Freudenthal dual} & & \text{Generalised} \Big\downarrow \text{Jordan dual} \\
\text{dual 4D black hole } ax + b\tilde{x} & \xrightarrow[\text{4D/5D lift}]{} & \text{dual 5D black hole } \alpha\tilde{B} + \beta A^{\star}
\end{array}$$