Scaling Law of Coherent Synchrotron Radiation in a Rectangular Chamber

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CSR Impedance of Parallel-Plates

An impedance with scaling property is given by

$$\left(\frac{\rho}{h}\right)\left(\frac{Z(n)}{n}\right)_{para} = (2\pi^2)(\frac{4\pi}{c})[\frac{\hat{k}}{2}]^{-4/3}\sum_{p(odd)\geq 1}^{kh/\pi}[Ai'(u)(Ai'(u)-iBi'(u))+uAi(u)(Ai(u)-iBi(u))]$$

where h is the distance between two plates, n=kρ, Ai and Bi are Airy functions, and their argument u is defined as

$$u = \frac{\pi^2 p^2}{2^{2/3}}\hat{k}^{-4/3}$$

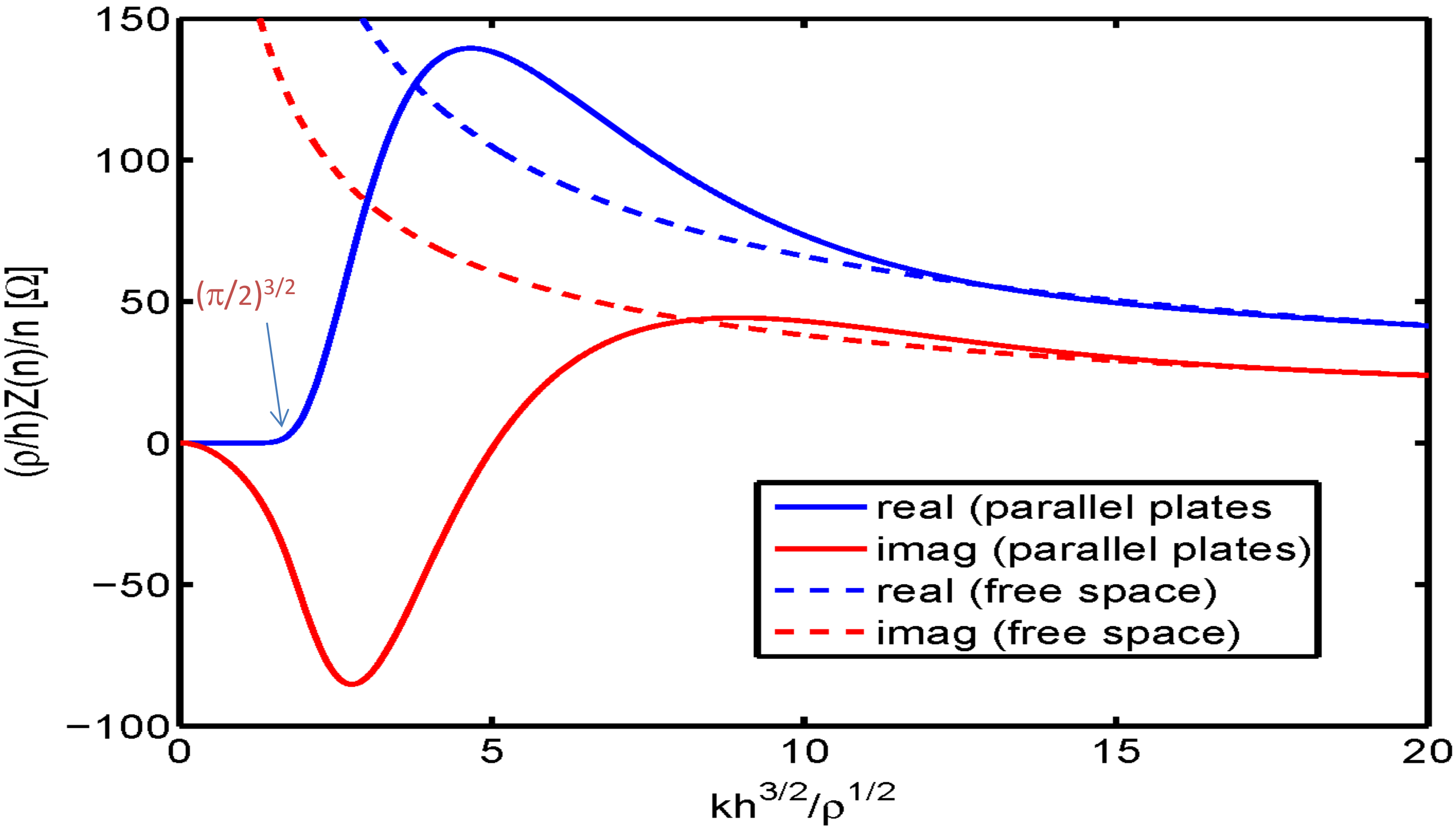
Dependence of n is all through

$$\hat{k} = n\frac{h^{3/2}}{\rho^{3/2}}$$

In fact, this scaling property holds for the CSR impedance in free space, formally

$$\left(\frac{\rho}{h}\right)\left(\frac{Z(n)}{n}\right)_{free} = (\frac{4\pi}{c})(\frac{\Gamma(2/3)}{3^{1/3}}(\frac{\sqrt{3}}{2}+\frac{i}{2})[n(\frac{h}{\rho})^{3/2}]^{-2/3}$$

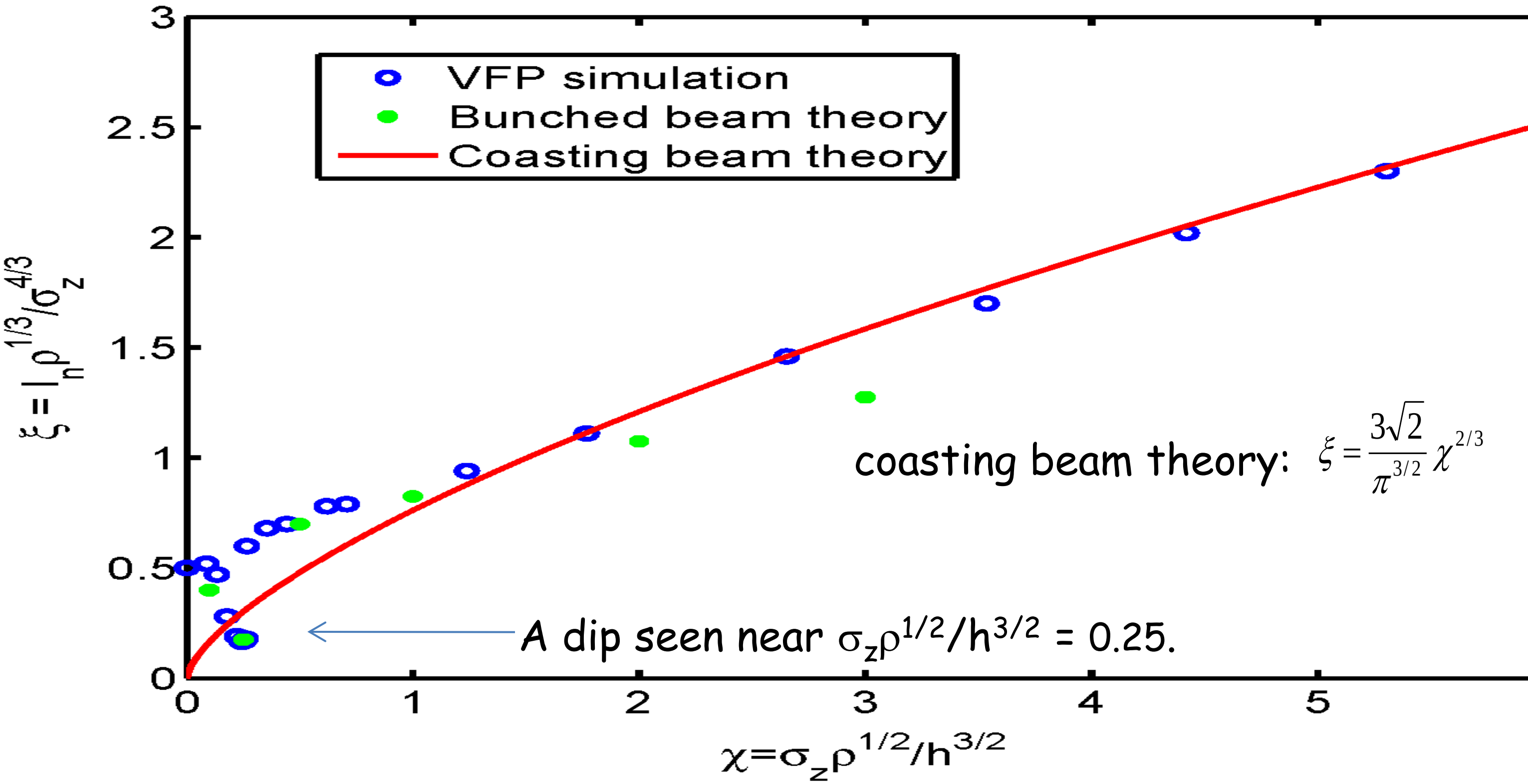
Scaling and Asymptotic Properties



The scale defines the strength of impedance and the location of the peak defines where the shielding effects start.

Threshold of Instability

for CSR of Parallel Metal Plates



Threshold ξ^{th} becomes a function of the shielding parameter $\chi = \sigma_z\rho^{1/2}/h^{3/2}$. Simulation was carried out by Bane, Cai, and Stupakov, PRSTAB **13**, 104402 (2010). For a long bunch, the coasting beam theory agrees well with the VFP simulation.

Summary of the Comparisons

Machine	σ_z [mm]	Radius ρ [m]	Height h [cm]	χ	ξ^{th} (theory)	ξ^{th} (meas.)
BESSY II	2.6	4.23	5.0	0.48	0.67	0.89
MLS	2.6	1.53	5.0	0.29	0.60	0.39
ANKA	1.0	5.56	3.2	0.42	0.64	0.50
SSRL	1.0	8.14	3.4	0.46	0.66	?
Diamond	0.7	7.13	3.8	0.25	0.17 ?	0.33

We have used

$$\xi^{th}(\chi) = 0.5 + 0.34\chi$$

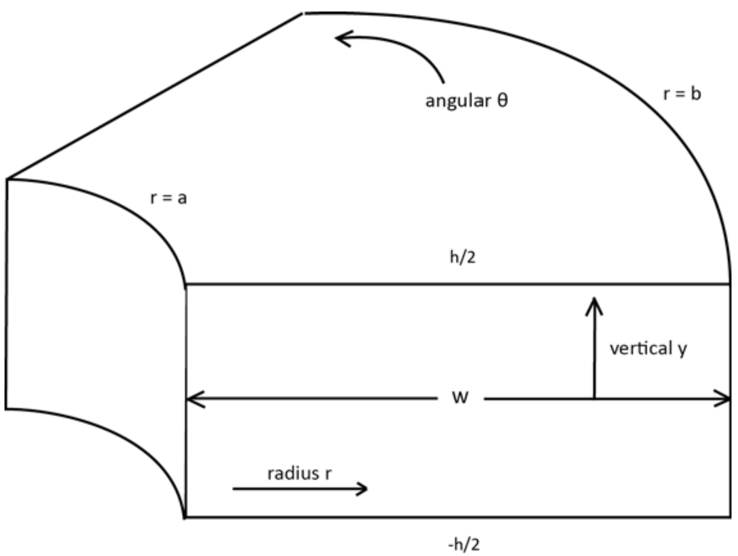
where $\chi = \sigma_z \rho^{1/2}/h^{3/2}$ is the shielding parameter. This simple relation was first obtained by fitting to the result of simulations (Bane, Cai, and Stupakov, PRSTAB **13**, 104402 (2010)).

Since the MLS's shielding parameter is very close to the dip. That may be a reason of its lower threshold.

Statements of Problem and Solution

- *Solve the Maxwell equation with a circulating charge inside a perfect conducting chamber*
- *Express the longitudinal impedance in terms of Bessel functions*
- *Approximate Bessel functions with Airy functions under the uniform asymptotic expansion*

A section of vacuum chamber



For detail read: SLAC-PUB-15875, January 2014

CSR Impedance of Rectangular Chamber

An impedance with scaling property is given by

$$\left(\frac{\rho}{h}\right)\left(\frac{Z(n)}{n}\right)_{rect} = (-i2\pi^2)(\frac{4\pi}{c})[\frac{\hat{k}}{2}]^{-4/3}\sum_{p(odd)\geq 1}^{kh/\pi}\frac{\hat{s}(u,u^+)\hat{s}(u,u^-)}{\hat{s}(u^+,u^-)} + u\frac{\hat{p}(u,u^+)\hat{p}(u,u^-)}{\hat{p}(u^+,u^-)}]$$

where w is the width of chamber and h the height, n=kρ, p and s hats are products of Airy functions and their derivatives. Similar to the parallel plates, one of arguments u is

$$u = \frac{\pi^2 p^2}{2^{2/3}}\hat{k}^{-4/3}$$

Dependence of n is all through

$$\hat{k} = n\frac{h^{3/2}}{\rho^{3/2}}$$

In addition, for aspect ratio of A = w/h, the other two arguments are

$$u^{\pm} = \frac{1}{2^{2/3}}(\pi^2 p^2 \hat{k}^{-4/3} \pm A\hat{k}^{2/3})$$

The Cross Products

of the Bessel functions

$$p_n(x,y) = J_n(x)Y_n(y) - Y_n(x)J_n(y),$$
$$s_n(x,y) = J_n'(x)Y_n'(y) - Y_n'(x)J_n'(y)$$

of the Airy functions

$$\hat{p}(x,y) = Ai(x)Bi(y) - Bi(x)Ai(y),$$
$$\hat{s}(x,y) = Ai'(x)Bi'(y) - Bi'(x)Ai'(y)$$

The uniform asymptotic expansion:

$$J_n(n\sqrt{1-x^2}) \approx (\frac{2}{n})^{1/3} Ai((\frac{2}{n})^{2/3} x^2),$$
$$Y_n(n\sqrt{1-x^2}) \approx -(\frac{2}{n})^{1/3} Bi((\frac{2}{n})^{2/3} x^2),$$
$$J_n'(n\sqrt{1-x^2}) \approx -(\frac{2}{n})^{2/3} Ai'((\frac{2}{n})^{2/3} x^2),$$
$$Y_n'(n\sqrt{1-x^2}) \approx (\frac{2}{n})^{2/3} Bi'((\frac{2}{n})^{2/3} x^2)$$

Scaling Property of Point-Charge Wakefield

The integrated wake can be written as

$$W[\hat{z}] = \frac{\rho}{h^2}\sum_j \hat{k}_j \frac{R_s}{Q}(\hat{k}_j)[\theta(-\hat{z})\cos(\hat{k}_j\hat{z}) - \frac{\text{sgn}(\hat{z})}{2}\exp(-\hat{k}_j|\hat{z}|)]$$

where ρ is bending radius and h height of chamber. We have defined a dimensionless longitudinal position:

$$\hat{z} = z\rho^{1/2}/h^{3/2}$$

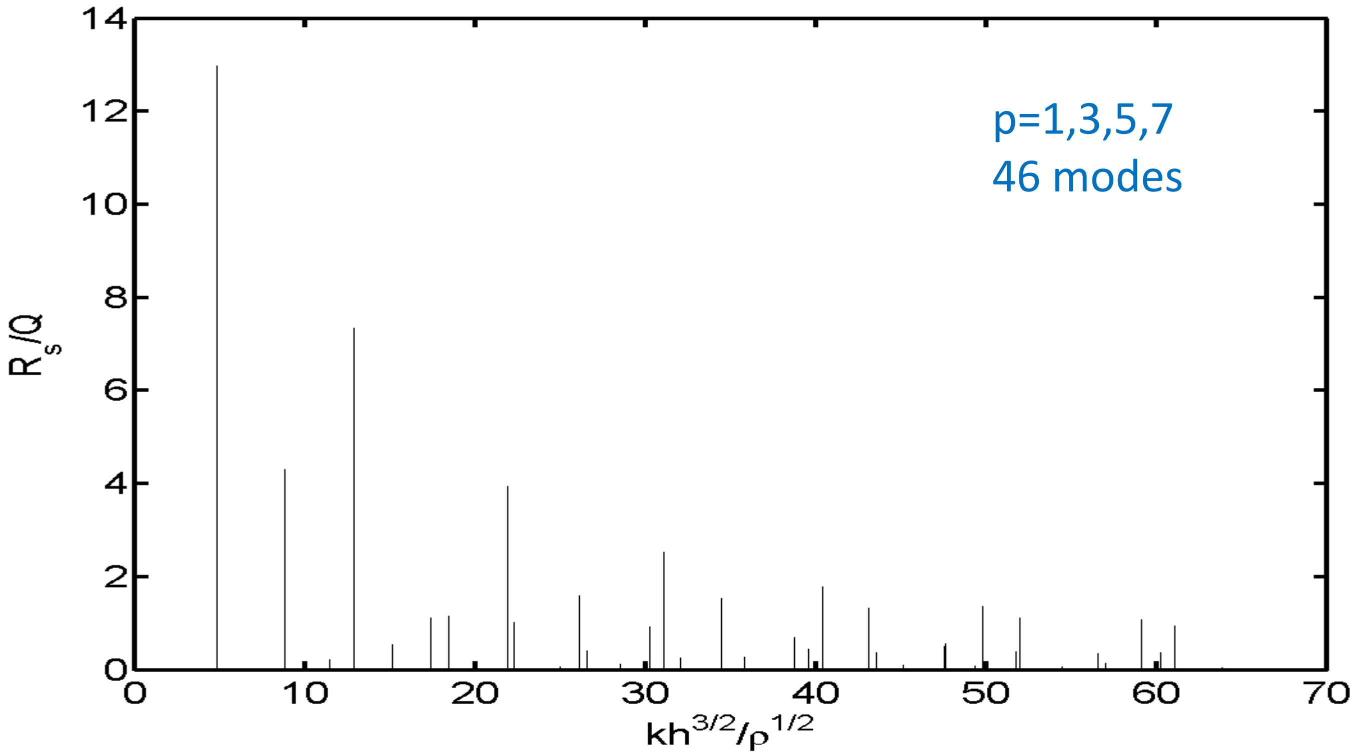
which naturally leads to the shielding parameter

$$\chi = \sigma_z\rho^{1/2}/h^{3/2}$$

if we introduce the normalized coordinate $q=z/\sigma_z$.

- An LRC resonance with infinite quality factor
- Decay term is due to the curvature

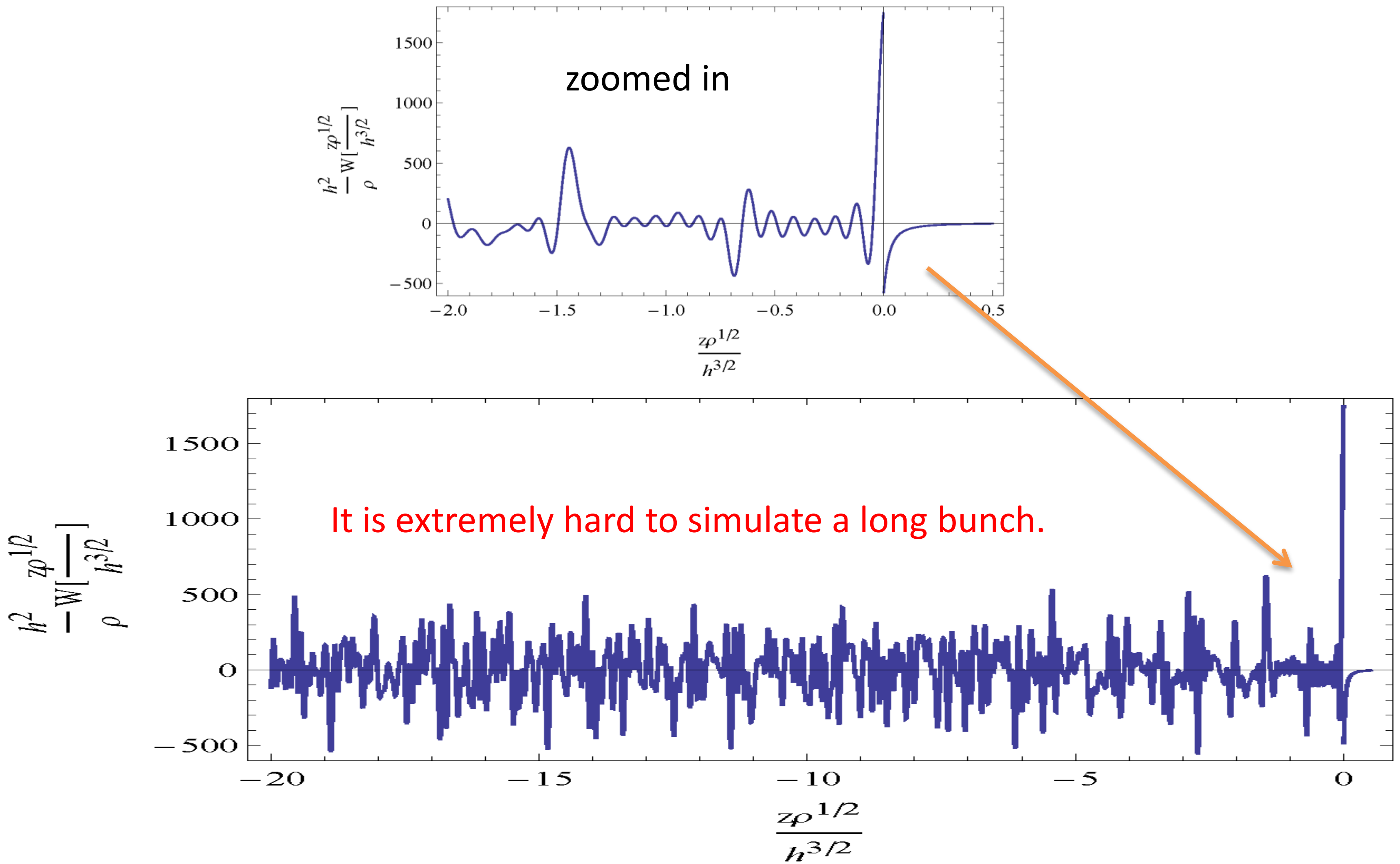
Dimensionless R_s/Q (A=1)



It can be computed by

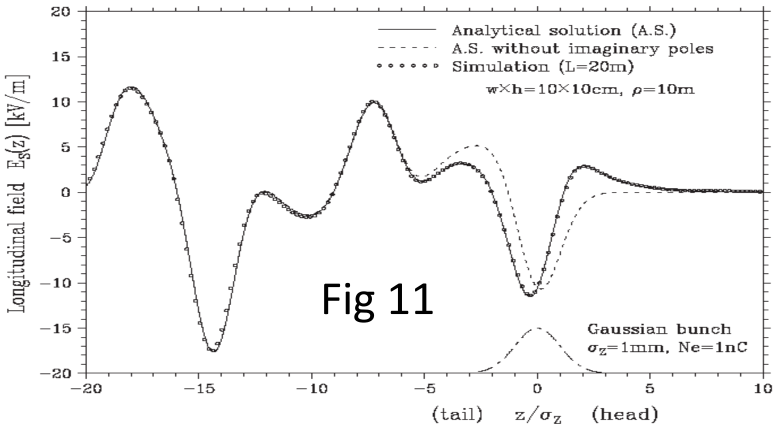
$$-32\pi^3 2^{1/3}[\hat{k}_j]^{-4/3}\frac{\hat{s}(u_j,u_j^+)\hat{s}(u_j,u_j^-)}{d\hat{s}(u_j^+,u_j^-)/d\hat{k}} \quad \text{or} \quad -32\pi^3 2^{1/3}[\hat{k}_j]^{-4/3}u_j\frac{\hat{p}(u_j,u_j^+)\hat{p}(u_j,u_j^-)}{d\hat{p}(u_j^+,u_j^-)/d\hat{k}}$$

Point-Charge Wakefield (A=1)

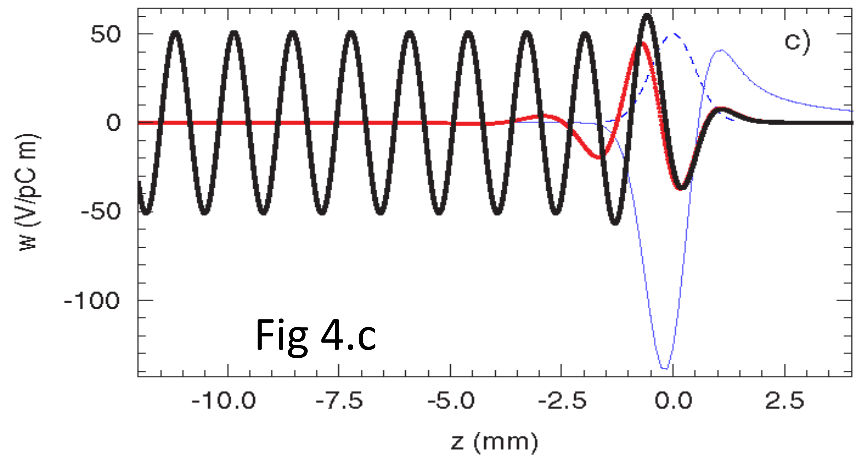


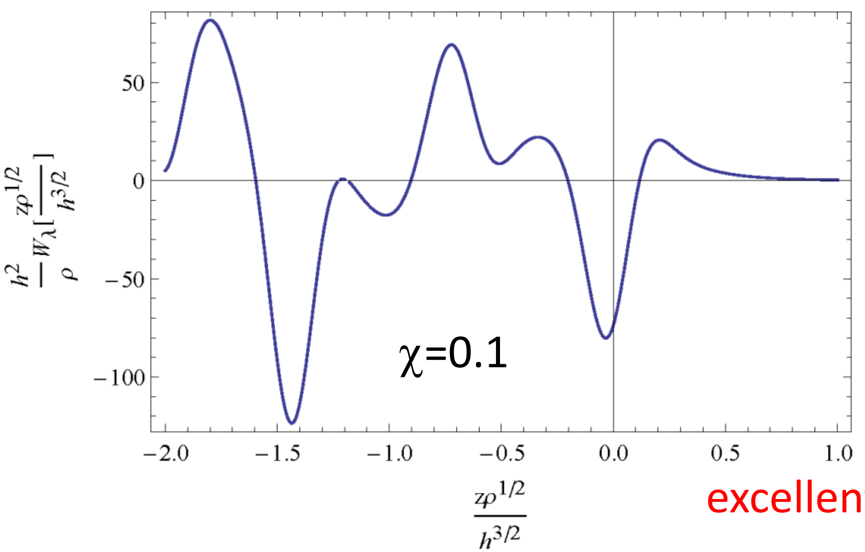
Comparisons of Bunch Wake

Agoh, PRSTAB, **12**, 094402 (2009)

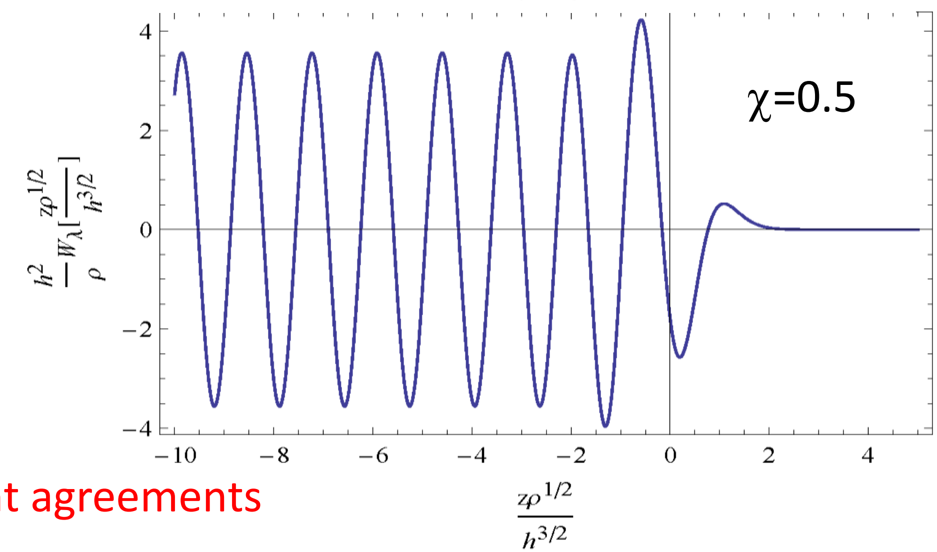


Stupakov & Kotelnikov, PRSTAB, **12** 104401 (2009)





excellent agreements



Longitudinal Beam Dynamics

Hamiltonian is given as

$$H = \frac{1}{2}(q^2 + p^2) - I_n \int_{-\infty}^{q} dq'' \int_{-\infty}^{\infty} dq' \lambda(q') W(q'' - q')$$

where I_n is the normalized current introduced by Oide and Yokoya (1990)

$$I_n = \frac{r_e N_b}{2\pi \nu_s \gamma \sigma_\delta}$$

$q=z/\sigma_z$, $p=-\delta/\sigma_\delta$ and W(q) is the integrated wake per turn (convention used in Alex Chao's book). The independent variable is $\theta=\omega_s t$.

Vlasov-Fokker-Planck equation is written as

$$\frac{\partial \Psi}{\partial \theta} - \{H, \Psi\} = 2\beta \frac{\partial}{\partial p}(p\Psi + \frac{\partial \Psi}{\partial p})$$

where $\Psi(q,p;\theta)$ is the beam density in the phase space and $\beta = 1/\omega_s \tau_d$.
A robust numerical solver was developed by Warnock and Ellison (2000).

Scaling Law of the Threshold

In the normalized coordinate, it can be shown easily

$$I_n W[q] = \xi \chi^{4/3} \sum_j \hat{k}_j \frac{R_s}{Q}(\hat{k}_j)[\theta(-q)\cos(\hat{k}_j \chi q) - \frac{\operatorname{sgn}(q)}{2}\exp(-\hat{k}_j \chi |q|)]$$

where $\xi = I_n \rho^{1/3} / \sigma_z^{4/3}$ is the dimensionless current and $\chi = \sigma_z \rho^{1/2} / h^{3/2}$ the shielding parameter. We redefine a dimensionless wake as

$$W_\xi[q] = \chi^{4/3} \sum_j \hat{k}_j \frac{R_s}{Q}(\hat{k}_j)[\theta(-q)\cos(\hat{k}_j \chi q) - \frac{\operatorname{sgn}(q)}{2}\exp(-\hat{k}_j \chi |q|)]$$

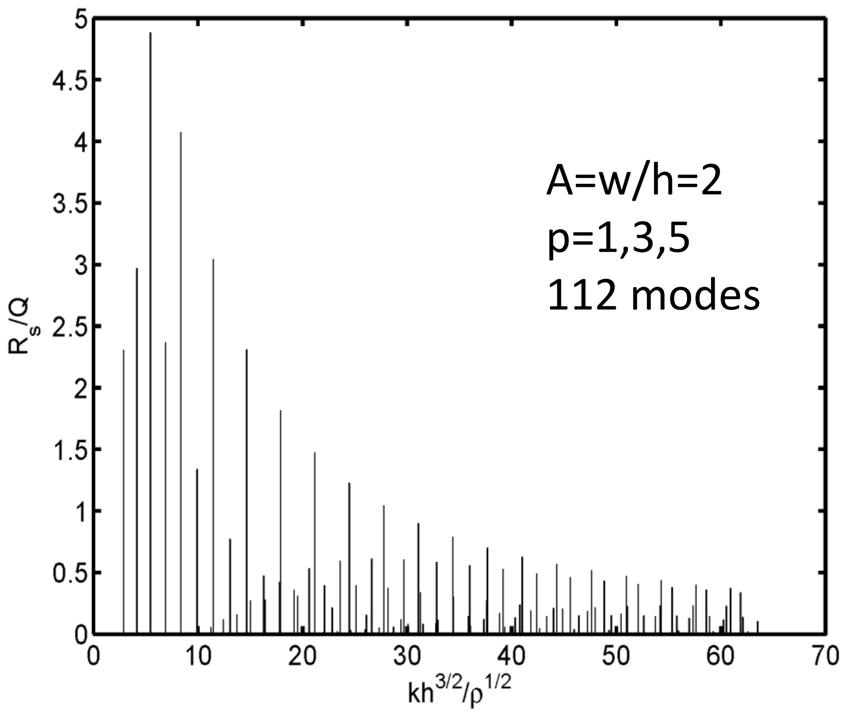
Clearly, this wake depends on χ and aspect ratio A through the values of $\hat{k}_j$

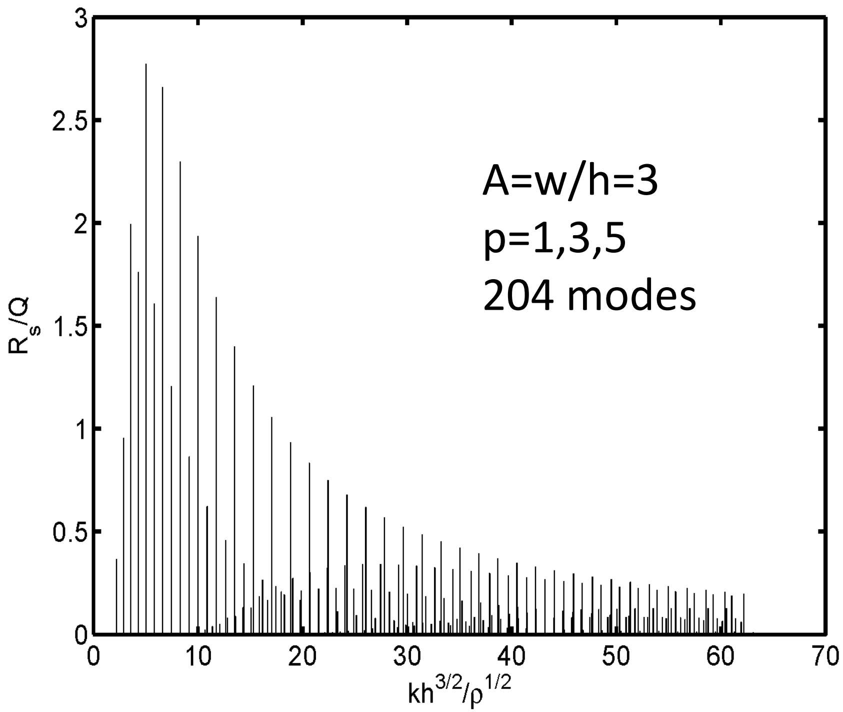
Based on the VFP equation. we conclude that the dimensionless current

$$\xi = \xi^{th}(\chi, A, \beta)$$

is a function of other three dimensionless parameters.

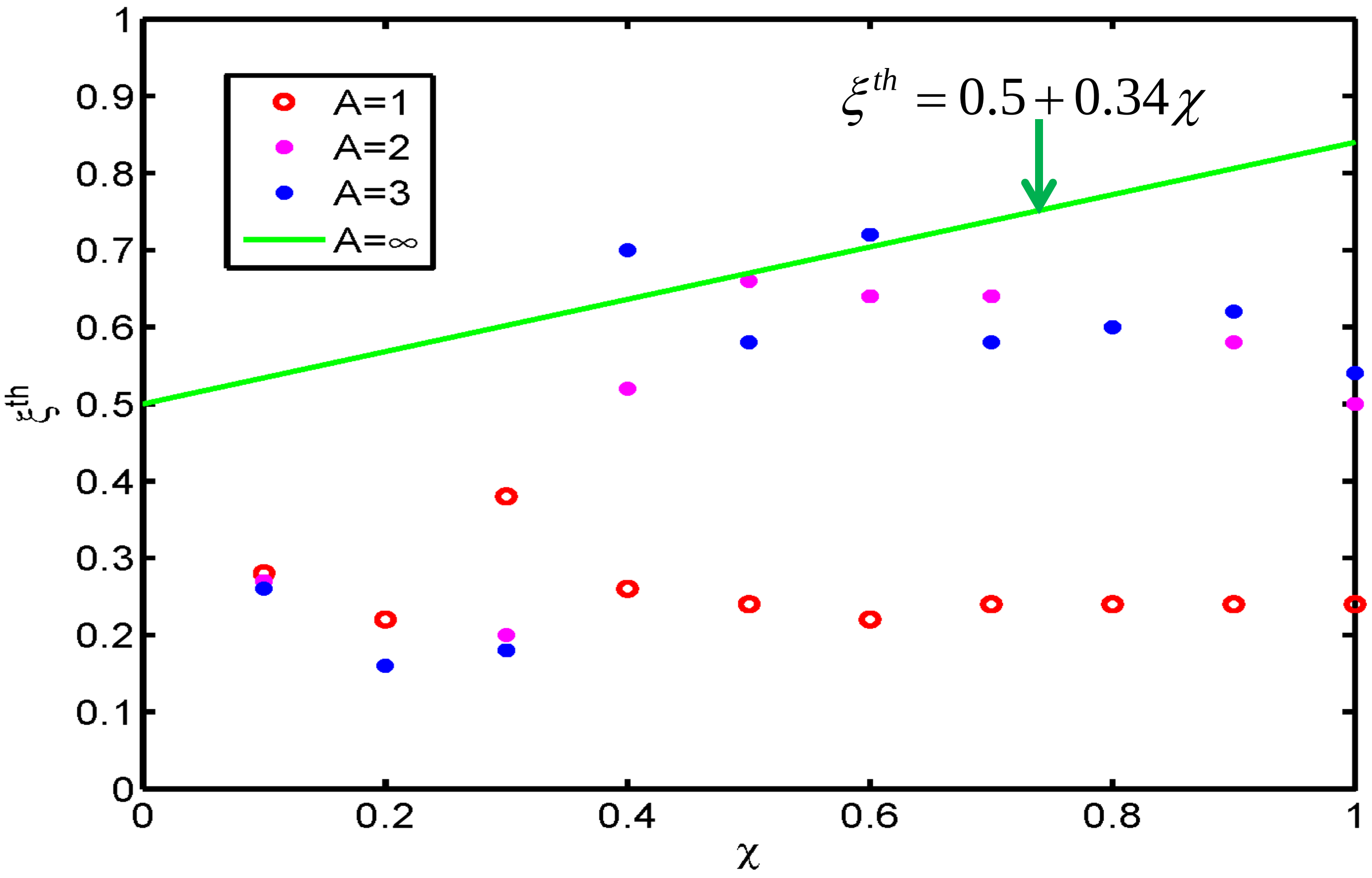
R_s/Q for Rectangular Chamber





Envelopes are very similar, also similar to the real part of Z(n)/n of the parallel model shown previously.

VFP Simulation



A square chamber has a much lower threshold: ξ^{th}=1/4.

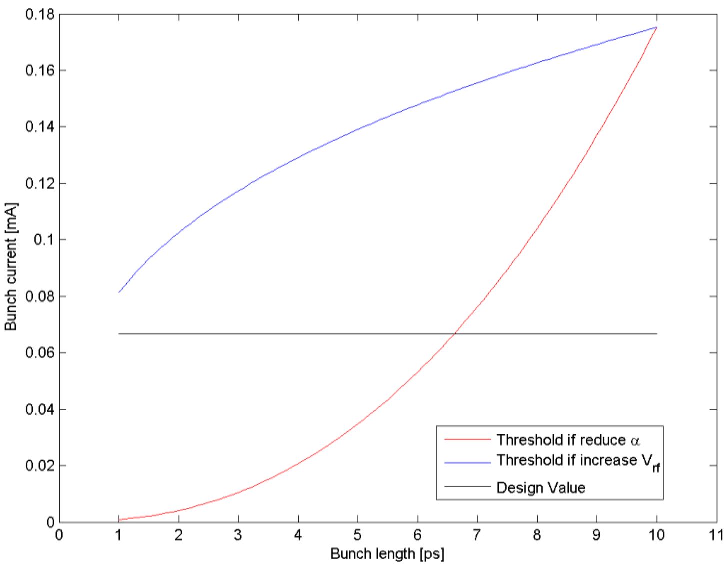
Mitigation of CSR Effects

Rewrite the threshold in terms of some practical parameters. The beam becomes unstable if

$$I_b > \frac{8\pi^2 \xi^{th} \sigma_z^{7/3} (V_{rf} f_{rf}) f_{rev} \cos\varphi_s}{c^2 Z_0 \rho^{1/3}}$$

- Extremely unfavorable scaling in terms of shortening bunch
- Stronger longitudinal focusing is very helpful
- Superconducting RF at higher frequency

Shorten the bunch to 1 ps in PEP-X



Conclusion

- The scaling law found in the parallel-plate mode is extended to the rectangular chamber by adding another parameter: A (aspect ratio of the chamber)
- The threshold of a square chamber is lower by a factor than the one of a rectangular chamber with A > 2
- More effective mean to shorten a bunch to a ps scale is to use superconducting RF at higher frequency

Important and Relevant References

- R. Warnock and P. Morton, "Fields Excited by a Beam in a Smooth Toroidal Chamber," SLAC-PUB-5462 March (1988); Also Pat. Accel.
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- K. Bane, Y. Cai, and G. Stupakov, "Threshold studies of microwave instability in electron storage rings," Phys. Rev. ST Accel. Beams **13**, 104402 (2010)

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Transverse Impedance

An impedance with scaling property is given by

$$Z^{\perp}(k) = \left(\frac{-i8\pi^2}{h}\right)\left(\frac{4\pi}{c}\right)\sum_{p(even)\geq 2}^{kh/\pi} u\left[\frac{\hat{s}(u,u^+)\hat{s}(u,u^-)}{\hat{s}(u^+,u^-)} + u\,\frac{\hat{p}(u,u^+)\hat{p}(u,u^-)}{\hat{p}(u^+,u^-)}\right]$$

where w is the width of chamber and h the height, n=kρ, p and s hats are products of Airy functions and their derivatives. Similar to the parallel plates, one of arguments u is

$$u = \frac{\pi^2 p^2}{2^{2/3}}\hat{k}^{-4/3}$$

Dependence of n is all through

$$\hat{k} = n\frac{h^{3/2}}{\rho^{3/2}}$$

In addition, for aspect ratio of A = w/h, the other two arguments are

$$u^{\pm} = \frac{1}{2^{2/3}}(\pi^2 p^2 \hat{k}^{-4/3} \pm A\hat{k}^{2/3})$$

Transverse Wakefield

The integrated wake can be written as

$$W_1[\hat{z}] = \frac{\rho^{1/2}}{h^{5/2}} \sum_j \frac{cR_s}{Q}(\hat{k}_j)[\theta(-\hat{z})\sin(\hat{k}_j \hat{z}) + \frac{1}{2}\exp(-\hat{k}_j |\hat{z}|)]$$

where ρ is bending radius and h height of chamber. We have defined a dimensionless longitudinal position:

$$\hat{z} = z\rho^{1/2} / h^{3/2}$$

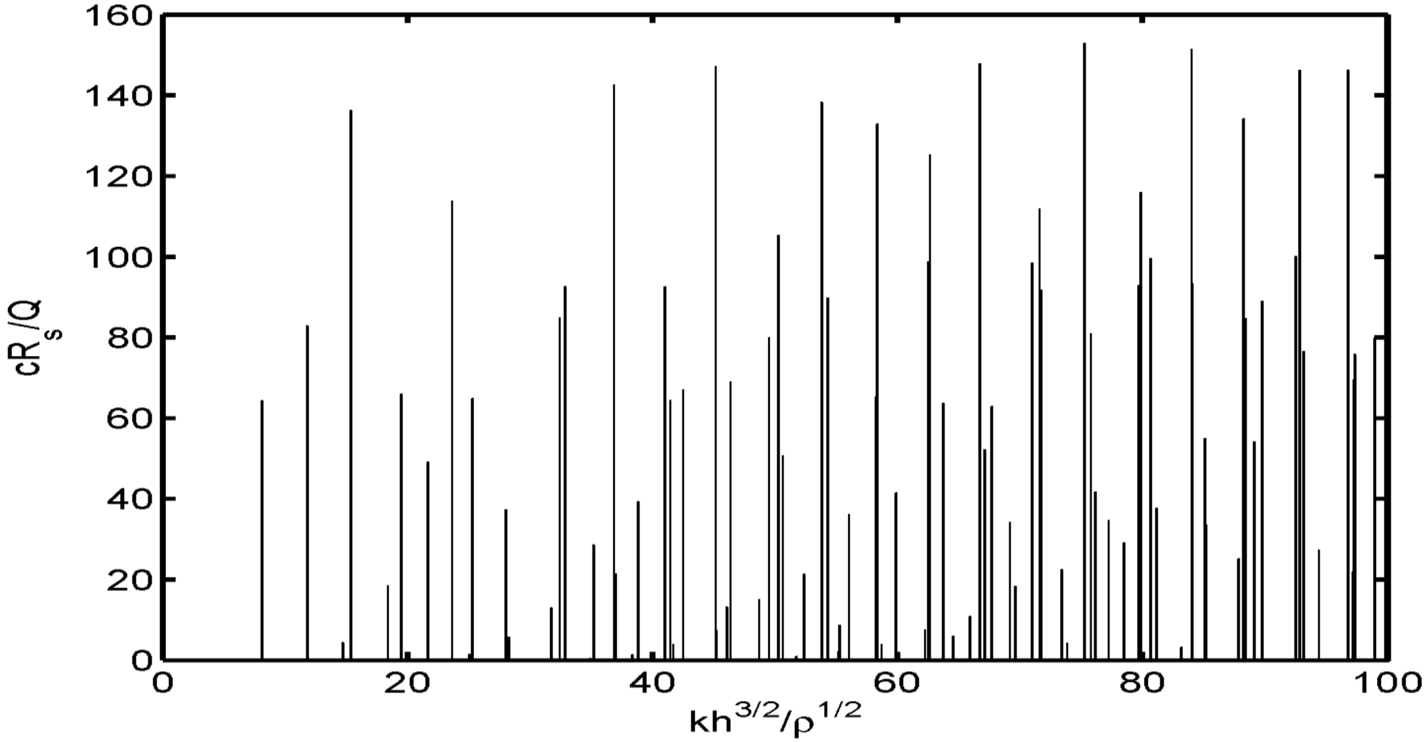
which naturally leads to the shielding parameter

$$\chi = \sigma_z \rho^{1/2} / h^{3/2}$$

if we introduce the normalized coordinate $q=z/\sigma_z$.

- An LRC resonance with infinite quality factor
- Decay term is due to the curvature

Dimensionless cR_s/Q (A=1)



It can be computed by

$$-64\pi^3 u_j \frac{\hat{s}(u_j,u_j^+)\hat{s}(u_j,u_j^-)}{d\hat{s}(u_j^+,u_j^-)/d\hat{k}} \quad \text{or} \quad -64\pi^3 u_j^2 \frac{\hat{p}(u_j,u_j^+)\hat{p}(u_j,u_j^-)}{d\hat{p}(u_j^+,u_j^-)/d\hat{k}}$$

Transverse Instability

In the normalized coordinates, Hamiltonian is given by

$$H = \frac{1}{2}(\frac{k_\beta}{k_s})(\hat{y}^2 + \hat{p}_y^2) + \frac{1}{2}(q^2 + p^2) - \frac{\hat{y}\hat{F}_y(q,\theta)}{E_0 k_\beta k_s},$$

where the force is written as

$$\frac{\hat{F}_y}{E_0} = -\frac{r_e N}{\gamma}\int_{-\infty}^{\infty} dq'\lambda(q')\hat{D}(q')(\frac{W_1(q-q')}{2\pi\rho})$$

If we define a dimensionless current

$$I = (\frac{r_e N}{\gamma})(\frac{1}{h^{5/2}\rho^{1/2}})(\frac{1}{k_\beta k_s})$$

we derive a scaling law

$$I = I^{th}(\chi, A, \frac{k_\beta}{k_s})$$