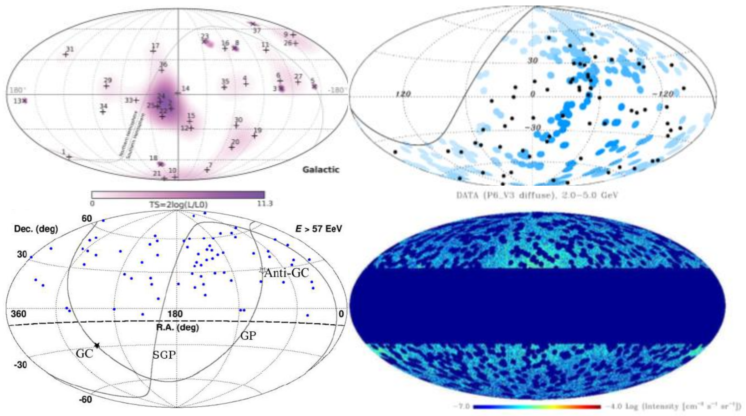

Model Independent Measurements of Angular Power Spectra

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CCAPP The Ohio State University's Center for Cosmology and AstroParticle Physics

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Observing "Points" in the Sky

- High-Energy Radiation Events
  - Gamma-Rays
  - Cosmic Ray Shower Events
  - Cosmic Neutrinos

Inference radiation sources, cosmic ray acceleration, ray propagation, etc.

- Celestial Objects
  - Galaxies
  - AGN
  - X-ray Clusters
  - …

Inference cosmic expansion history, large scale structure, galaxy formation, etc.

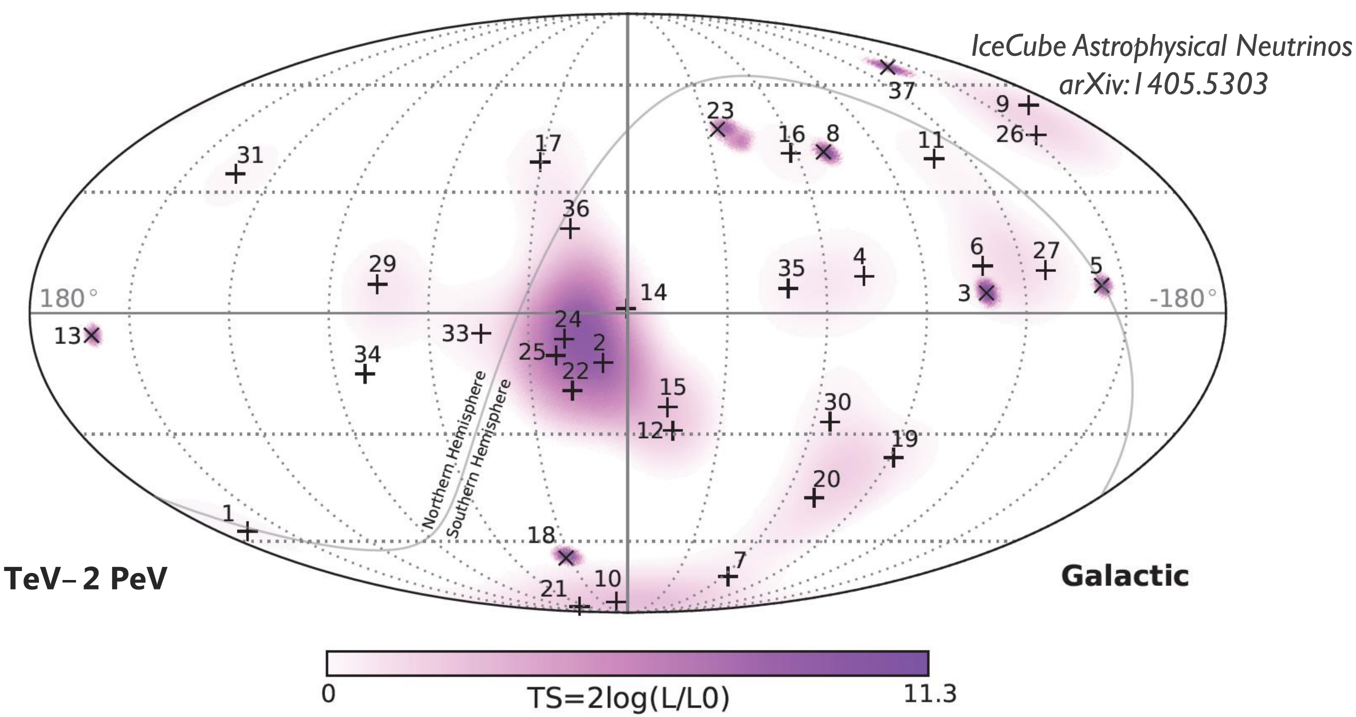
Potential radiation sources!

Specify distribution of a class of events/objects in the sky.

- objects in a redshift range, radiation events in an energy bin, etc.

Angular Distribution Methods

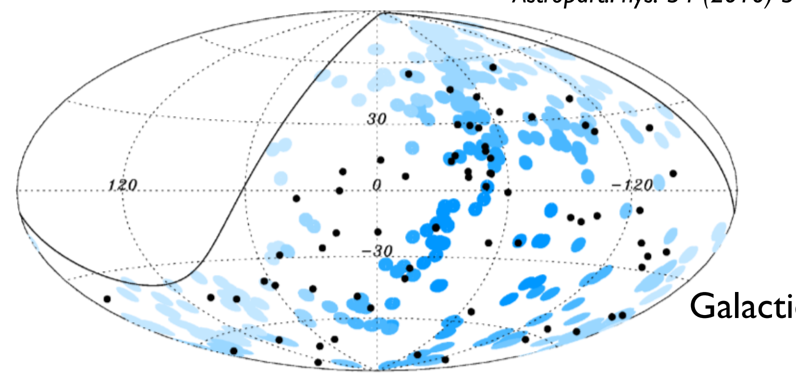
- When point sources cannot be resolved,
  - the angular distribution of **observed events** approaches the angular distribution of **sources** (messenger-propagated and projected) on our sky (full **skymap**).



Resolving Large Scale Structures

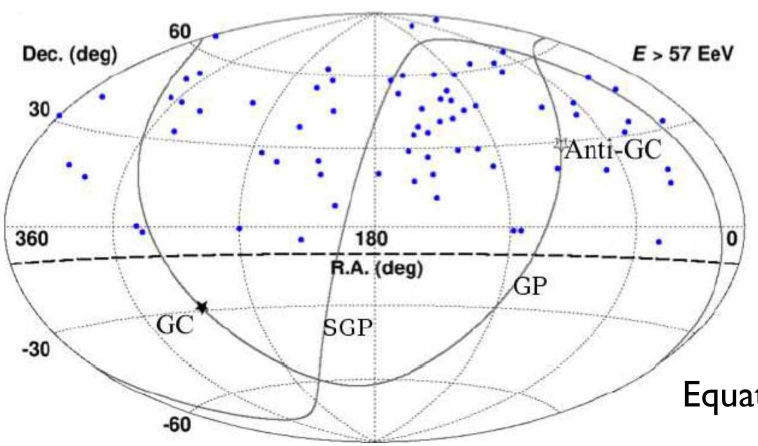
- single nearby source?
- sources pattern?
- structure in propagation medium?

Astropart.Phys. 34 (2010) 314

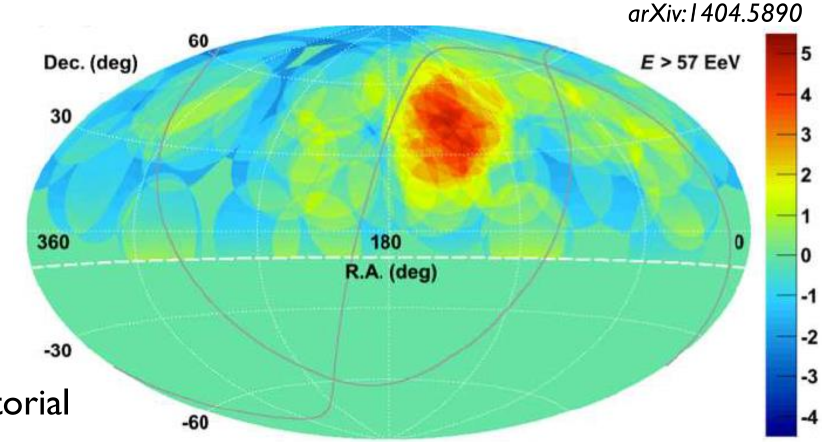


Pierre Auger Cosmic Ray Events (black dots), E>55 EeV.
Compared with VCV AGN catalog (blue dots).

arXiv:1404.5890



Equatorial

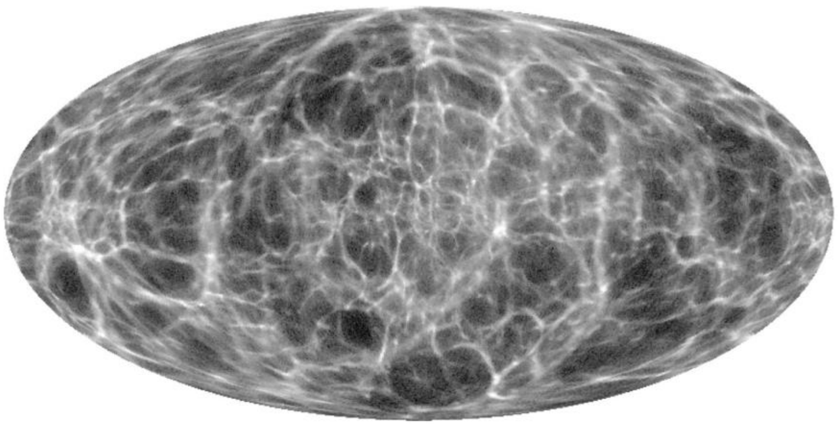


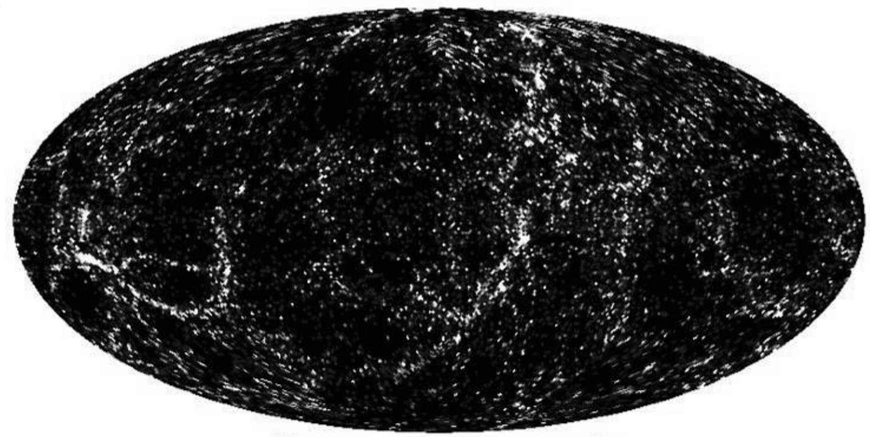
Telescope Array Cosmic Ray Events, E>57 EeV.

Clustering Significance

Distinguishing Dense vs. Sparse

Francisco-Shu Kitaura et al., MNRAS 427, L35 (2012)





Dense Distributions, e.g.,

- radio galaxies
- dark matter annihilation

All events from different source.

Sparse Distributions, e.g.,

- active galactic nuclei
- local extragalactic structure

More sources with multiple events.

Given N events, what can we infer about the full **skymap**?

A Popular Measure of Angular Distribution: The Angular Power Spectrum

Intensity Angular Power Spectrum C_ℓ

$$I(E,\boldsymbol{n}) - \langle I(E)\rangle = \sum_{\ell,m} a_{\ell m}(E) Y_\ell^m(\boldsymbol{n}) \qquad C_\ell(E) = \frac{1}{2\ell+1}\sum_m |a_{\ell m}(E)|^2$$

- Absolute intensity fluctuations.
- Monotonically increases as sources are added.

Fluctuation Angular Power Spectrum $\widetilde{C_\ell}$

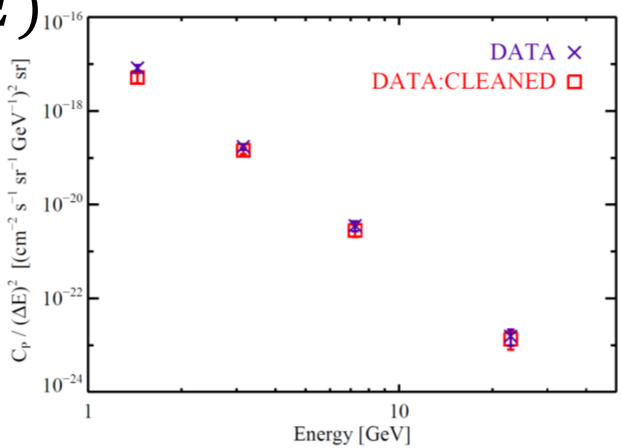
$$\frac{I(E,\boldsymbol{n}) - \langle I(E)\rangle}{\langle I(E)\rangle} = \sum_{\ell,m} \tilde{a}_{\ell m}(E) Y_\ell^m(\boldsymbol{n}) \qquad \widetilde{C_\ell}(E) = \frac{1}{2\ell+1}\sum_m |\tilde{a}_{\ell m}(E)|^2$$

- Relative intensity fluctuations.
- Constant for universal spectrum sources at fixed redshift.

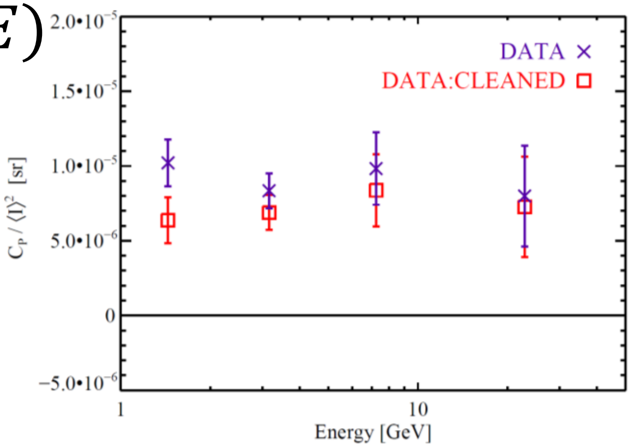
Measurement of Diffuse Gamma-Ray C_ℓ

- First 22 months of Fermi-LAT data.

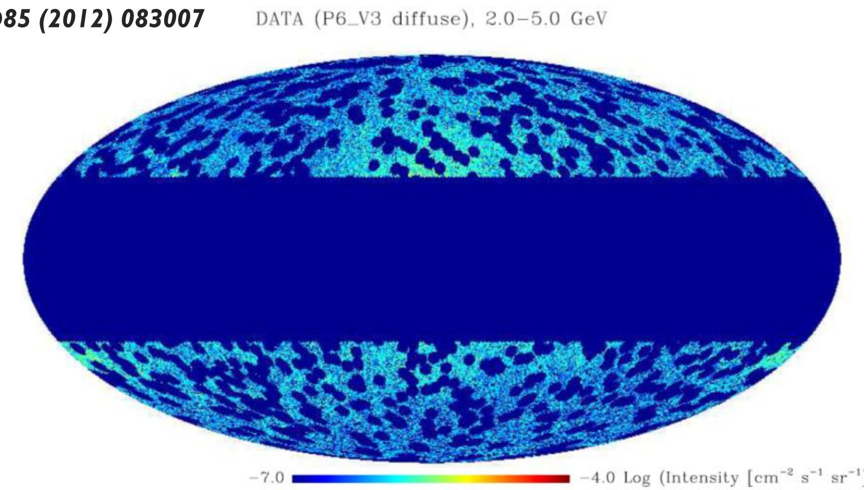
$C_\ell(E)$



$\tilde{C}_\ell(E)$



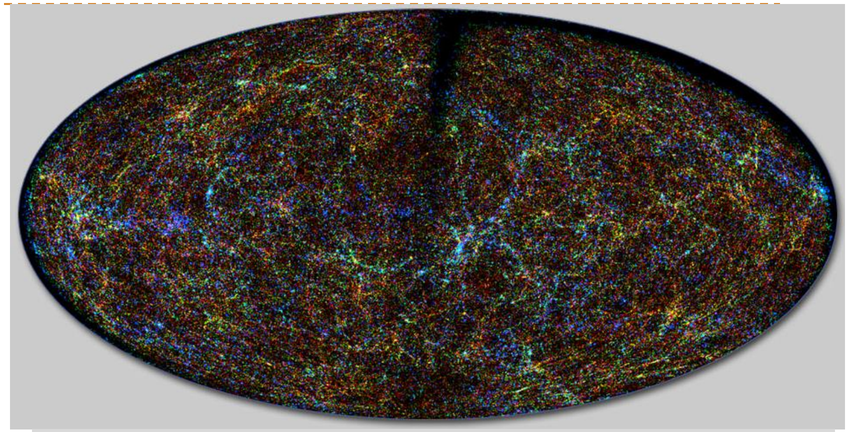
Phys.Rev. D85 (2012) 083007



- Error-weighted means over $155 \leq \ell \leq 504$.
- This already places constraints on models of unresolved gamma-ray point sources.
- Level of precision means it is now important to carefully ensure all effects are properly taken into account.

The Problem

- Let $\tilde{C}_\ell$ be the fluctuation (normalized) APS of a **skymap**– what we are trying to measure.
- Receive N events at random, weighted by the sky map.
- Assume full sky observations with uniform exposure.



A hypothetical projected **skymap** of sources.
The 2 micron sky courtesy of the 2MASS collaboration, http://www.ipac.caltech.edu/2mass/.

- What is the angular power spectrum of the N events, $\tilde{C}_{\ell,N}$, from a full sky map with distribution $\tilde{C}_\ell$?
  - mean of $\tilde{C}_{\ell,N}$?
  - variance of $\tilde{C}_{\ell,N}$?

Angular Clustering of the Source Skymap

- Positive, real function on the sphere $F(\boldsymbol{n})$.
- Normalize: Let $S(\boldsymbol{n}) = \frac{F(\boldsymbol{n})}{\langle F \rangle} - 1$.
- Normalized spherical transform:

$$\tilde{a}_{\ell m} = \int d\boldsymbol{n}\, Y^*_{\ell m}(\boldsymbol{n}) S(\boldsymbol{n})$$

- Angular power spectrum:

$$\tilde{C}_\ell = \sum_{m=-\ell}^{\ell} |\tilde{a}_{\ell m}|^2$$

- Angular bispectrum:

$$\tilde{B}_{\ell_1 \ell_2 \ell_3} = \sum_{m_1, m_2, m_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tilde{a}_{\ell_1 m_1} \tilde{a}_{\ell_2 m_2} \tilde{a}_{\ell_3 m_3}$$

Special Case: Pure Isotropic Source

- Receive N events at uniformly random positions.

$$\tilde{a}_{\ell m,N} = \frac{4\pi}{N} \sum_{i=1}^{N} Y_{\ell m}^{*}(\hat{n}_i) \qquad \tilde{C}_{\ell,N} = \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} \left|\tilde{a}_{\ell m,N}\right|^2$$

$$\langle \tilde{C}_{\ell,N} \rangle = \tilde{C}_{P,N} = \frac{4\pi}{N}$$ Shot noise/Poisson noise.

$$\sigma_{\tilde{C}_{\ell,N}} = \sqrt{\frac{2}{2\ell+1} \frac{4\pi}{N}}$$

Error Estimate with Anisotropic Source

- Lesson from CMB: Cosmic Variance

- The dominant statistical uncertainty in CMB anisotropy.
  - Cosmic Variance ⟺ Unknown Initial Conditions

- Assuming the signal is randomly Gaussian distributed, then our estimator for $\tilde{C}_\ell$ is the maximum likelihood estimator with uncertainty:

$$\sigma_{\tilde{C}_\ell} = \sqrt{\frac{2}{2\ell + 1}}\tilde{C}_\ell$$

“Rule of Thumb” Stat. Uncertainty Est.

- Angular power spectrum from “events”.
- Assume sources are approximately Gaussian distributed.
- Shot noise is a bias to be subtracted from estimator.

$$\hat{\tilde{C}}_{\ell,N} = \frac{1}{2\ell+1}\sum_{m=-\ell}^{\ell}\left|\frac{4\pi}{N}\sum_{i=1}^{N}Y_{\ell m}^{*}(\boldsymbol{n}_i)\right|^2 - \frac{4\pi}{N}$$

$$\sigma_{\hat{\tilde{C}}_{\ell,N}} = \sqrt{\frac{2}{2\ell+1}}\left(\frac{4\pi}{N} + \tilde{C}_\ell\right)$$

Knox, PRD52, 4307 (1995)

- The goal is to check these standard estimates.

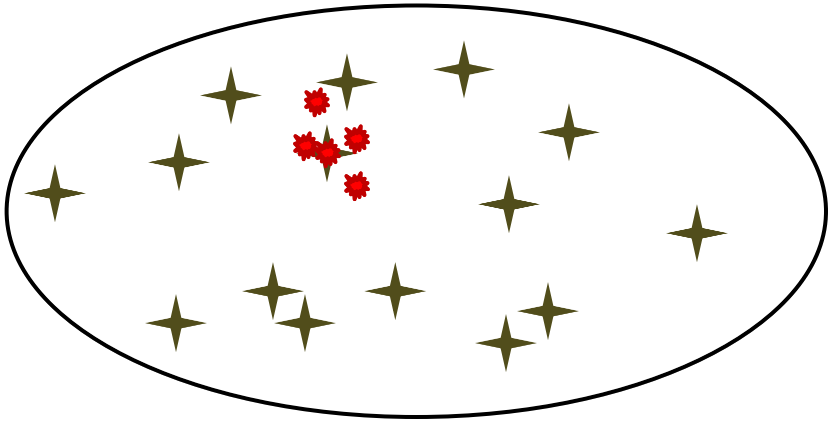
Improving Our Understanding of the Statistical Variance

- Some conceptual difficulties with using the cosmic variance as we did.
  - Cosmic variance is a theoretical error, which applies when making physical inferences about our models based on data.
  - The angular power spectrum measurement should be able to be made independently of any model.
  - We should not need to assume the signal is Gaussian-distributed.

- Investigations have led to a new formula for the model-independent statistical variance of the angular power spectrum of events from a background distribution.

Strategy for Calculation

Consider each event observed at position $\hat{n}'$ but originated from position $\hat{n}$.



1) For fixed source positions $\hat{n}_i$, average over event position $\hat{n}_i{}'$, via the instrument point spread function.

Result of this step: what is being measured is the sky map convolved with the instrument PSF.

2) Average the N events source positions, weighted by the skymap.

Statistical Mean

- The average measurement of $\tilde{C}_{\ell,N}$ from a random sample:

$$\left\langle \tilde{C}_{\ell,N} \right\rangle = \frac{4\pi}{N} + \left(1 - \frac{1}{N}\right) \tilde{C}_{\ell}$$

$\tilde{C}_{\ell}$ is now APS of source skymap, convolved with instrument PSF.

  - Angular power spectrum of events is a *biased* estimator of the source distribution.

- Therefore, an unbiased estimator $\hat{\tilde{C}}_{\ell,N}$ with $\left\langle \hat{\tilde{C}}_{\ell,N} \right\rangle = \tilde{C}_{\ell}$:

$$\hat{\tilde{C}}_{\ell,N} = \frac{1}{1 - \frac{1}{N}} \left[\tilde{C}_{\ell,N} - \frac{4\pi}{N} \right]$$

  - In agreement with previous estimates.

Statistical Variance of $\hat{\hat{C}}_\ell$

$$\sigma^2_{\hat{\hat{C}}_{\ell,N}} = \frac{(4\pi)^2}{N(N-1)}\left[\frac{2}{2\ell+1} + 2\tilde{C}_\ell^{(2)} + 4(N-2)\frac{1}{4\pi}\left(\frac{\tilde{C}_\ell}{2\ell+1} + \tilde{C}_\ell^{(3)}\right) - (4N-6)\left(\frac{\tilde{C}_\ell}{4\pi}\right)^2\right]$$

$$\tilde{C}_\ell^{(2)} = \sum_{\ell'=0}^{2\ell} \frac{2\ell'+1}{4\pi}\begin{pmatrix}\ell & \ell & \ell' \\ 0 & 0 & 0\end{pmatrix}^2 \tilde{C}_{\ell'}$$

$$\tilde{C}_\ell^{(3)} = \frac{1}{2\ell+1}\sum_{\ell'=0}^{2\ell}\sqrt{\frac{2\ell'+1}{4\pi}}\begin{pmatrix}\ell & \ell & \ell' \\ 0 & 0 & 0\end{pmatrix}\tilde{B}_{\ell\ell\ell'}$$

Analytic Work Generated Higher Order Angular Spectra

$$\tilde{C}_\ell^{(2)} = \sum_{\ell'=0}^{2\ell} \frac{2\ell'+1}{4\pi} \begin{pmatrix} \ell & \ell & \ell' \\ 0 & 0 & 0 \end{pmatrix}^2 \tilde{C}_{\ell'}$$

$$\tilde{C}_\ell^{(3)} = \frac{1}{2\ell+1} \sum_{\ell'=0}^{2\ell} \sqrt{\frac{2\ell'+1}{4\pi}} \begin{pmatrix} \ell & \ell & \ell' \\ 0 & 0 & 0 \end{pmatrix} \tilde{B}_{\ell\ell\ell'}$$

$$\tilde{C}_\ell^{(4)} = \tilde{C}_\ell^2$$

I know two ways to see that $\tilde{C}_\ell$ is the first order angular spectrum, and that these comprise the complete set of 2nd order spectra.

Higher Order Spectra: Tensor Picture

- First and Second Rank Spherical Harmonic Transforms of *S*:

$$\tilde{a}_{\ell m} = \int d\boldsymbol{n}\; Y^*_{\ell m}(\boldsymbol{n})\, S(\boldsymbol{n}), \qquad \tilde{a}_{\ell m_1 m_2} = \int d\boldsymbol{n}\, Y^*_{\ell m_1}(\boldsymbol{n}) Y^*_{\ell m_2}(\boldsymbol{n})\, S(\boldsymbol{n})$$

- Raised Azimuthal Indices generated by ${Y_\ell}^m = (-1)^m\, Y^*_{\ell,-m}$:

$$\tilde{a}_{\ell m_1}{}^{m_2} = \int d\boldsymbol{n}\, Y^*_{\ell m_1}(\boldsymbol{n}) {Y_\ell}^{m_2}(\boldsymbol{n})\, S(\boldsymbol{n}) = (-1)^{m_2}\, \tilde{a}_{\ell, m_1, -m_2}$$

- Create rank 0 (rotation invariant) tensors by contracting azimuthal indices:

$$\tilde{C}_\ell = \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} \tilde{a}_\ell{}^m \tilde{a}_{\ell m}$$

Higher Order Spectra: Tensor Picture

- All possible rank 0 tensors from rank 1 and 2 transforms.

$$\tilde{C}_\ell^{(2)} = \frac{1}{(2\ell+1)^2} \sum_{m_1,m_2} \tilde{a}_\ell{}^{m_1 m_2}\ \tilde{a}_{\ell m_1 m_2}$$

$$\tilde{C}_\ell^{(3)} = \frac{1}{(2\ell+1)^2} \sum_{m_1,m_2} \tilde{a}_\ell{}^{m_1 m_2}\ \tilde{a}_{\ell m_1}\ \tilde{a}_{\ell m_2}$$

$$\tilde{C}_\ell^{(4)} = \tilde{C}_\ell^2 = \frac{1}{(2\ell+1)^2} \sum_{m_1,m_2} \tilde{a}_\ell{}^{m_1} \tilde{a}_{\ell m_1} \tilde{a}_\ell{}^{m_2} \tilde{a}_{\ell m_2}$$

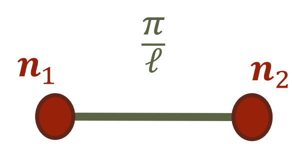
Higher Order Spectra: Field Theory Pic.

- Use the Spherical Harmonic Addition Theorem:

$$\frac{1}{2\ell+1}\sum_m Y_\ell{}^m(\boldsymbol{n}_1)\, Y^*_{\ell m}(\boldsymbol{n}_2) = \frac{1}{4\pi} P_\ell(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2)$$

- Angular Power Spectrum is like 2 field configurations connected by a "correlator".

$$\tilde{C}_\ell = 4\pi \int \frac{d\boldsymbol{n}_1}{4\pi}\frac{d\boldsymbol{n}_2}{4\pi}\, S(\boldsymbol{n}_1) P_\ell(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2)\, S(\boldsymbol{n}_2)$$



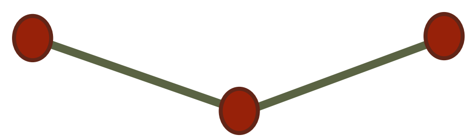
Higher Order Spectra: Field Theory Pic.

- All possible diagrams with 2 correlators.

$$\tilde{C}_\ell^{(2)} = \int \frac{d\boldsymbol{n}_1}{4\pi}\frac{d\boldsymbol{n}_2}{4\pi}\, S(\boldsymbol{n}_1) P_\ell^2(\boldsymbol{n}_1\cdot\boldsymbol{n}_2)\, S(\boldsymbol{n}_2)$$

"Composite Angular Power Spectrum"

$$\tilde{C}_\ell^{(3)} = 4\pi \int \frac{d\boldsymbol{n}_1}{4\pi}\frac{d\boldsymbol{n}_2}{4\pi}\frac{d\boldsymbol{n}_3}{4\pi}\, S(\boldsymbol{n}_1) P_\ell(\boldsymbol{n}_1\cdot\boldsymbol{n}_2)\, S(\boldsymbol{n}_2) P_\ell(\boldsymbol{n}_2\cdot\boldsymbol{n}_3)\, S(\boldsymbol{n}_3)$$



"Open Angular Bispectrum"

$$\tilde{C}_\ell^{(4)} = \tilde{C}_\ell^2$$

"Disjoint Angular Trispectrum"

Statistical Variance of $\hat{\hat{\tilde{C}}}_\ell$

$$\sigma^2_{\hat{\hat{\tilde{C}}}_{\ell,N}} = \frac{(4\pi)^2}{N(N-1)}\left[\frac{2}{2\ell+1} + 2\tilde{C}_\ell^{(2)} + 4(N-2)\frac{1}{4\pi}\left(\frac{\tilde{C}_\ell}{2\ell+1} + \tilde{C}_\ell^{(3)}\right) - (4N-6)\left(\frac{\tilde{C}_\ell}{4\pi}\right)^2\right]$$

$$\approx \left(\frac{4\pi}{N}\right)^2\left[\frac{2}{2\ell+1} + 2\tilde{C}_\ell^{(2)} + \frac{4N}{2\ell+1}\frac{\tilde{C}_\ell}{4\pi} + 4N\frac{\tilde{C}_\ell^{(3)}}{4\pi} - 4N\left(\frac{\tilde{C}_\ell}{4\pi}\right)^2\right],$$

$$\boxed{N \gg 1}$$

Compare to Gaussian Cosmic Variance

- Old method with shot noise + Gaussian cosmic variance:

$$\sigma^2_{\hat{\tilde{C}}_{\ell,N}} = \frac{2}{2\ell+1}\left(\frac{4\pi}{N} + \tilde{C}_\ell\right)^2$$
$$\simeq \left(\frac{4\pi}{N}\right)^2 \left[\frac{2}{2\ell+1} + \frac{4N}{2\ell+1}\frac{\tilde{C}_\ell}{4\pi} + \frac{2N^2}{2\ell+1}\left(\frac{\tilde{C}_\ell}{4\pi}\right)^2\right]$$

- New variance formula:

$$\sigma^2_{\hat{\tilde{C}}_{\ell,N}} \simeq \left(\frac{4\pi}{N}\right)^2 \left[\frac{2}{2\ell+1} + 2\tilde{C}_\ell^{(2)} + \frac{4N}{2\ell+1}\frac{\tilde{C}_\ell}{4\pi} + 4N\frac{\tilde{C}_\ell^{(3)}}{4\pi} - 4N\left(\frac{\tilde{C}_\ell}{4\pi}\right)^2\right]$$

- The new formula agrees surprisingly well with the traditional estimate, with dominant contributions for a weak signal in precise agreement.
- New terms important at large N. Note no N-independent terms!

Gaussian-Distributed Sky Map

- Our results do not assume Gaussianity.
- If the sky map is Gaussian, then higher order spectra are determined from $\tilde{C}_\ell$ as follows:

$$\left\langle \tilde{C}_\ell^{(2)} \right\rangle = \sum_{\ell'=0}^{2\ell} \frac{2\ell'+1}{4\pi} \begin{pmatrix} \ell & \ell & \ell' \\ 0 & 0 & 0 \end{pmatrix}^2 \left\langle \tilde{C}_{\ell'} \right\rangle$$

$$\left\langle \tilde{C}_\ell^{(3)} \right\rangle = 0$$

$$\left\langle \tilde{C}_\ell^{(4)} \right\rangle = \frac{2\ell+3}{2\ell+1} \left\langle \tilde{C}_\ell \right\rangle^2$$

Estimating the Statistical Variance of $\hat{\tilde{C}}_\ell$

$$\sigma^2_{\hat{\tilde{C}}_{\ell,N}} = \frac{(4\pi)^2}{N(N-1)}\left[\frac{2}{2\ell+1} + 2\tilde{C}_\ell^{(2)} + 4(N-2)\frac{1}{4\pi}\left(\frac{\tilde{C}_\ell}{2\ell+1} + \tilde{C}_\ell^{(3)}\right) - (4N-6)\left(\frac{\tilde{C}_\ell}{4\pi}\right)^2\right]$$

Unbiased estimators for all these spectra were determined for N events.

$$\widehat{\sigma^2_{\hat{\tilde{C}}_{\ell,N}}} = \frac{(4\pi)^2}{N(N-1)}\left[\frac{2}{2\ell+1} + 2\hat{\tilde{C}}_{\ell,N}^{(2)} + 4(N-2)\frac{1}{4\pi}\left(\frac{\hat{\tilde{C}}_{\ell,N}}{2\ell+1} + \hat{\tilde{C}}_{\ell,N}^{(3)}\right) - (4N-6)\frac{\widehat{\tilde{C}^2_{\ell,N}}}{(4\pi)^2}\right]$$

Consequences of Findings

- Experiments using Monte Carlo to estimate error already take into account these new effects automatically.
- Experiments using Gaussian Cosmic Variance may be missing higher orders in the uncertainty of angular power.
  - Fermi-LAT anisotropy measurement should check estimators of these terms for possible corrections to their uncertainties.
  - Small χ^2 suggests either their errors should be smaller (possibly due to some more subtle effects) or energy bins are somehow correlated.
- This error analysis must also take into account effects of:
  - non-uniform exposure,
  - sky masking,
  - other observational bias or instrumental effects.

Conclusions

- A new analytic error analysis of angular power spectra of radiation events (γ-rays, etc.) or survey targets (galaxies, AGN, etc.) is presented.
- The unbiased estimator of the source's angular power spectrum is in agreement with usual estimates.
- The uncertainty has the usual shot noise and first order signal contributions, but gives new higher order anisotropy contributions important for large N.
- These results do not assume Gaussianity of signal/sources.
  - Results apply to any event distribution.